23.06.2020

9.4. $f(z) = z^7 - 12iz^5 + 15z^4 + i$

$B_2(0)$: $|z| = 2$ $\quad f_1(z) = -12iz^5$, $\quad f_2(z) = z^7 + 15z^4 + i$

$|f_1(z)| = 12 \cdot 2^5 = 12 \cdot 32 = 384$

$|f_2(z)| \leq |z^7| + 15|z|^4 + |i| = 128 + 15 \cdot 16 + 1 = 369 < 384.$

$\#$ Nullstellen von f in $B_2(0)$ $=$ $\#$ Nullstellen von f_1 in $B_2(0)$

$= 5$

$B_1(0)$: $|z| = 1 \Rightarrow |15z^4| = 15$

$|z^7 - 12iz^5 + i| \leq 1 + 12 + 1 = 14 < 15$

$\#$ Nullstellen von f in $B_1(0) = 4$

$B_2(0) \setminus B_1(0)$: $\quad 5 - 4 = 1.$

9.5 $f(z) = z(z^3 - 4) + e^{z/2} + \frac{1}{4z-6}$

$B_2(0)$: $f_1(z) = z^4$

$f_2(z) = -4z + e^{z/2} + \frac{1}{4z-6}$

$|z| = 2 \Rightarrow |f_1(z)| = 16$

$|f_2(z)| \leq |-4z| + |e^{z/2}| + \left|\frac{1}{4z-6}\right|$

$= 8 + e^{\mathrm{Re}\frac{z}{2}} + \frac{1}{|4z-6|}$

$\leq 8 + e^{\frac{|z|}{2}} + \frac{1}{2}$

$= 8 + e + \frac{1}{2} < 8 + 3 + 1 = 12 < 16$

$\forall \alpha, \beta \in \mathbb{C}$
$||\alpha \pm \beta| \geq ||\alpha| - |\beta||$

$|4z-6| \geq 4|z| - 6 = 2$

$a, b \in \mathbb{R}$
$a \leq b \Rightarrow e^a \leq e^b$

$\mathrm{Re}\,\alpha \leq |\alpha|$
$\forall \alpha \in \mathbb{C}$

Rouché $\Rightarrow$

$$\underbrace{\#\text{Nullstelle von } f \text{ in } B_2(0)} - \underbrace{\#\text{Polstelle von } f \text{ in } B_2(0)}_{= 1} = \underbrace{\#\text{Nullstelle von } f_1 \text{ in } B_2(0)}_{= 4} - \underbrace{\#\text{Polstelle von } f_1 \text{ in } B_2(0)}_{= 0}$$

f hat einen Pol in $\frac{3}{2} \in B_2(0)$

$f_1(z) = z^4$

$$\Rightarrow \#\text{Nullstelle von } f \text{ in } B_2(0) = 5$$

$B_1(0)$: $\quad f_1(z) = -4z$

$$f_2(z) = z^4 + e^{z/2} + \frac{1}{4z-6}$$

$$|z| = 1 \implies |f_1(z)| = 4$$

$$|f_2(z)| \leq |z|^4 + |e^{z/2}| + \left|\frac{1}{4z-6}\right| \leq 1 + e^{\frac{1}{2}} + \frac{1}{2} < 4$$

Rouché $\implies$ f, f_1 sind holomorph auf $B_1(0)$,

$\#$ Nullstelle von f in $B_1(0)$ = $\#$ Nullstelle von f_1 in $B_1(0) = 1$

$\implies$ $\#$ Nullstellen von f in $B_2(0) \setminus B_1(0) = 5 - 1 = 4.$

9.3 b)

$$I = \int_0^{+\infty} \frac{\sqrt[4]{x}}{x^2+2x+4}\,dx$$

Method 4 in Vorlesung 08.2.

