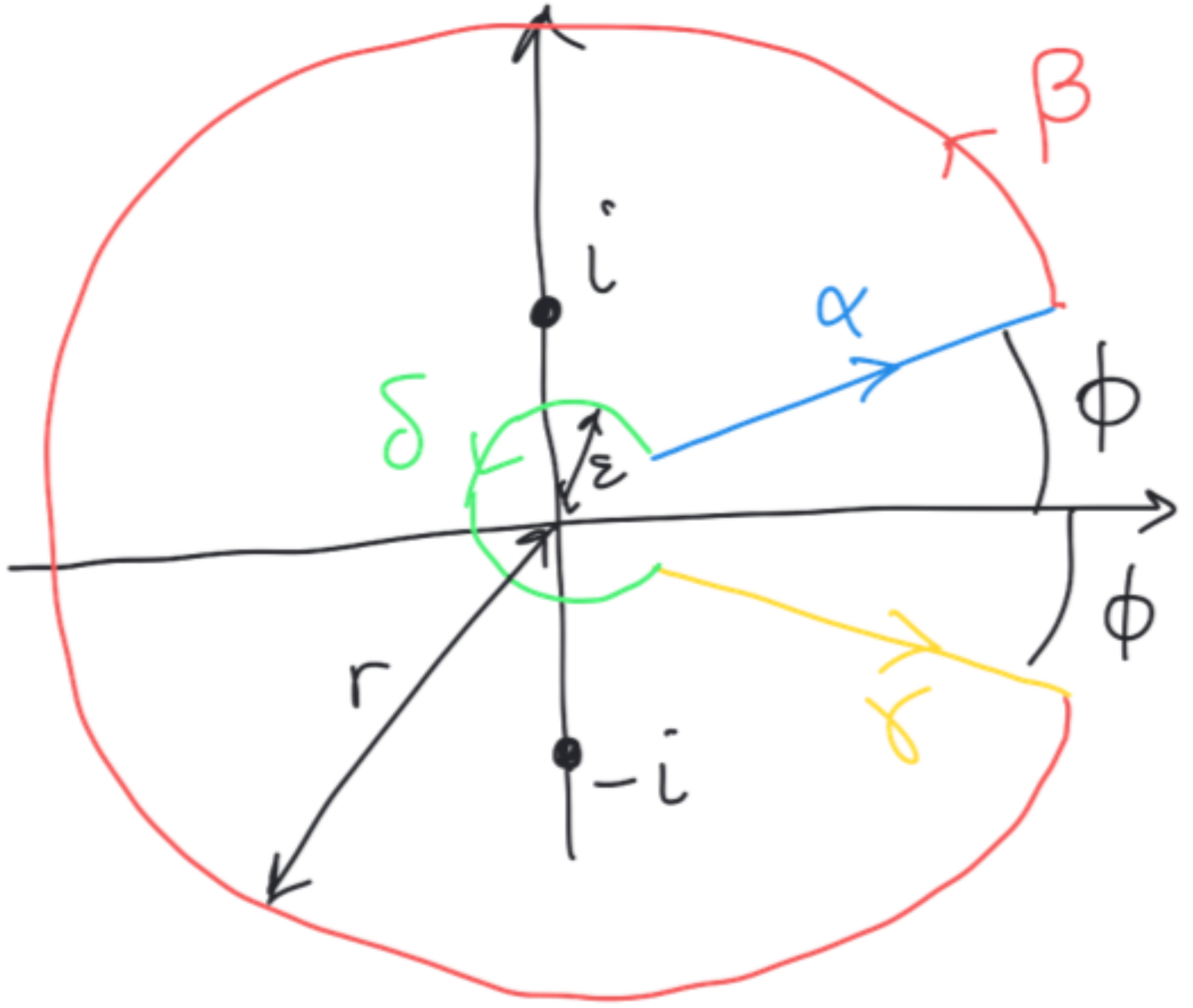
18.06.2020

8.5

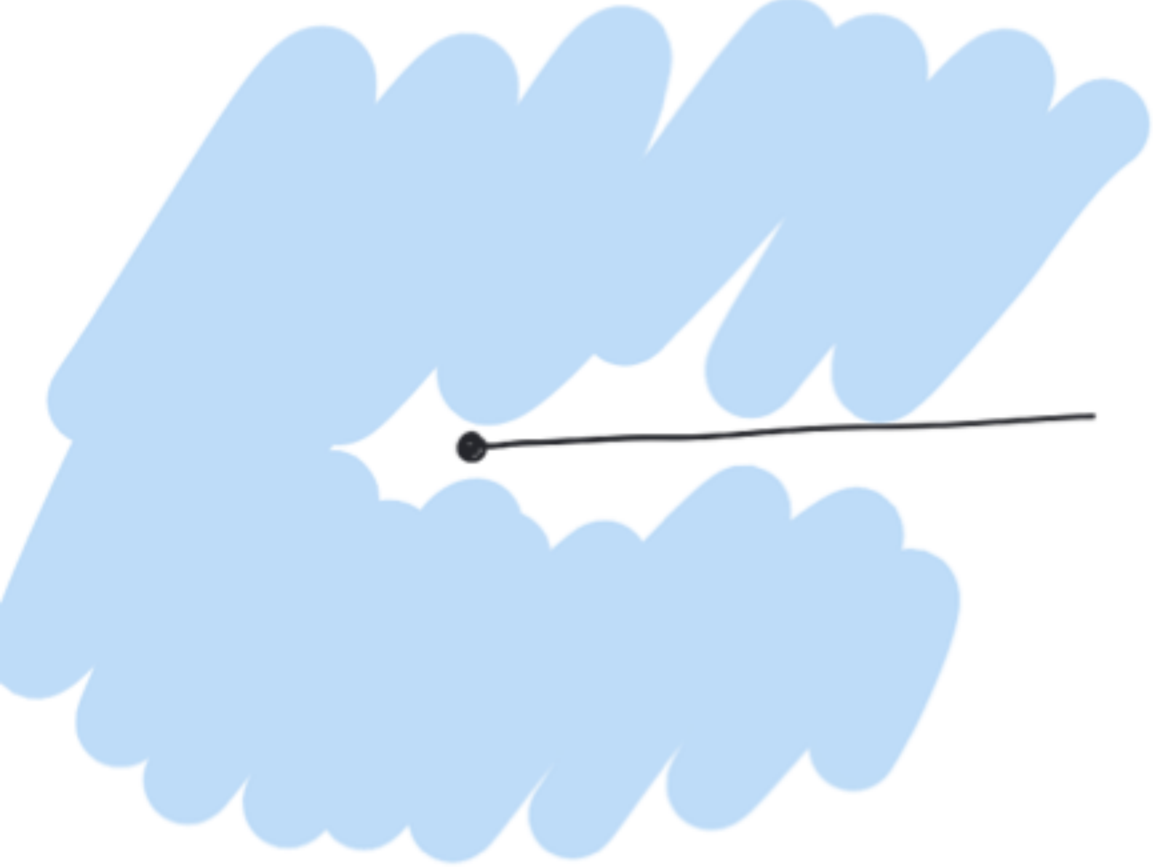
$$I = \int_0^{+\infty} \frac{\sqrt[n]{x}}{x^2+1}\, dx \qquad n \geq 2 \qquad \zeta = e^{i\frac{\pi}{2n}}$$

i) $0<\varepsilon<1$, $r>1$, $0<\phi<\frac{\pi}{2}$



$\alpha * \beta * \iota(\gamma) * \iota(\delta)$ ist eine Schleife.

8.5. ii) $U = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$



$\forall z \in U$, $\exists!\, r > 0$ and $\theta \in (0, 2\pi)$
s.d. $z = re^{i\theta}$

$$g : U \longrightarrow \mathbb{C}$$
$$re^{i\theta} \longmapsto \sqrt[n]{r}\, e^{i\frac{\theta}{n}} \qquad \forall r > 0,\ \forall \theta \in (0, 2\pi)$$

g ist wohldefiniert.

$$v : U \longrightarrow \mathbb{R}$$
$$x + iy \longmapsto \begin{cases} \frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & y > 0 \\ \pi + \arctan\left(\frac{y}{x}\right) & x < 0 \\ \frac{3}{2}\pi - \arctan\left(\frac{x}{y}\right) & y < 0 \end{cases} \qquad \forall x, y \in \mathbb{R} \text{ mit } x + iy \in U$$

$l : U \to \mathbb{C}$

$x+iy \longmapsto \frac{1}{2}\log(x^2+y^2) + i\,\sigma(x,y) \quad \forall x,y \in \mathbb{R}$ mit $x+iy \in U$

l erfüllt C.R. $\Rightarrow$ l holomorph.

