11.06.2020

$$f = \sum_{n \in \mathbb{Z}} a_n T^n$$ Laurentreihe

$$f' = \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n n T^{n-1}$$ has no term in degree -1.

- $\operatorname{ord}(f) = -\infty \Rightarrow \operatorname{ord}(f') = -\infty$
- $\operatorname{ord}(f) \in \mathbb{Z} \setminus \{0\} \Rightarrow \operatorname{ord}(f') = \operatorname{ord}(f) - 1$
- $\operatorname{ord}(f) = +\infty \ (f = 0) \Rightarrow \operatorname{ord}(f') = +\infty$.
- $\operatorname{ord}(f) = 0$

$f = 1 \Rightarrow f' = 0 \quad \operatorname{ord}(f') = +\infty$

$f = 1 + T \Rightarrow f' = 1 \quad \operatorname{ord}(f') = 0$

$f = 1 + T^2 \Rightarrow f' = 2T \quad \operatorname{ord}(f') = 1$

$\Downarrow$

$\operatorname{ord}(f') \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$

7.5j $\forall \varepsilon > 0, \quad \{\exp(z^{-1}) \mid z \in \mathbb{C},\ 0 < |z| < \varepsilon\} = \mathbb{C}^*.$

$$\begin{aligned} \mathbb{C}^* &\longrightarrow \mathbb{C} \\ z &\longmapsto \exp(z^{-1}) \end{aligned} \quad \text{holomorph}$$

Casorati-Weierstrass $\Rightarrow$ diese Funktion hat eine wesentliche Singularität in 0.

$\subseteq$) trivial.

$\supseteq$) $w \in \mathbb{C}^*, \quad w = |w| e^{i\theta}, \ \theta \in \mathbb{R}.$

$\forall n \in \mathbb{Z} \quad a_n = \log|w| + i(\theta + 2\pi n) \in \mathbb{C}$

$$e^{a_n} = e^{\log|w| + i(\theta + 2\pi n)} = |w| e^{i\theta} = w. \quad \forall n \in \mathbb{Z}$$

$$|a_n| = \sqrt{(\log|w|)^2 + (\theta + 2\pi n)^2}$$

$$\lim_{n \to +\infty} |a_n| = +\infty$$

$\Rightarrow$ man kann $N \in \mathbb{Z}$ finden, sodass $|a_N| > \frac{1}{\varepsilon}$.

$$z = (a_N)^{-1} \qquad |z| = \frac{1}{|a_N|} < \varepsilon$$

$$\exp(z^{-1}) = \exp(a_N) = w.$$

7.5 b $f \in \mathcal{O}(\mathbb{C})$

$$\forall n \geq 1 \quad \int_{\partial B_2(0)} \frac{f(z)}{(z - n^{-2}e^{in})^2}\,dz = 0.$$

$z_n = n^{-2}e^{in}$, $\quad \lim_{n\to+\infty} z_n = 0$

$|z_n| = \frac{1}{n^2} \leq 1 < 2$

$W(\partial B_2(0), z_n) = 1 \quad \forall n \geq 1.$

Potenzreihenentwicklung von f in z_n

$f'(z_n) =$ Koeffizient von T in der Potenzreihenentw. von f in z_n $= \frac{1}{2\pi i} \int_{\partial B_2(0)} \frac{f(z)}{(z-z_n)^2}\,dz = 0$

$\forall n \geq 1 \quad f'(z_n) = 0.$

$$\left.\begin{array}{l} f' : \mathbb{C} \to \mathbb{C} \text{ stetig} \\ f'(z_n) = 0 \quad \forall n \geq 1 \\ \lim\limits_{n \to +\infty} z_n = 0 \end{array}\right\} \Rightarrow \quad f'(0) = 0$$

f' ist null auf $\{z_n \mid n \geq 1\} \cup \{0\}$, das nicht diskret ist.

$f' : \mathbb{C} \to \mathbb{C}$ analytisch

Analytic continuation

$\Rightarrow f' = 0$ auf $\mathbb{C}$.

$\Rightarrow f$ konstant.

7.5.e $\mathbb{C}$ und $B_1(0)$ sind nicht biholomorph:

By contradiction $\exists f: \mathbb{C} \to B_1(0)$ Biholomorphismus.