$U = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, $\quad g: U \longrightarrow \mathbb{C}$

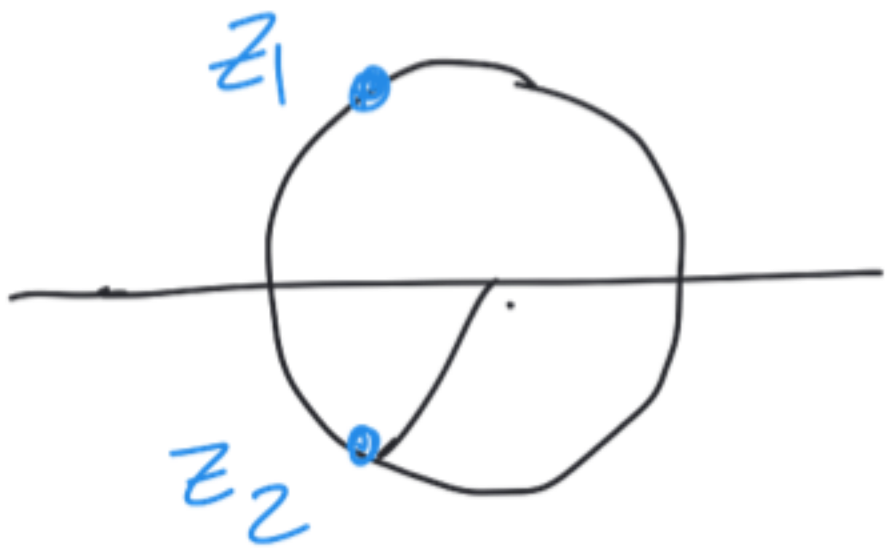
$\rho e^{i\theta} \longmapsto \sqrt[4]{\rho}\, e^{i\frac{\theta}{4}} \qquad \rho > 0,\ \theta \in (0, 2\pi)$

$f \in \mathcal{M}(U) \qquad f(z) = \dfrac{g(z)}{z^2+2z+4}$

$z^2+2z+4=0 \qquad (z+1)^2+3=0 \qquad z+1 = \pm\sqrt{3}\,i \qquad z = -1 \pm \sqrt{3}\,i$

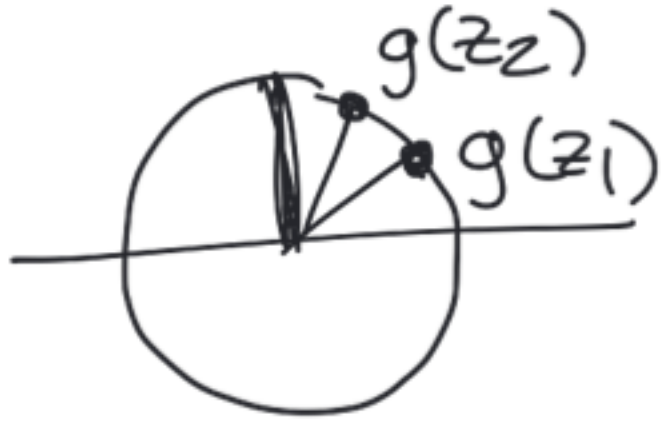
$z_1 = -1+\sqrt{3}i = 2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2e^{\frac{2\pi}{3}i}$

$z_2 = -1-\sqrt{3}i = 2e^{\frac{4\pi}{3}i}$



$$g(z_1) = \sqrt[4]{2}\, e^{i\frac{\pi}{6}} = \sqrt[4]{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$g(z_2) = \sqrt[4]{2}\, e^{i\frac{\pi}{3}} = \sqrt[4]{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$



$$f(z) = \frac{g(z)}{z^2+2z+4} \qquad \operatorname{ord}(f, z_1) = \operatorname{ord}(f, z_2) = -1$$

$$\operatorname{Res}(f, z_1) = \left.\frac{g(z)}{2z+2}\right|_{z=z_1} = \frac{g(z_1)}{2(-1+\sqrt{3}i)+2} = \sqrt[4]{2}\,\frac{\frac{\sqrt{3}}{2}+i\frac{1}{2}}{2\sqrt{3}i}$$

$$\operatorname{Res}(f, z_2) = \left.\frac{g(z)}{2z+2}\right|_{z=z_2} = \sqrt[4]{2}\,\frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{-2\sqrt{3}i}$$