$g(z) = e^{\frac{1}{n} l(z)}$ composition of holomorphic functions

$\forall z \in U$

$\Rightarrow$ g ist holomorph.

$$g(i) = g(e^{i\frac{\pi}{2}}) = e^{i\frac{\pi}{2n}} = \zeta$$

$$g(-i) = g(e^{-i\frac{\pi}{2}}) \neq e^{-i\frac{\pi}{2n}}$$

$$g(-i) = g(e^{i\frac{3}{2}\pi}) = e^{i\frac{3\pi}{2n}} = \zeta^3$$

8.5.iii) $0<\varepsilon<1$, $r>1$, $0<\phi<\frac{\pi}{2}$.

$$\eta_{\varepsilon,r,\phi} = \alpha_{\varepsilon,r,\phi} * \beta_{\varepsilon,r,\phi} * \iota(\gamma_{\varepsilon,r,\phi}) * \iota(\delta_{\varepsilon,r,\phi})$$

Schleife in $U\setminus\{i,-i\}$, nullhomolog in $U=\mathbb{C}\setminus\mathbb{R}_{\geq 0}$

$\omega = \frac{g(z)}{z^2+1}dz$ lokal exakt auf $U\setminus\{i,-i\}$

$$\omega = \frac{i}{2}\frac{g(z)}{z+i}dz - \frac{i}{2}\frac{g(z)}{z-i}dz$$

Cauchyische Integralformel $\Longrightarrow$

$$W(\eta_{\varepsilon,r,\phi}, i) = W(\eta_{\varepsilon,r,\phi}, -i) = 1$$

$$\int_{\eta_{\varepsilon,r,\phi}} \omega = \frac{i}{2}\cdot 2\pi i\cdot g(-i) - \frac{i}{2}2\pi i\, g(i) = \pi(\zeta-\zeta^3).$$

<u>Def</u> X Menge, (Y,d) met. Raum.

$(f_n : X \to Y)_{n \geq 0}$, $f : X \to Y$.

- (f_n) konvergiert PUNKTWEISE zu f, wenn

  $\forall x \in X, \ \lim\limits_{n \to +\infty} f_n(x) = f(x)$, d.h.

  $\forall x \in X, \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \ d(f_n(x), f(x)) < \varepsilon$

  $\forall \varepsilon > 0, \forall x \in X, \exists N \in \mathbb{N} : \forall n \geq N, \ d(f_n(x), f(x)) < \varepsilon.$

- (f_n) konvergiert GLEICHMÄßIG zu f, wenn

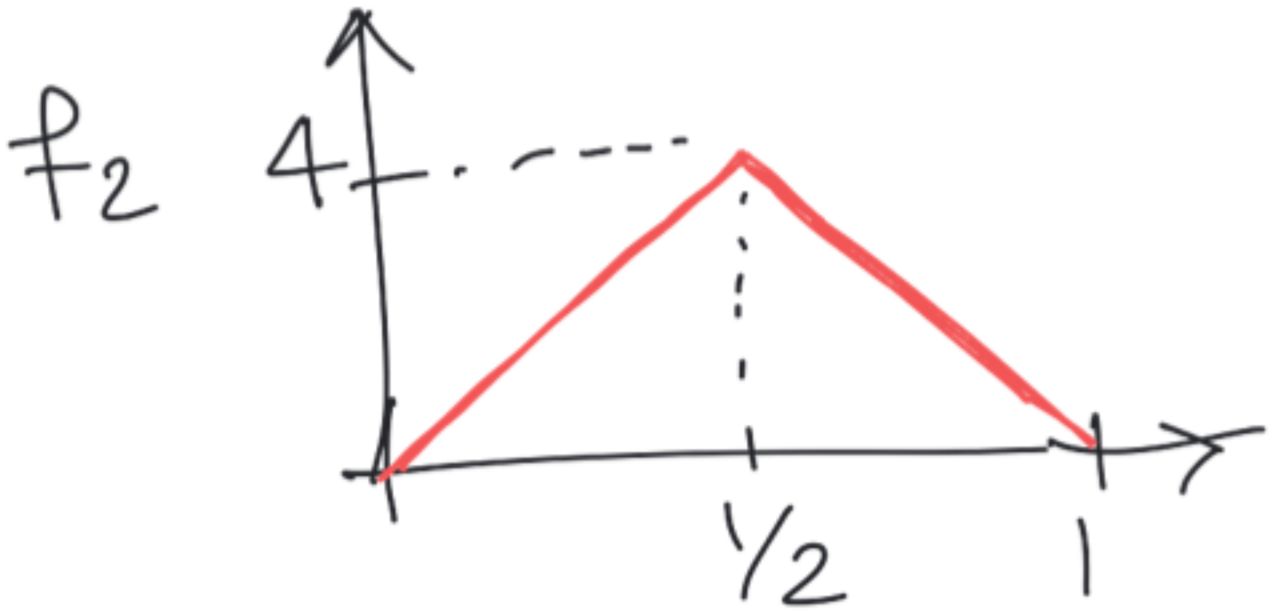
  $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall x \in X, \forall n \geq N, \ d(f_n(x), f(x)) < \varepsilon$

