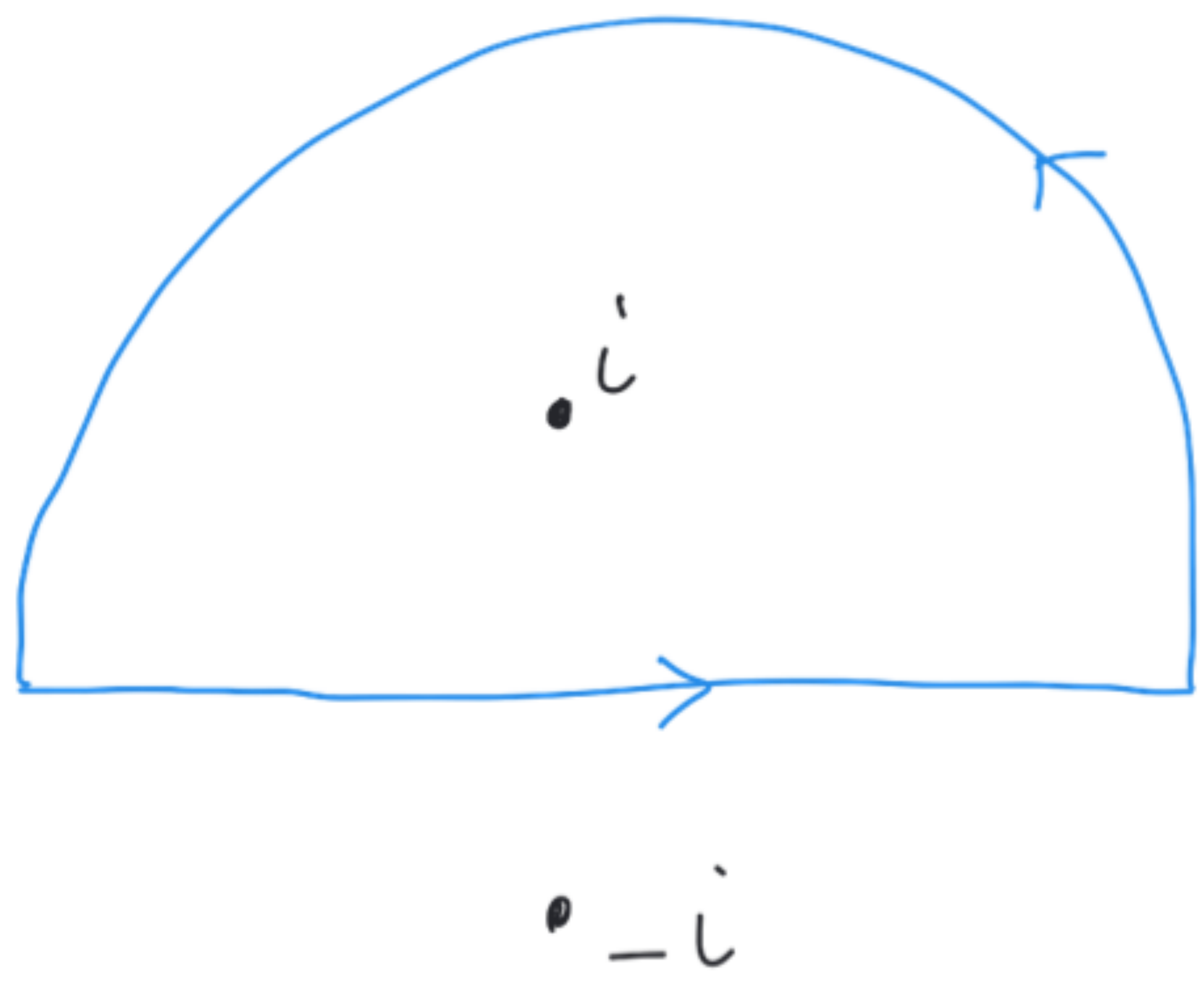
7.7.2020

11.1 $I = \int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+1)^2}\,dx$ $\qquad f(z) = \frac{e^{iz}}{(z^2+1)^2}$ meromorph auf $\mathbb{C}$