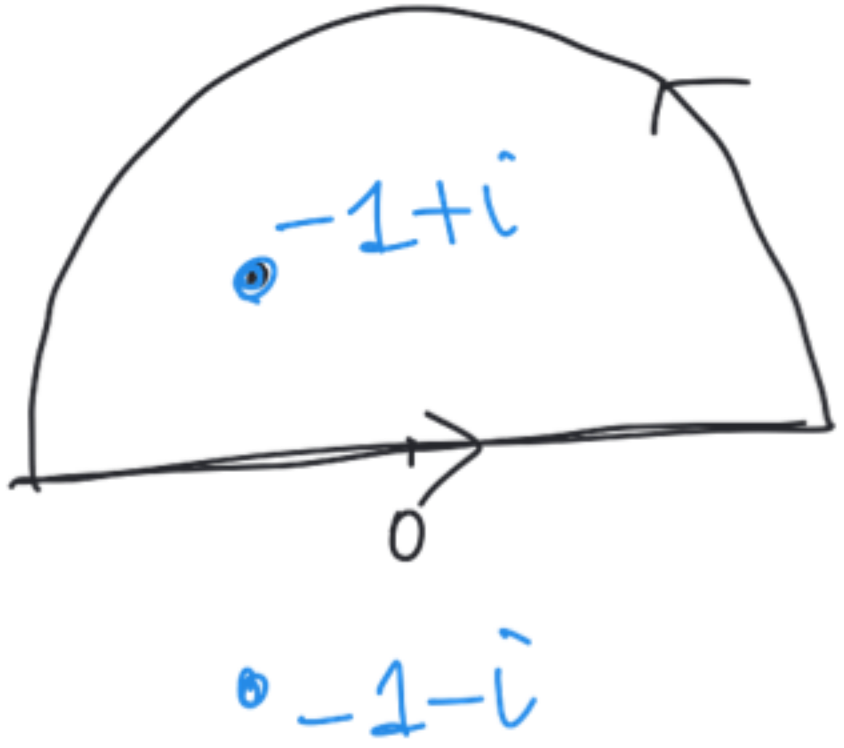
$$(1-i)\cdot I = 2\pi i\cdot\left[\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right]$$

$$= \cancel{2}\pi \cancel{i}\,\frac{\sqrt[4]{2}}{\cancel{2}\sqrt{3}\cancel{i}}\left[\frac{\sqrt{3}}{2} + i\frac{1}{2} - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right] = \overbrace{\frac{\pi\sqrt[4]{2}}{\sqrt{3}}\cdot\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)}^{I}(1-i)$$

9.2 ii & iii)

$$I_1 = \int_{-\infty}^{+\infty} \frac{\cos x}{x^2+2x+2}\,dx \quad , \quad I_2 = \int_{-\infty}^{+\infty} \frac{\sin x}{x^2+2x+2}\,dx$$

$$f \in \mathcal{M}(\mathbb{C}) \qquad f(z) = \frac{e^{iz}}{z^2+2z+2}$$



$$I_1 + i I_2 = \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+2x+2}\,dx = 2\pi i \operatorname{Res}(f, -1+i)$$

$$= 2\pi i \left. \frac{e^{iz}}{2z+2} \right|_{z=-1+i}$$

$$= 2\pi i \cdot \frac{e^{i(-1+i)}}{2i} = \pi e^{-1} e^{-i} = \frac{\pi}{e}(\cos 1 - i \sin 1)$$

$$\Rightarrow I_1 = \frac{\pi}{e}\cos 1 , \qquad I_2 = -\frac{\pi}{e}\sin 1 .$$

9.1 a) $f(z) = \dfrac{1}{(z^2+1)(z-i)^3} = \dfrac{1}{(z-i)^4 \cdot (z+i)}$

$\operatorname{ord}(f, i) = -4$

$\operatorname{ord}(f, -i) = -1$

$$\operatorname{Res}(f, -i) = \lim_{z \to -i} (z+i) \cdot f(z) = \left.\frac{1}{(z-i)^4}\right|_{z=-i} = \frac{1}{(-2i)^4} = \frac{1}{16}$$