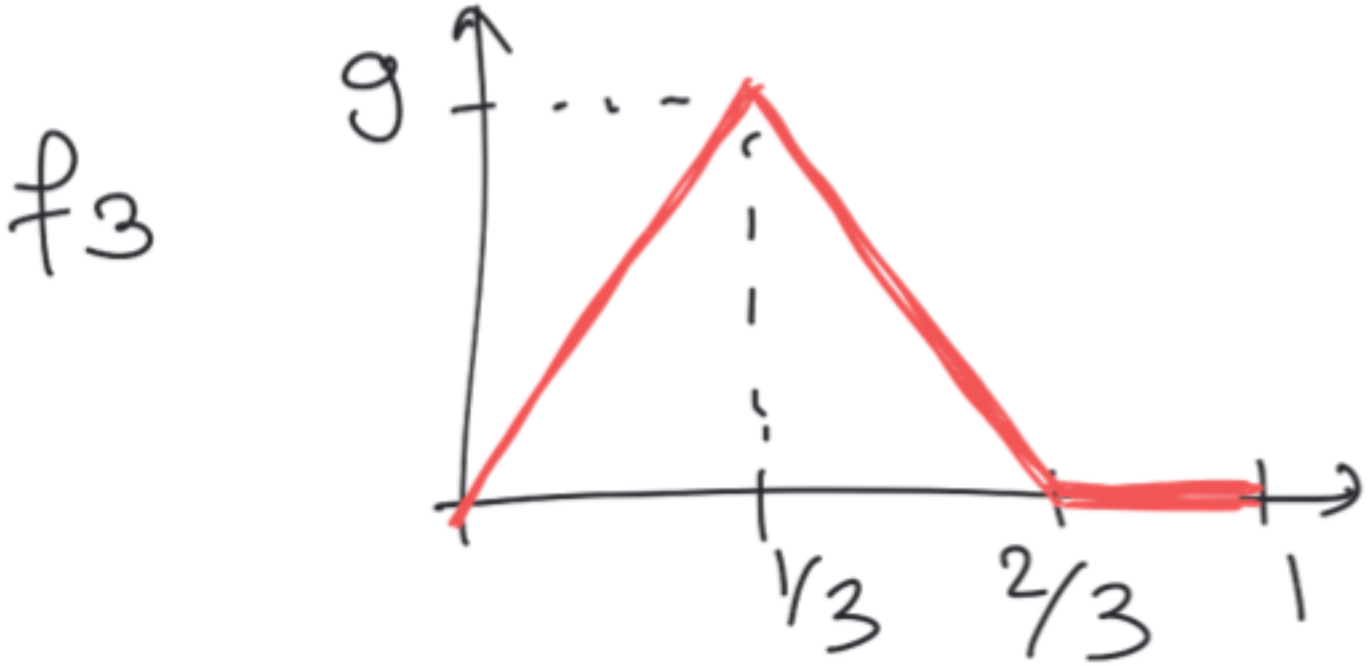
  $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in X, \ d(f_n(x), f(x)) < \varepsilon$