$f(\mathbb{C}) = B_1(0) \Rightarrow \forall z \in \mathbb{C}, |f(z)| < 1 \Rightarrow f$ beschränkt

$\overset{\text{Liouville}}{\Longrightarrow}$ f konstant $\Rightarrow$ f nicht bijektiv. $\lightning$

7.5.g $\frac{dz}{z+1}$ $\quad B_1(0) = \{z \in \mathbb{C} \mid |z| < 1\}$ einfach zusammenhängend

$\Rightarrow \frac{dz}{z+1}$ exakt auf $B_1(0)$.

$\Rightarrow \frac{dz}{z+1}$ exakt auf $B_1(0) \setminus \{0\}$

7.2c $f \in \mathcal{O}(\mathbb{C})$ nicht konstant $\Longrightarrow$ $f(\mathbb{C})$ dicht in $\mathbb{C}$.

By contradiction, $\exists w \in \mathbb{C},\ r>0$: $f(\mathbb{C}) \cap B_r(w) = \emptyset$.

$\forall z \in \mathbb{C},\quad f(z) \notin B_r(w) \qquad |f(z)-w| \geq r.$

$g: \mathbb{C} \to \mathbb{C}$
$z \mapsto \dfrac{1}{f(z)-w}$ holomorph $\quad \forall z \in \mathbb{C} \quad |g(z)| = \dfrac{1}{|f(z)-w|} \leq \dfrac{1}{r}$

$\Longrightarrow$ g beschränkt $\overset{\text{Liouville}}{\Longrightarrow}$ g konstant : $\exists a \in \mathbb{C}^*$:
$\forall z \in \mathbb{C},\ a = g(z) = \dfrac{1}{f(z)-w}$

$f(z) - w = a^{-1}$

$\forall z \in \mathbb{C},\ f(z) = w + a^{-1}$

$\Longrightarrow$ f konstant $\quad \lightning$

7.2.d $f, g \in \mathcal{O}(\mathbb{C})$ nicht-konstant $\Longrightarrow$
$g \circ f$ nicht beschränkt.

$g \circ f \in \mathcal{O}(\mathbb{C})$. By Liouville it is enough to prove that $g \circ f$ is not constant.

1. By contradiction $g \circ f$ is constant.

$\exists c \in \mathbb{C} : \forall z \in \mathbb{C} \quad g(f(z)) = c.$

$\forall z \in \mathbb{C}, \quad f(z) \in g^{-1}(c)$

$\underbrace{f(\mathbb{C})}_{\text{connected}} \subseteq \underbrace{g^{-1}(c)}_{\text{discrete}}$

discrete, because g is non constant
Analytic continuation

Consider the zeroes of
$\mathbb{C} \longrightarrow \mathbb{C}$
$z \longmapsto g(z) - c$

$\Rightarrow f(\mathbb{C})$ is a point $\Rightarrow$ f constant $\lightning$

2 f not constant $\overset{7.2.c}{\Longrightarrow}$ $f(\mathbb{C})$ dense in $\mathbb{C}$.

By contradiction $g \circ f$ is constant $\Rightarrow$ $g(f(\mathbb{C})) = \{c\}$ for some $c \in \mathbb{C}$.

$$\left.\begin{array}{l} g \text{ continuous} \\ f(\mathbb{C}) \text{ is dense in } \mathbb{C} \\ \forall w \in f(\mathbb{C}),\ g(w) = c \end{array}\right\} \Rightarrow \forall w \in \mathbb{C},\ g(w) = c \Rightarrow g \text{ constant } \lightning$$

every point in $\mathbb{C}$ is a limit of points in $f(\mathbb{C})$.

7.3 a)

<u>Def</u> $U \subseteq \mathbb{C} = \mathbb{R}^2$ offen. $f: U \to \mathbb{C}$ stetig. f hat MWE, wenn whenever $\overline{B_r(z_0)} \subset U$