$h(z) = (z-i)^4 \cdot f(z) = \dfrac{1}{z+i}$ $\qquad \operatorname{Res}(f, i) = \dfrac{h^{(3)}(i)}{3!}$

$$\int_{\partial B_r(0)} f(z)\,dz = 2\pi i \cdot \left[\operatorname{Res}(f, -i) + \operatorname{Res}(f, i)\right]$$
$$= 2\pi i \left(\frac{1}{16} + \operatorname{Res}(f, i) \right)$$

$$f(z) = \frac{1}{(z-i)^4 (z+i)}$$

$$\left| \int_{\partial B_r(0)} f(z)\,dz \right| \leq \max_{\partial B_r(0)} |f| \cdot 2\pi r \leq \frac{1}{(r-1)^5} \cdot 2\pi r$$

$$r \to +\infty \implies \operatorname{Res}(f, i) + \operatorname{Res}(f, -i) = 0$$

$$\implies \operatorname{Res}(f, i) = -\operatorname{Res}(f, -i) = -\frac{1}{16}$$

9.1 c)

$$f(z) = \exp\left(\frac{z+1}{z-1}\right) = \exp\left(\frac{z-1+2}{z-1}\right) =$$

$$= \exp\left(1 + \frac{2}{z-1}\right) = \exp(1) \cdot \exp\left(\frac{2}{z-1}\right)$$

$$\exp(\alpha+\beta) = \exp(\alpha) \cdot \exp(\beta)$$

$$\exp(w) = \sum_{n \geq 0} \frac{1}{n!} w^n \leadsto$$

$$n = -k$$

$$f(z) = e \cdot \sum_{n \geq 0} \frac{1}{n!} \left(\frac{2}{z-1}\right)^n = e \cdot \sum_{n \geq 0} \frac{1}{n!} \frac{2^n}{(z-1)^n} = \sum_{k \leq 0} \frac{e\, 2^{-k}}{(-k)!} (z-1)^k$$

$$\mathrm{Res}(f, 1) = 2e$$

$$f(z) = \exp\left(\frac{z+1}{z-1}\right)$$

$$\left.\begin{matrix} w = z-1 \\ z = w+1 \end{matrix}\right\} \Rightarrow \operatorname{Res}(f,1) = \operatorname{Res}\left(\exp\left(\frac{w+2}{w}\right), 0\right)$$

$$\exp\left(\frac{w+2}{w}\right) = \exp\left(1+\frac{2}{w}\right) = \sum_{n\geq 0} \frac{1}{n!}\left(1+\frac{2}{w}\right)^n =$$

$$= \sum_{n\geq 0} \frac{1}{n!} \cdot \sum_{j=0}^{n} \binom{n}{j}\left(\frac{2}{w}\right)^j$$

$$\operatorname{Res}(f,1) = \operatorname{Res}\left(\exp\left(\frac{w+2}{w}\right), 0\right) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\substack{0\leq j\leq n \\ j=1}} \binom{n}{1} 2$$

$$= \sum_{n\geq 1} \frac{1}{n!}\, n\cdot 2 = \sum_{n\geq 1} \frac{1}{(n-1)!} 2 = e\cdot 2$$

03.3_analytisch2

<u>Satz</u> $U \subseteq \mathbb{C}$ zusammenhängend offen, $f \in \mathcal{O}(U)$.
Die folgenden Aussagen sind äquivalent:

1) $\exists z_0 \in U : \forall n \geq 0 \quad f^{(n)}(z_0) = 0$
2) $\exists r > 0$ und $z_0 \in U$ mit $B_r(z_0) \subseteq U$ und $f|_{B_r(z_0)} = 0$.
2') $f^{-1}(0) = \{$Nullstellen von $f\}$ hat nicht leere innere Kern
3) $f = 0 : \quad \forall z \in U \quad f(z) = 0$
4) $f^{-1}(0)$ ist nicht diskret
4'') $\exists z^* \in U$ und eine Folge (z_n) in $U \setminus \{z^*\}$:
$\lim_{n \to +\infty} z_n = z^*$ und $\forall n \geq 1 \quad f(z_n) = 0$.

<u>Kor</u> $U \subseteq \mathbb{C}$ offen, zusammenhängend, $f, g \in \mathcal{O}(U)$.
Die folgenden Aussagen sind äquivalent:

1) $\exists z_0 \in U : \forall n \geq 0 \quad f^{(n)}(z_0) = g^{(n)}(z_0)$

2) $\exists z_0 \in U$ und $r > 0$: $B_r(z_0) \subseteq U$ und $f|_{B_r(z_0)} = g|_{B_r(z_0)}$

2') $\{z \in \mathbb{C} \mid f(z) = g(z)\}$ hat eine nicht leere innere Kern.

3) $f = g$

4) $\{z \in \mathbb{C} \mid f(z) = g(z)\}$ ist nicht diskret

4'') $\exists z^* \in U$ und eine Folge (z_n) in $U \setminus \{z^*\}$ sodass
$\lim_{n \to +\infty} z_n = z^*$ und $\forall n$, $f(z_n) = g(z_n)$.

<u>Bws</u> $f - g$

$U \subseteq \mathbb{C}$ offen.

$z_0 \in U$ $\quad f \in \mathcal{O}(U \setminus \{z_0\})$

$\exists \tilde{f} \in \mathcal{O}(U)$ mit $\tilde{f}|_{U \setminus \{z_0\}} = f$ $\iff$ f hat eine hebbare Singularität in z_0