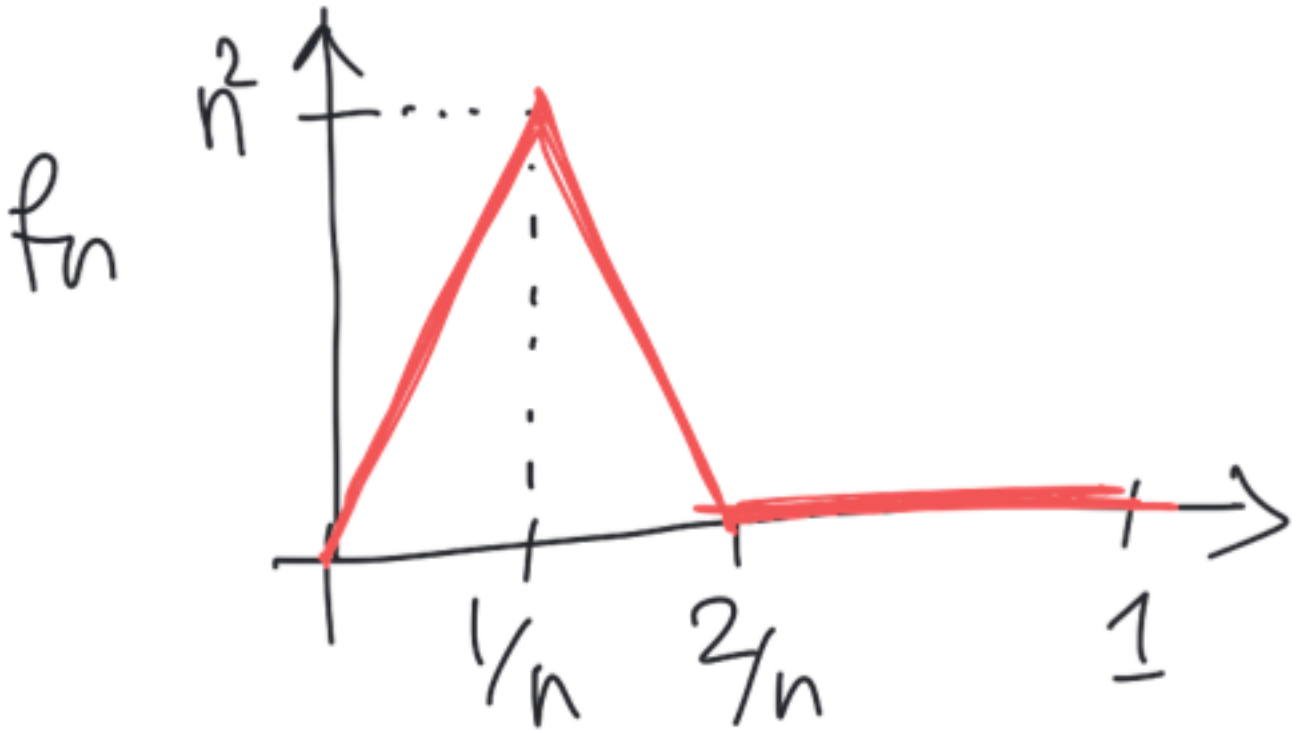
  $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \ \sup\limits_{x \in X} d(f_n(x), f(x)) \leq \varepsilon$

  $\lim\limits_{n \to +\infty} \sup\limits_{x \in X} d(f_n(x), f(x)) = 0.$

Beispiel $f_n : [0,1] \longrightarrow \mathbb{R}$ $\quad n \geq 2$







$$f_n(x) = \begin{cases} n^3 x & 0 \leq x \leq \frac{1}{n} \\ -n^3 x + 2n^2 & \frac{1}{n} \leq x \ \frac{2}{n} \\ 0 & x \geq \frac{2}{n} \end{cases}$$

f_n stetig

(f_n) konvergiert punktweise nach 0:

- $\forall n \geq 2 \quad f_n(0) = 0$
- $\forall x \in (0,1]$, $\exists N \in \mathbb{N} : \forall n \geq N$, $\quad f_n(x) = 0$

$$\frac{2}{N} \leq x \quad N \geq \frac{2}{x}$$

$\Rightarrow \lim\limits_{n \to +\infty} f_n(x) = 0 \; \forall x \in X.$

(f_n) konvergiert gleichmäßig nicht zu 0:

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = n^2$$

$$\lim_{n \to +\infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| \neq 0.$$

$$\int_0^1 f_n(x)\,dx = n$$

Satz $f_n : [a,b] \to \mathbb{R}$ stetig, konvergiert gleichmäßig zu $f: [a,b] \to \mathbb{R}$

$\Rightarrow f$ ist stetig und $\int_a^b f_n(t)\,dt \xrightarrow{n \to +\infty} \int_a^b f(t)\,dt$

Bws

$$\left| \int_a^b f(t)\,dt - \int_a^b f_n(t)\,dt \right| \leq \int_a^b |f_n(t) - f(t)|\,dt \leq$$

$$\leq \int_a^b \sup_{x \in [a,b]} |f_n(x) - f(x)| \cdot dt = (b-a) \cdot \underbrace{\sup_{x \in [a,b]} |f_n(x) - f(x)|}_{\xrightarrow{n \to +\infty}\, 0}$$

8.5.iv $0<\varepsilon<1<r, \quad 0<\phi<\frac{\pi}{2}$

$$f_\phi : [\varepsilon, r] \longrightarrow \mathbb{C}, \quad t \longmapsto \frac{\sqrt[n]{t}\, e^{i\frac{\phi}{n}}}{t^2 e^{2i\phi}+1}$$

$$f : [\varepsilon, r] \longrightarrow \mathbb{C}, \quad t \longmapsto \frac{\sqrt[n]{t}}{t^2+1}$$

$$\forall t \in [\varepsilon, r], \quad \lim_{\phi \to 0^+} f_\phi(t) = f(t) \quad :$$ konvergiert punktweise

konvergiert gleichmäßig?