$z_0 \in U$
$r > 0$

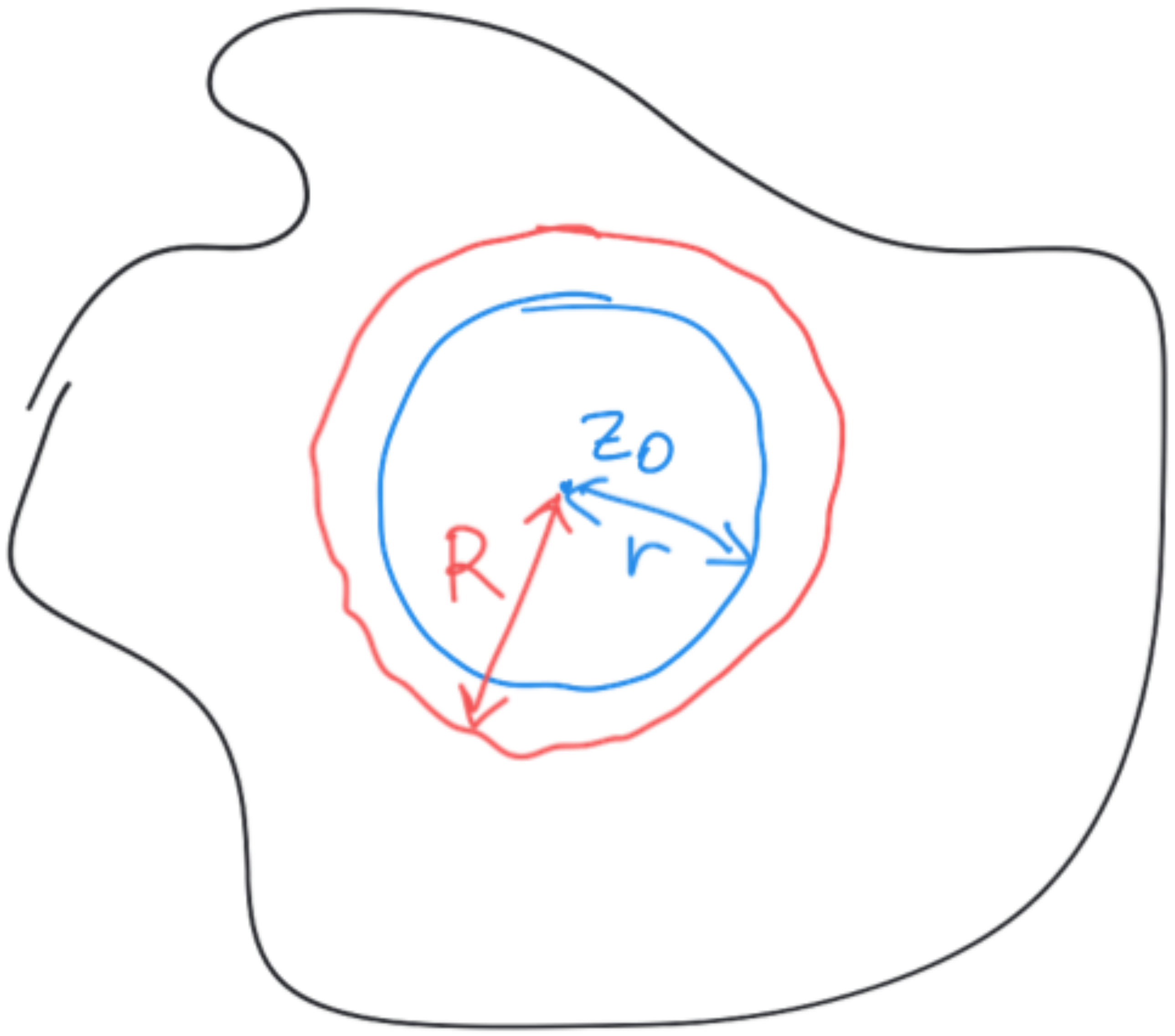
$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$
$$= \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{z - z_0}\, dz$$

- $f \in \mathcal{O}(U) \Rightarrow f$ MWE

- $f: U \to \mathbb{C}$ stetig, dann:
  f MWE $\Longleftrightarrow$ $\mathrm{Re} f$ und $\mathrm{Im} f$ haben MWE

$U \subseteq \mathbb{C}$ open, $u : U \to \mathbb{R}$ harmonic.
u is C^2 (C^∞) and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on U.

$z_0 \in U$, $r > 0$, $\overline{B_r(z_0)} \subset U$



Can find $R > r$ s.t.

$$B_R(z_0) \subseteq U.$$

$B_R(z_0)$ simply connected
$\overset{6.3c}{\Longrightarrow}$ $\exists f \in \mathcal{O}(B_R(z_0))$ s.t.
$\mathrm{Re}\, f = u|_{B_R(z_0)}$.

$\Rightarrow$ $u|_{B_R(z_0)}$ has MWE $\Rightarrow$

$$\Rightarrow u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt.$$

Now I give a proof of the yellow statement.