$U \subseteq \mathbb{C}$ offene zusammenhängend

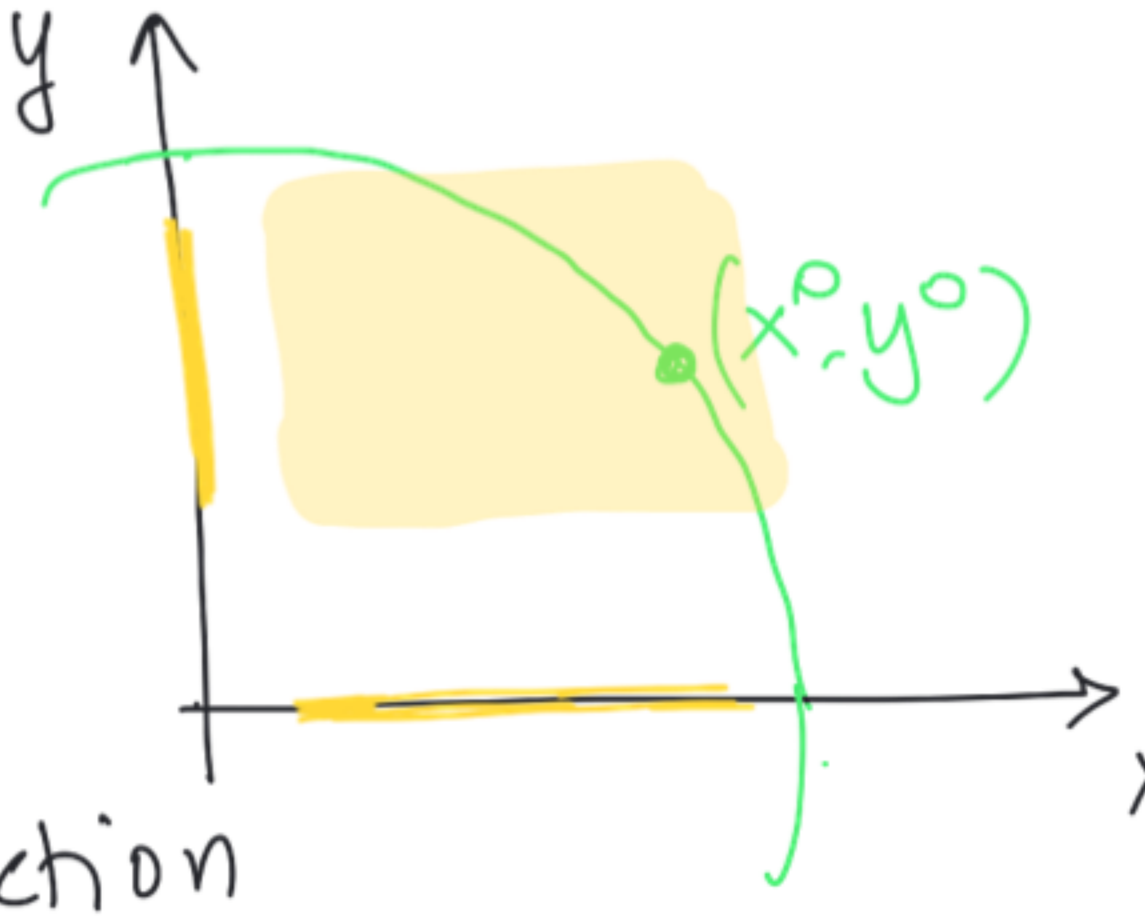
$f \in \mathcal{O}(U)$. Die folgenden Aussagen sind äquivalent:

1) $\exists z_0 \in U : \forall n \geq 1 \quad f^{(n)}(z_0) = 0$

2) $\exists z_0 \in U$ und $r > 0$: $B_r(z_0) \subseteq U$ und $f|_{B_r(z_0)}$ konstant ist

3) f konstant

$F: \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad (x,y) \longmapsto x^2+y^2-1$



$(x^0,y^0) \in F^{-1}(0) \qquad 0<x_0<1$

$\qquad\qquad\qquad\qquad\qquad 0<y_0<1$

$J_F(x^0,y^0) = (2x^0 \;\; 2y^0)$

satz der impliziten Funktion

$\frac{\partial F}{\partial y}(x^0,y^0) \neq 0 \Longrightarrow \exists A$ Umg. von x^0 in $\mathbb{R}$ | $x^0 \in A \subseteq (0,1)$