$\forall t \in [\varepsilon, r]$

$$|f_\phi(t) - f(t)| = \left| \frac{\sqrt[n]{t}\, e^{i\frac{\phi}{n}} t^2 + \sqrt[n]{t}\, e^{i\frac{\phi}{n}} - \sqrt[n]{t}\, e^{2i\phi} t^2 - \sqrt[n]{t}}{(t^2 e^{2i\phi}+1)(t^2+1)} \right| \leq$$

$$\underset{\substack{\uparrow \\ t \geq \varepsilon}}{\leq} \frac{\sqrt[n]{t} \cdot t^2 \cdot \left| e^{i\frac{\phi}{n}} - e^{2i\phi} \right| + \sqrt[n]{t} \left| e^{i\frac{\phi}{n}} - 1 \right|}{(1-\varepsilon^2)^2} \leq$$

$$\underset{\substack{\uparrow \\ t \leq r}}{\leq} \frac{\sqrt[n]{r} \cdot r^2 \cdot \left| e^{i\frac{\phi}{n}} - e^{2i\phi} \right| + \sqrt[n]{r} \left| e^{i\frac{\phi}{n}} - 1 \right|}{(1-\varepsilon^2)^2}$$

$$\sup_{t \in [\varepsilon, r]} |f_\phi(t) - f(t)| \leq \underbrace{\frac{\sqrt[n]{r}\, r^2 \left| e^{i\frac{\phi}{n}} - e^{2i\phi} \right| + \sqrt[n]{r} \left| e^{i\frac{\phi}{n}} - 1 \right|}{(1-\varepsilon^2)^2}}_{\substack{\downarrow \text{ wenn } \phi \to 0^+ \\ 0}}$$

$$\Longrightarrow \lim_{\phi \to 0^+} \sup_{t \in [\varepsilon, r]} |f_\phi(t) - f(t)| = 0. \qquad \int_\varepsilon^r f_\phi(t)\,dt \xrightarrow{\phi \to 0^+} \int_\varepsilon^r f(t)\,dt$$

$f_n : [a,b] \longrightarrow \mathbb{R}$ konvergiert punktweise nach f

f_n stetig

f stetig

$\forall t \in [a,b]$, $\forall n \geq 0$, $f_n(t) \leq f_{n+1}(t)$

$\Rightarrow$ (f_n) konvergiert gleichmäßig.

8.5.v

$$\int_{\alpha_{\varepsilon,r,\phi}} \omega = \int_\varepsilon^r f_\phi(t)\,dt \xrightarrow[\uparrow \atop 8.5.\text{iv}.]{\phi\to 0^+} \int_\varepsilon^r \frac{\sqrt[n]{t}}{t^2+1}\,dt.$$

8.5.vi $\quad \gamma_{\varepsilon,r,\phi}(t) = te^{-i\phi} \qquad \varepsilon\le t\le r$

$$\int_{\gamma_{\varepsilon,r,\phi}} \omega = \int_\varepsilon^r \frac{g(te^{-i\phi})}{(te^{-i\phi})^2+1}\,e^{-i\phi}\,dt = \int_\varepsilon^r \frac{g(t\,e^{i(2\pi-\phi)})}{(te^{-i\phi})^2+1}\,e^{-i\phi}dt$$

$$= e^{-i\phi}\cdot\int_\varepsilon^r \frac{\sqrt[n]{t}\,e^{i\frac{2\pi-\phi}{n}}}{(t^2e^{-i\phi})^2+1}\,dt = e^{-i\phi}\cdot e^{i\frac{2\pi}{n}}\cdot\int_\varepsilon^r \frac{\sqrt[n]{t}\,e^{-i\frac{\phi}{n}}}{(t^2e^{-i\phi})^2+1}\,dt$$

$$\xrightarrow{\phi\to 0^+} e^{i\frac{2\pi}{n}}\cdot\int_\varepsilon^r \frac{\sqrt[n]{t}}{t^2+1}\,dt = \zeta^4\cdot\int_\varepsilon^r \frac{\sqrt[n]{t}}{t^2+1}\,dt$$

8.5.vii : einfach.