<u>Lemma</u> (X,d) metric space. $Y \subseteq X$ closed.

$g : X \longrightarrow [0,+\infty)$ "distance from Y"

$x \longmapsto \inf\{ d(x,y) \mid y \in Y \}$

Then: i) g is continuous

ii) $\forall x \in X$, $g(x) = 0 \iff x \in Y$.

<u>Pf</u> ii) because Y is closed in X.

i) g is Lipschitz: $\forall x_1, x_2 \in X$ $|g(x_1) - g(x_2)| \leq d(x_1, x_2)$

$x_1, x_2 \in X$. $\varepsilon > 0$: by the property of inf

$\exists y \in Y$: $g(x_1) + \varepsilon \geq d(x_1, y)$

$\Rightarrow g(x_2) \leq d(x_2, y) \leq d(x_2, x_1) + d(x_1, y) \leq d(x_2, x_1) + g(x_1) + \varepsilon$

$\Rightarrow g(x_2) - g(x_1) \leq d(x_2, x_1) + \varepsilon$.

ε arbitrary $\Rightarrow g(x_2) - g(x_1) \leq d(x_2, x_1)$. Now swap x_1 and x_2.

$\square$

$U \subseteq \mathbb{C}$ open, $z_0 \in U$, $r > 0$, $\overline{B_r(z_0)} \subset U$.

Consider the distance from $\mathbb{C} \setminus U$, which is closed in $\mathbb{C}$

$$g : \mathbb{C} \to [0, +\infty)$$
$$z \mapsto \inf \{ |z - w| \mid w \in \mathbb{C} \setminus U \}$$

g is continuous

$\overline{B_r(z_0)} \cap (\mathbb{C} \setminus U) = \emptyset \Rightarrow \forall z \in \overline{B_r(z_0)}, \; g(z) > 0$

$\overline{B_r(z_0)}$ compact $\Rightarrow g|_{\overline{B_r(z_0)}}$ has a minimum > 0

Can find $\varepsilon > 0$ s.t. $\forall z \in \overline{B_r(z_0)} \quad g(z) \geq \varepsilon$.

$\forall z \in \overline{B_r(z_0)}, \forall w \in \mathbb{C} \setminus U \quad |z - w| \geq \varepsilon$

$\ldots \Rightarrow \quad B_{r+\varepsilon}(z_0) \subseteq U \qquad R = r + \varepsilon. \qquad \square$

contractible
zusammenziehbar
(homotopieäquivalent
zu einem Punkt)

property of topological
space

(*)