B " " y^0 in $\mathbb{R}$ | $y^0 \in B \subseteq (0,1)$

$\underbrace{\forall x \in A, \; \exists!\, y \in B : F(x,y)=0}$

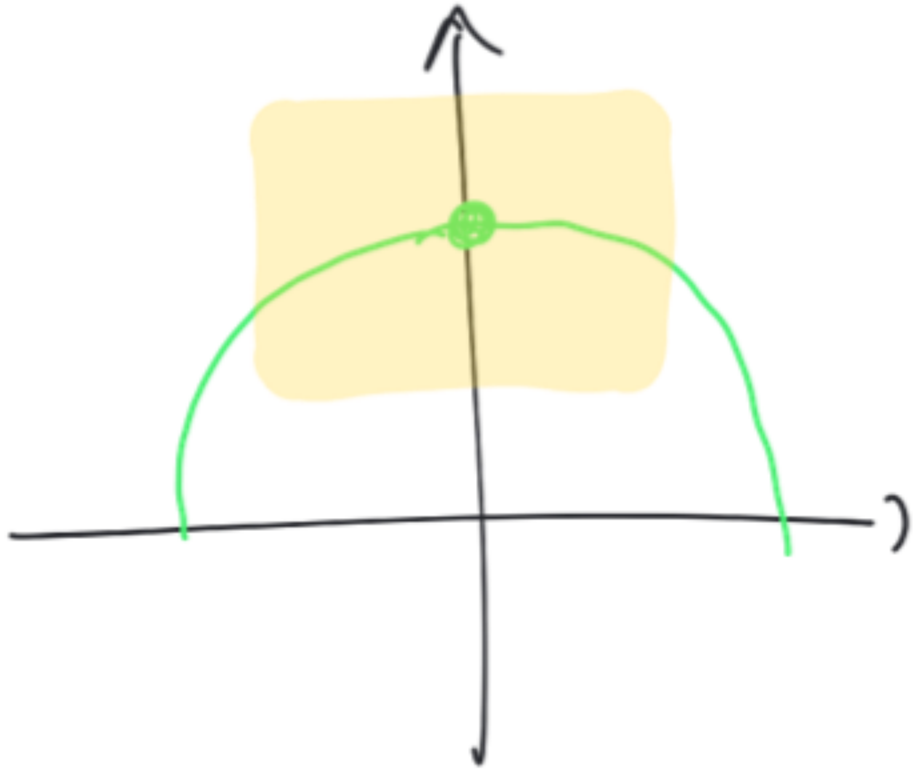
the intersection of $F^{-1}(0)$ with the rectangle $A \times B$ $\Big] =$ the graph of a function

$g: A \longrightarrow B$

$x \longmapsto g(x)$

$F: \mathbb{R}^2 \longrightarrow \mathbb{R} \quad (x,y) \longmapsto x^2+y^2-1$

$(x^0,y^0)=(0,1) \qquad \frac{\partial F}{\partial x}(x^0,y^0)=0 \;,\; \frac{\partial F}{\partial y}(x^0,y^0)=2\neq 0$



there is an implicit function

$$g: A \longrightarrow B, \quad x \longmapsto y$$

but there is no implicit function $y \mapsto x$

$\forall U$ nbd of 1 in $\mathbb{R}$, $\forall V$ nbd of 0 in $\mathbb{R}$,

$\forall y \in U$, there exist two $x \in V$ with $F(x,y)=0$

the intersection of $F^{-1}(0)$ with a rectangle in (x^0,y^0) is not the graph of a function $y \longmapsto x$.

$$F(x,y) = x^2 + y^2 - 1$$

$$(x^0, y^0) = (0, 1) \qquad y = \sqrt{1 - x^2}$$

$x = \pm\sqrt{1 - y^2}$ is not a function

$$(x^0, y^0) = (0, -1) \qquad y = -\sqrt{1 - x^2}$$

$U \subseteq \mathbb{C}$ zusammenhängend, offen

$$D : \mathcal{O}(U) \longrightarrow \mathcal{O}(U)$$
$$f \longmapsto f'$$

$$\partial : \mathcal{O}(U) \longrightarrow H^1(U, \mathbb{C})$$
$$f \longmapsto \left[\begin{array}{l} [f(z)dz] : \pi_1(U) \longrightarrow \mathbb{C} \\ \qquad [\gamma] \longmapsto \int_\gamma f(z)dz \end{array} \right.$$

$$\mathbb{C} \xrightarrow{\text{konst}} \mathcal{O}(U)$$
$$a \longmapsto \left[\begin{array}{l} U \longrightarrow \mathbb{C} \\ z \longmapsto a \end{array} \right]$$

$$\left. \begin{array}{l} \operatorname{im} D = \ker \partial \\ \ker D = \operatorname{im} \text{konst} \\ \ker \text{konst} = 0 \end{array} \right\} \Longrightarrow 0 \to \mathbb{C} \xrightarrow{\text{konst}} \mathcal{O}(U) \xrightarrow{D} \mathcal{O}(U) \xrightarrow{\partial} H^1(U, \mathbb{C})$$