8.5. viii : $0<\varepsilon<1$, $r>1$ fest.

$$\lim_{\phi\to 0^+} \int_{\beta_{\varepsilon,r,\phi}} \omega \ , \quad \lim_{\phi\to 0^+} \int_{\delta_{\varepsilon,r,\phi}} \omega \quad \text{existieren}$$

$$\lim_{\phi\to 0^+} \int_{\beta_{\varepsilon,r,\phi}} \omega = \lim_{\phi\to 0^+} \int_{\phi}^{2\pi-\phi} \frac{\sqrt[n]{r}\, e^{i\frac{t}{n}}}{r^2 e^{2it}+1}\, r e^{it}\, i\, dt$$

$$= \int_0^{2\pi} \frac{\sqrt[n]{r}\, e^{i\frac{t}{n}}}{r^2 e^{2it}+1}\, r e^{it}\, i\, dt$$

8.5. ix einfach

8.5 x $$\int_{\alpha_{\varepsilon,r,\phi}} \omega + \int_{\beta_{\varepsilon,r,\phi}} \omega - \int_{\gamma_{\varepsilon,r,\phi}} \omega - \int_{\delta_{\varepsilon,r,\phi}} \omega = \pi(\zeta-\zeta^3)$$

$\phi \to 0^+$:

$$\int_\varepsilon^r \frac{\sqrt[n]{t}}{t^2+1}\,dt + \int_0^{2\pi} \cdots - \zeta^4 \cdot \int_\varepsilon^r \frac{\sqrt[n]{t}}{t^2+1}\,dt - \int_0^{2\pi} \cdots = \pi(\zeta-\zeta^3)$$

$\varepsilon \to 0^+$, $r \to +\infty$

$$(1-\zeta^4)\cdot\int_0^{+\infty} \frac{\sqrt[n]{t}}{t^2+1}\,dt = \pi(\zeta-\zeta^3) \Longrightarrow \int_0^{+\infty} \frac{\sqrt[n]{t}}{t^2+1}\,dt = \pi\,\frac{\zeta-\zeta^3}{1-\zeta^4} =$$

$$= \pi\,\frac{\zeta\cancel{(1-\zeta^2)}}{\cancel{(1-\zeta^2)}(1+\zeta^2)} \overset{ix}{=} \pi\,\frac{1}{2\cos\frac{\pi}{2n}}.$$

8.3 $f \in \mathcal{O}(\mathbb{C}^*)$, $\forall z \in \mathbb{C}^* \quad |f(z)| \leq 2|z|^{\frac{1}{2}} + 3|z|^{-\frac{1}{3}}$

$\mathbb{C}^*$ Kreisring $\Rightarrow$ f hat Laurententwicklung auf $\mathbb{C}^*$: $\exists (a_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$: $\forall z \in \mathbb{C}^* \quad f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$.

$$a_n = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(z)}{z^{n+1}} dz \quad \forall n \in \mathbb{Z} \quad \forall r > 0$$

$$|a_n| \leq \frac{1}{2\pi} \cdot \frac{\max_{\partial B_r(0)} |f|}{r^{n+1}} \cdot 2\pi r = \frac{\max_{\partial B_r(0)} |f|}{r^n} \leq$$

$$\leq \frac{2r^{\frac{1}{2}} + 3r^{-\frac{1}{3}}}{r^n} = 2r^{\frac{1}{2}-n} + 3r^{-\frac{1}{3}-n} \quad \forall r > 0 \quad \forall n \in \mathbb{Z}.$$

$\forall n \in \mathbb{Z}, \forall r > 0 \qquad |a_n| \leq 2r^{\frac{1}{2}-n} + 3r^{-\frac{1}{3}-n}.$

$n > 0: \quad r \to +\infty \implies |a_n| = 0 \implies a_n = 0.$

$n < 0: \quad r \to 0^+ \implies |a_n| = 0 \implies a_n = 0.$

$\forall n \in \mathbb{Z} \setminus \{0\} \quad a_n = 0$

$\implies \forall z \in \mathbb{C}^* \quad f(z) = a_0.$

8.4

Cauchy $\Longrightarrow$

$U \subseteq \mathbb{C}$ offen, $f \in \mathcal{O}(U)$, $z_0 \in U$, $r > 0$, $\overline{B_r(z_0)} \subset U$
$\Longrightarrow$ die Potenzreihenentwicklung von f in z_0 hat Konvergenzradius $\geq r$.