Schleifen / loops

pathhomotopic
weghomotop

$\Downarrow$

they have the
same base points

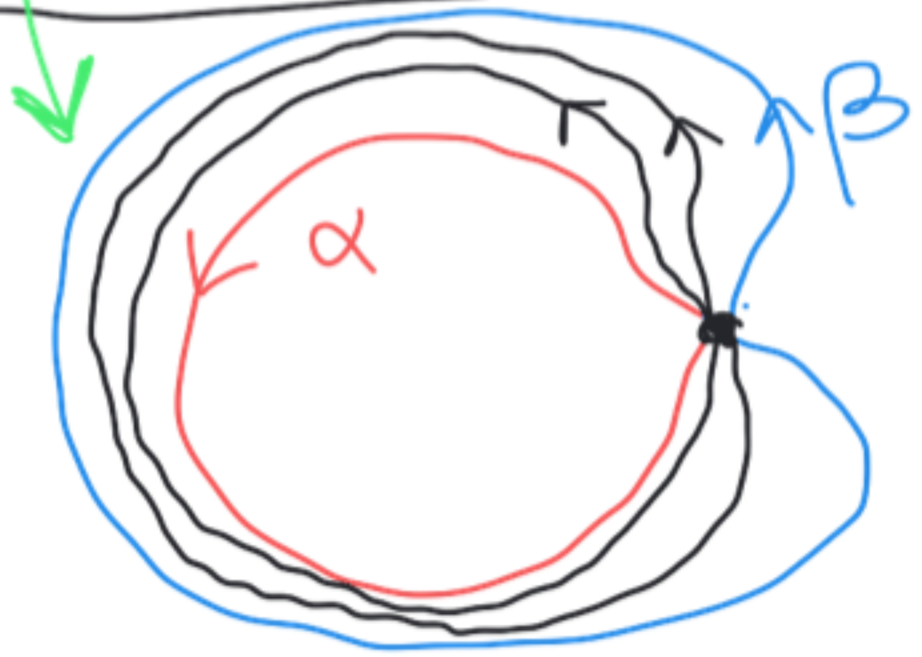
homotopic
homotop

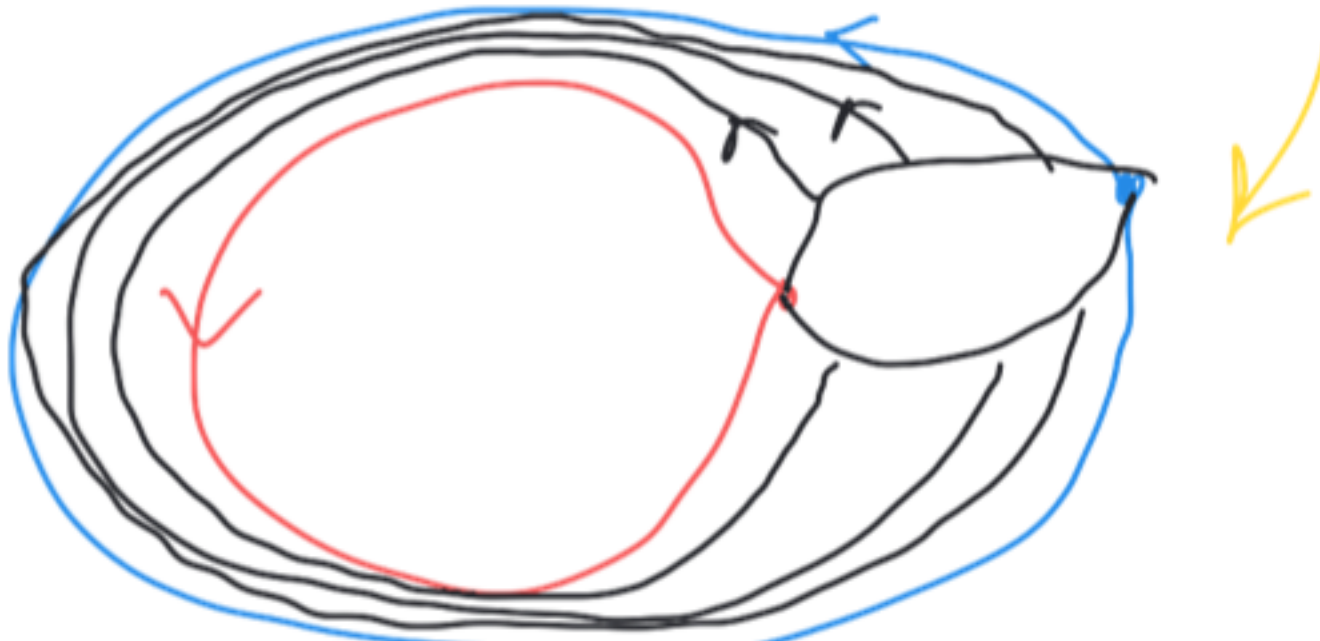
homologous
homolog

X top. space. $\alpha, \beta : [0,1] \to X$ Schleifen/loops
(Weg/path, $\alpha(0) = \alpha(1)$, $\beta(0) = \beta(1)$)

- α, β pathhomotopic if they are homotopic relatively to $\{0,1\}$:
  $\exists F : [0,1] \times [0,1] \to X$ cont. $F(\cdot, 0) = \alpha$, $F(\cdot, 1) = \beta$, $F(0, \cdot)$, $F(1, \cdot)$ constant
- α, β homotopic if
  $\exists F : [0,1] \times [0,1] \to X$ cont. $F(\cdot, 0) = \alpha$, $F(\cdot, 1) = \beta$