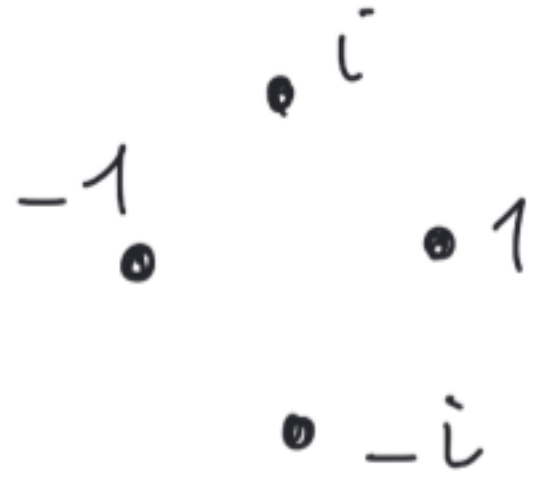
$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2+1)^2}\,dx = 2\pi i \cdot \operatorname{Res}(f, i)$$
$$= 2\pi i \cdot \frac{1}{e \cdot 2i} = \frac{\pi}{e}$$

$$I = \operatorname{Re} \int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2+1)^2}\,dx = \frac{\pi}{e}$$

$$\int_{-\infty}^{+\infty} \frac{\sin x}{(x^2+1)^2}\,dx = 0$$

11.2

A



B

C

D

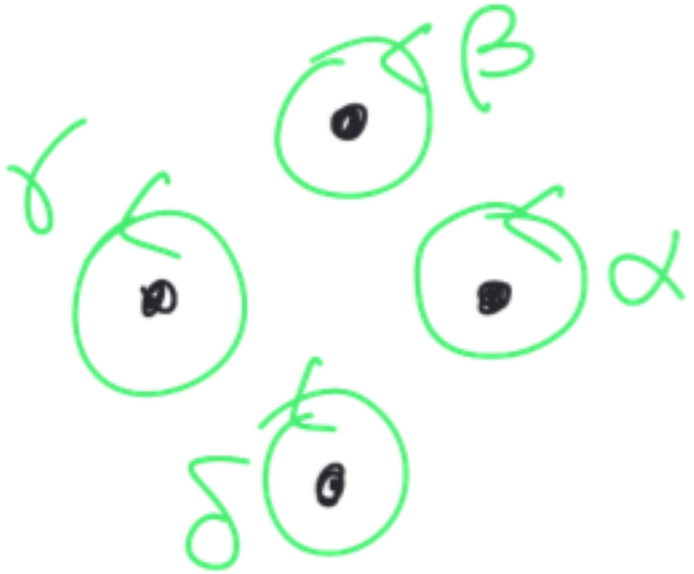
$$f(z) = \frac{pz^3+qz^2+rz+s}{z^4-1}$$ hat einfache Pole in $1, i, -1, -i$

$$\operatorname{Res}(f,1) = \left.\frac{pz^3+qz^2+rz+s}{4z^3}\right|_{z=1} = \frac{p+q+r+s}{4}$$

$$\operatorname{Res}(f,i) = \left.\frac{pz^3+qz^2+rz+s}{4z^3}\right|_{z=i} = \frac{-pi-q+ri+s}{-4i} = \frac{p-qi-r+si}{4}$$

$$\mathrm{Res}(f,-1) = \frac{p-q+r-s}{4} \qquad \mathrm{Res}(f,-i) = \frac{p+qi-r-si}{4}$$

a) When $f(z)\,dz$ exact on A?



$f(z)\,dz$ exakt auf A $\iff \begin{cases} \int_\alpha f(z)\,dz = 0 \\ \int_\beta f(z)\,dz = 0 \\ \int_\gamma f(z)\,dz = 0 \\ \int_\delta f(z)\,dz = 0 \end{cases} \iff \begin{cases} \mathrm{Res}(f,1) = 0 \\ \mathrm{Res}(f,i) = 0 \\ \mathrm{Res}(f,-1) = 0 \\ \mathrm{Res}(f,-i) = 0 \end{cases}$

linear system in p, q, r, s

Residue theorem

$$\int_\alpha f(z)\,dz = 2\pi i \cdot \mathrm{Res}(f,1)$$

$$p = q = r = s = 0$$

$$\left[\begin{array}{l} f, g \in \mathcal{O}(B_\varepsilon(z_0)) \qquad \mathrm{ord}(g, z_0) = 1 \\ \Rightarrow \quad \mathrm{Res}\left(\frac{f}{g}, z_0\right) = \dfrac{f(z_0)}{g'(z_0)} \end{array}\right]$$

- If $f(z_0) = 0$, then

$$f(z) = (z - z_0)^k \tilde{f}(z_0) \qquad k \geq 1$$
$$g(z) = (z - z_0)\, \tilde{g}(z_0)$$