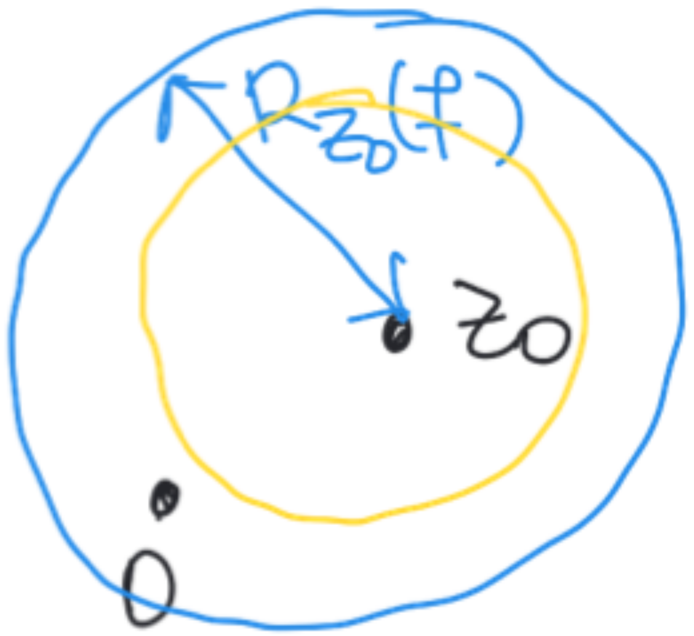
$U \subseteq \mathbb{C}$ offen, $f \in \mathcal{O}(U)$, $z_0 \in U$, $r > 0$, $B_r(z_0) \subseteq U$
$\Longrightarrow$ die Potenzreihenentwicklung von f in z_0 hat Konvergentradius $\geq r$.

a) $f: \mathbb{C}^* \longrightarrow \mathbb{C}$
$\quad z \longmapsto z^{-1}$

$R_{z_0}(f) :=$ Konvergenzradius von P.RE von f in z_0

$z_0 \in \mathbb{C}^*$ $\qquad B_{|z_0|}(z_0) \subseteq \mathbb{C}^* \implies R_{z_0}(f) \geq |z_0|$.

Wenn $R_{z_0}(f) > |z_0|$, dann:



sei $g \in \mathcal{O}(B_{R_{z_0}(f)}(z_0))$
definiert durch die PRE von f in z_0
$\forall z \in B_{|z_0|}(z_0)$, $g(z) = f(z)$

principle of analytic continuation $\implies$

$\forall z \in B_{R_{z_0}(f)}(z_0) \setminus \{0\}$, $\quad g(z) = f(z)$

$\implies f$ hat eine hebbare Singularität in 0 $\quad \lightning$

b) $f: \mathbb{C}^* \longrightarrow \mathbb{C}$, $z \longmapsto \frac{\sin z}{z}$ hat eine hebbare Singularität in 0

$\Rightarrow \exists g \in \mathcal{O}(\mathbb{C}) : g|_{\mathbb{C}^*} = f$

$\forall z_0 \in \mathbb{C}^*$ PRE von f in z_0 = PRE von g in z_0
hat Konvradius $+\infty$.
in jede z_0

d) $f: \cancel{B_{\frac{1}{2}}(i)} \longrightarrow \mathbb{C}$

$\mathbb{C} \setminus \{0,1,2\}$

$$f(z) = \frac{e^{2z} + e^{-2z} - 2}{z^2(z^2 - 3z + 2)} = \left(\frac{e^z - e^{-z}}{z}\right)^2 \cdot \frac{1}{(z-1)(z-2)}$$

$$\exists h \in \mathcal{O}(\mathbb{C}) : \forall z \in \mathbb{C}^* \quad h(z) = \frac{e^z - e^{-z}}{z}$$

$$\begin{aligned} g: \mathbb{C} \setminus \{1,2\} &\longrightarrow \mathbb{C} \\ z &\longmapsto h(z)^2 \cdot \frac{1}{(z-1)(z-2)} \end{aligned} \implies R_i(f) \geq |i-1| = \sqrt{2}$$

i

0 1 2

8.2 f $\quad z \cdot \sin\left(\frac{1}{z}\right)$

$$\sin w = w - \frac{w^3}{6} + o(w^4)$$

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{6}\left(\frac{1}{z}\right)^3 + \text{terms with order} \leq -5$$

$$z \cdot \sin \frac{1}{z} = 1 - \frac{1}{6} z^{-2} + \text{terms with order} \leq -4$$

8.2.e $\dfrac{z}{\sin z}$

$$\frac{\sin z}{z} = 1 - \frac{1}{6} z^2 - z^4 \cdot g(z)$$ g hol. in einer Umgebung von 0, $g(0) \neq 0$.

$$\frac{z}{\sin z} = \frac{1}{1 - \left(\frac{1}{6} z^2 + z^4 \cdot g(z)\right)} = \sum_{n \geq 0} \left(\frac{1}{6} z^2 + z^4 \cdot g(z)\right)^n$$

$$= 1 + \frac{1}{6} z^2 + z^4 \cdot g(z) + \frac{1}{36} z^4 + z^8 g(z) + \frac{1}{3} z^6 g(z) + \dots$$

$$= 1 + \frac{1}{6} z^2 + o(z^3)$$