$F(\cdot, s)$ might not be a loop





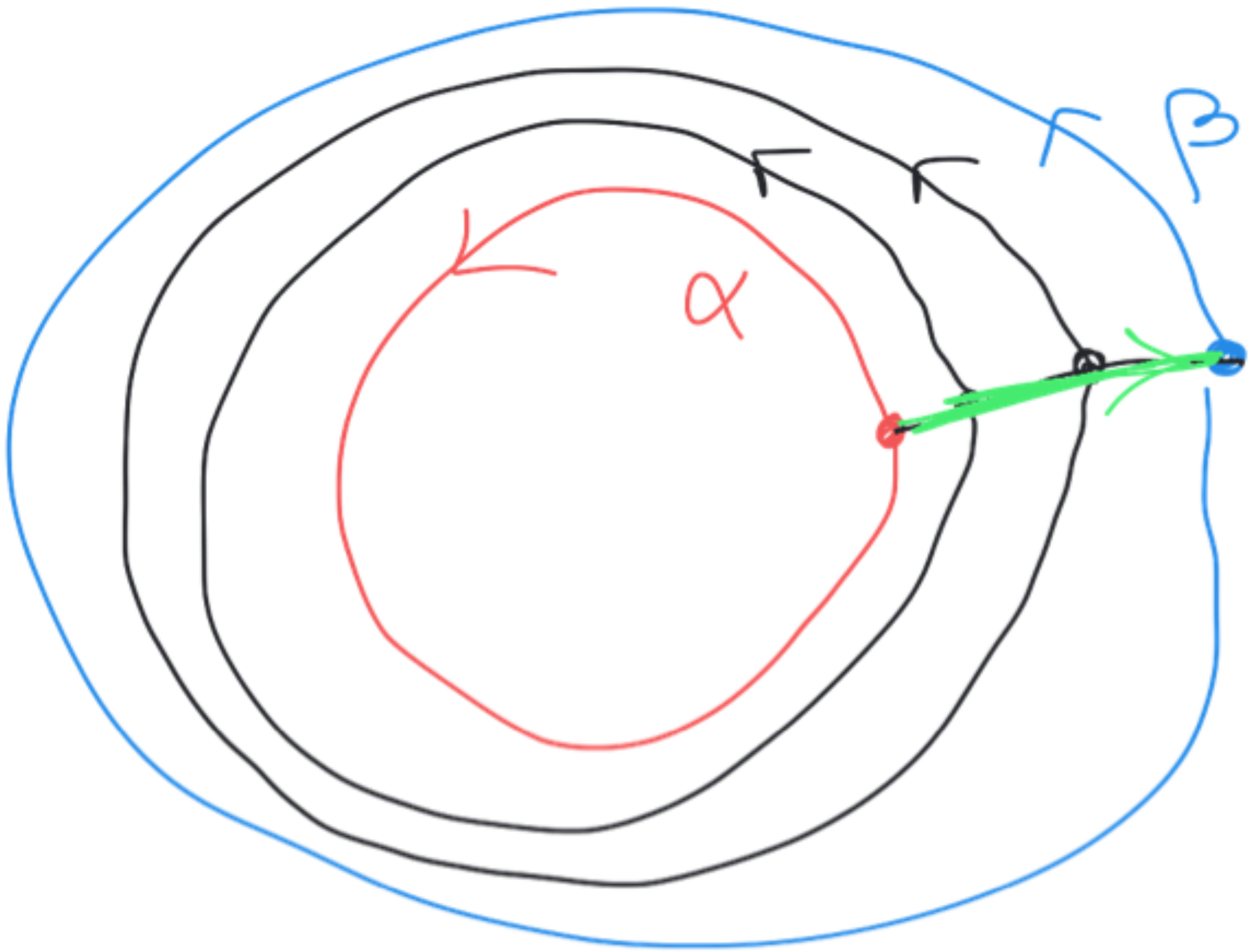
$\circledast$ $\exists F : [0,1] \times [0,1] \to X$ cont.