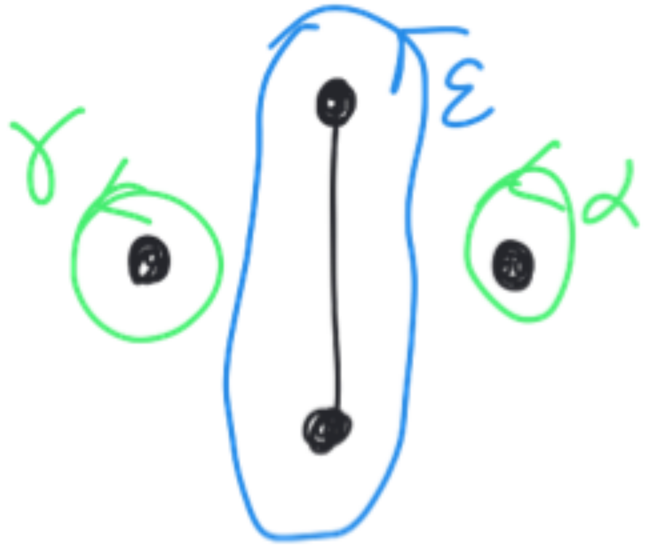
$\tilde{f}(z_0) \neq 0$

$\tilde{g}(z_0) \neq 0$

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-1} \frac{\tilde{f}(z_0)}{\tilde{g}(z_0)}$$ holomorphic in a nbd of z_0

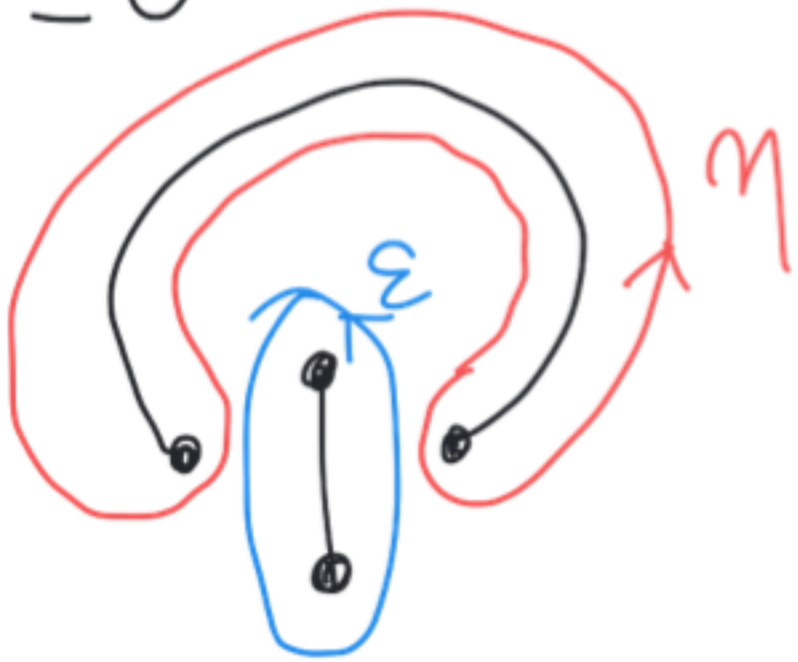
$$\Rightarrow \quad \mathrm{Res}\left(\frac{f}{g}, z_0\right) = 0.$$

- If $f(z_0) \neq 0$, $\mathrm{ord}\left(\frac{f}{g}, z_0\right) = -1$ Vorlesung

b) 

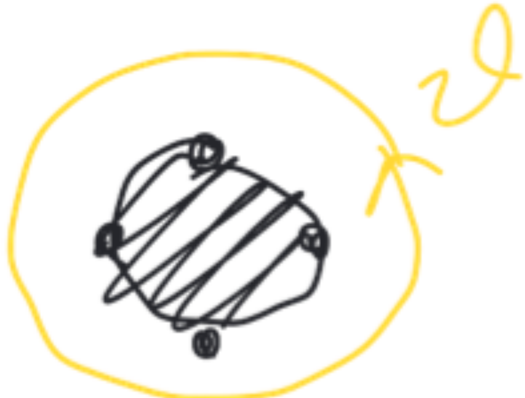
$f(z)dz$ exakt auf B $\iff \begin{cases} 0 = \int_\alpha f(z)dz = 2\pi i \operatorname{Res}(f,1) \\ 0 = \int_\varepsilon f(z)dz = 2\pi i (\operatorname{Res}(f,i) + \operatorname{Res}(f,-i)) \\ 0 = \int_\gamma f(z)dz = 2\pi i \operatorname{Res}(f,-1) \end{cases}$