$F(\cdot, 0) = \alpha$

$F(\cdot, 1) = \beta$

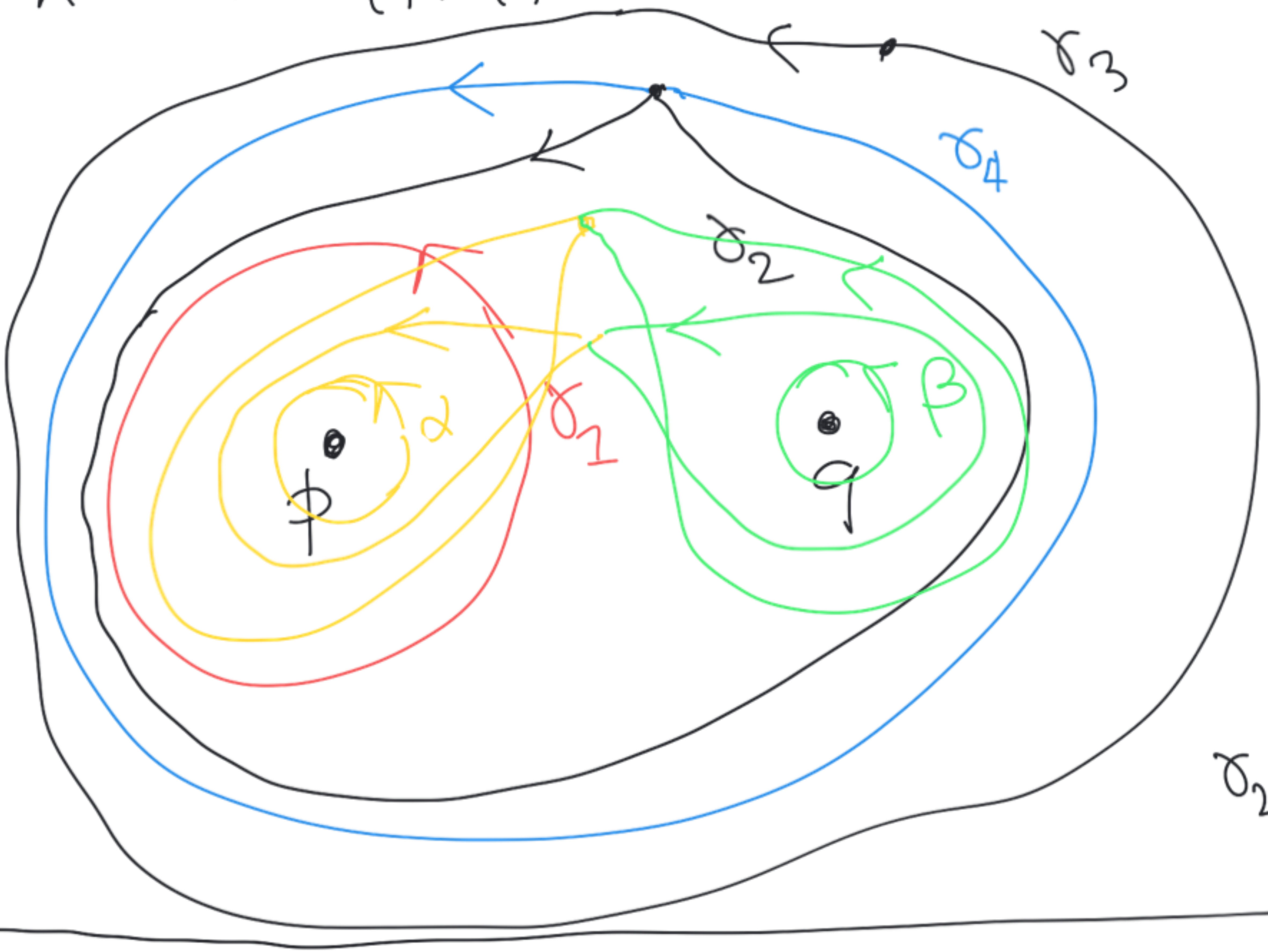
$\forall s \in [0,1]$ $F(\cdot, s)$ is a loop

$F(0,s) = F(1,s)$



$F(0, \cdot) = F(1, \cdot)$

$X = \mathbb{C} \setminus \{p, q\}$



Everything respect to X:

α and γ_1 $(*)$ $\Longrightarrow$ α and γ_1 are homologous

α and γ_1 are not path homotopic

γ_2 and γ_2 are pathhomotopi

γ_2 and γ_3 $(*)$ $\Longrightarrow$ γ_2 and γ_3 are homologou

γ_2 is homologous to $\alpha + \beta$

$\mathbb{C} \setminus \{p\}$: γ_4 and γ_1 are homologous.

$$\sin z = z - \frac{z^3}{6} + \underbrace{\frac{z^5}{?}}_{} + \cdots$$

Tips for 8.4: don't do calculations!

$f \in \mathcal{O}(B_r(z_0)) \Longrightarrow$ the convergence radius of the power series expansion of f in z_0 is $\geq r$

(this is a consequence of Cauchy).

$B \subset \overline{B}$ all functions have values in $\mathbb{R}$)

$$
\begin{array}{ccc}
C(\overline{B}) & \hookrightarrow & C(B) \\
 & & \cup\!\!| \\
 & & C^2(B)
\end{array}
$$