$\begin{cases} p = r = 0 \\ q + s = 0 \end{cases}$

c) 

$p = r = 0$

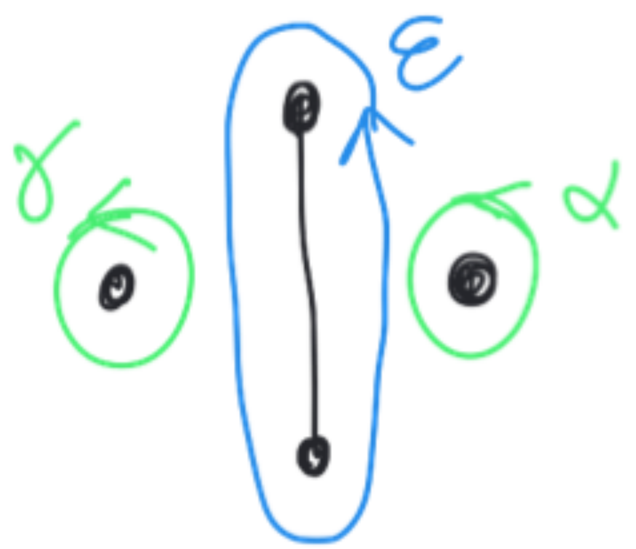
$A \supseteq B \supseteq C \supseteq D$

d) 

$p = 0$

$h(z) = (z-1)^k (z-i)^\ell (z+1)^m (z+i)^n \qquad h: A \longrightarrow \mathbb{C}^*$

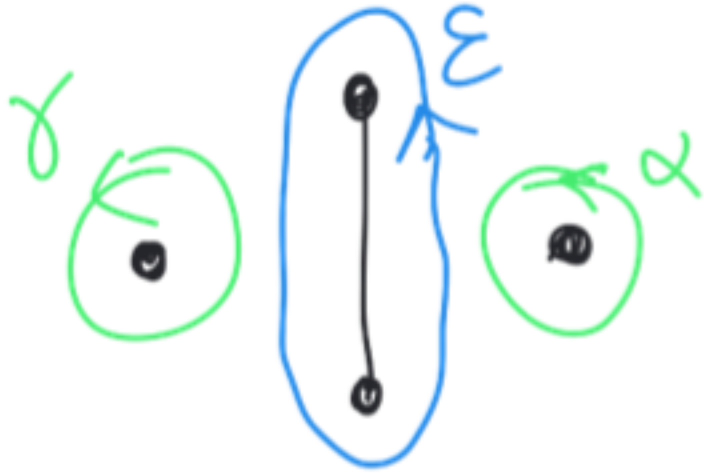
e) B



$h|_B$ hat einen Logarithmus $\iff \dfrac{h'(z)}{h(z)}dz$ exakt auf B $\iff \begin{cases} \operatorname{Res}\left(\frac{h'}{h}, 1\right) = 0 \\ \operatorname{Res}\left(\frac{h'}{h}, i\right) + \operatorname{Res}\left(\frac{h'}{h}, -i\right) = 0 \\ \operatorname{Res}\left(\frac{h'}{h}, -1\right) = 0 \end{cases}$

$\iff \begin{cases} k = 0 \\ \ell + n = 0 \\ m = 0 \end{cases}$

h) $h(z) = (z-1)^k (z-i)^\ell (z+1)^m (z+i)^n \qquad h: A \longrightarrow \mathbb{C}^*$

B 

$h|_B$ hat eine zweite Wurzel $\iff \begin{cases} \frac{1}{2\pi i}\int_\alpha \frac{h'(z)}{h(z)}dz \in 2\mathbb{Z} \\ \frac{1}{2\pi i}\int_\varepsilon \frac{h'(z)}{h(z)}dz \in 2\mathbb{Z} \\ \frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}dz \in 2\mathbb{Z} \end{cases} \iff \begin{cases} \operatorname{Res}\left(\frac{h'}{h}, 1\right) \in 2\mathbb{Z} \\ \operatorname{Res}\left(\frac{h'}{h}, i\right) + \operatorname{Res}\left(\frac{h'}{h}, -i\right) \in 2\mathbb{Z} \\ \operatorname{Res}\left(\frac{h'}{h}, -1\right) \in 2\mathbb{Z} \end{cases}$

$$\begin{cases} k \in 2\mathbb{Z} \\ \ell + n \in 2\mathbb{Z} \\ m \in 2\mathbb{Z} \end{cases}$$

B

$\cancel{\log z}$
$\cancel{\sqrt{z}}$
$\cancel{\sqrt[3]{z}}$

$h(z) = z^2 + 1$

no logarithm on B
has a square root on B

A

$h(z) = z^2 + 1$

no logarithm on A
has no square root on A

11.5

$f : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ Möbius transformation

$$z \in \mathbb{C} \setminus \{-1\} \longmapsto \frac{z-1}{iz+i} = i\,\frac{1-z}{1+z}$$

$$-1 \longmapsto \infty$$

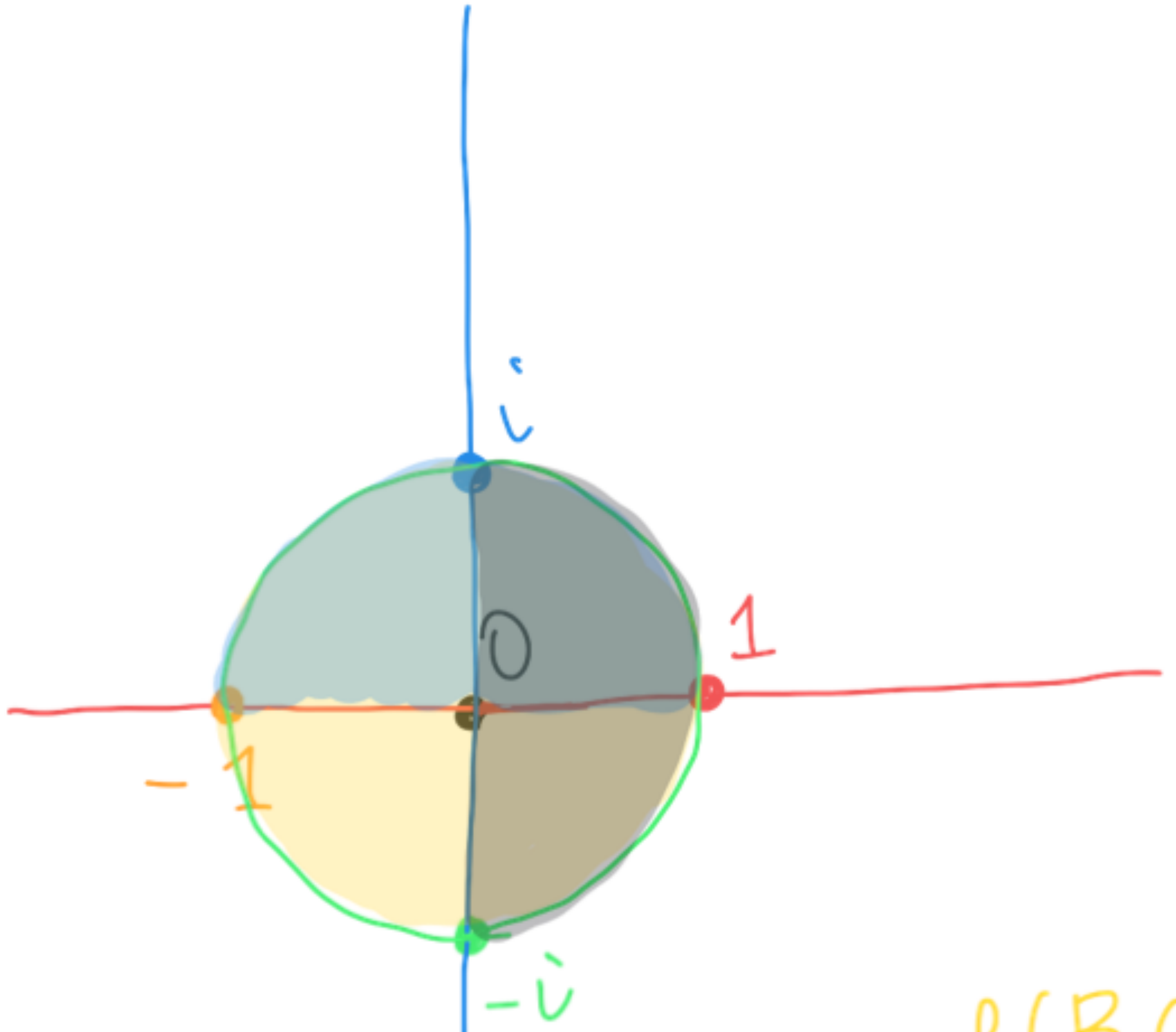
$$\infty \longmapsto -i$$

$$f = [A] \qquad A = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \qquad A^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

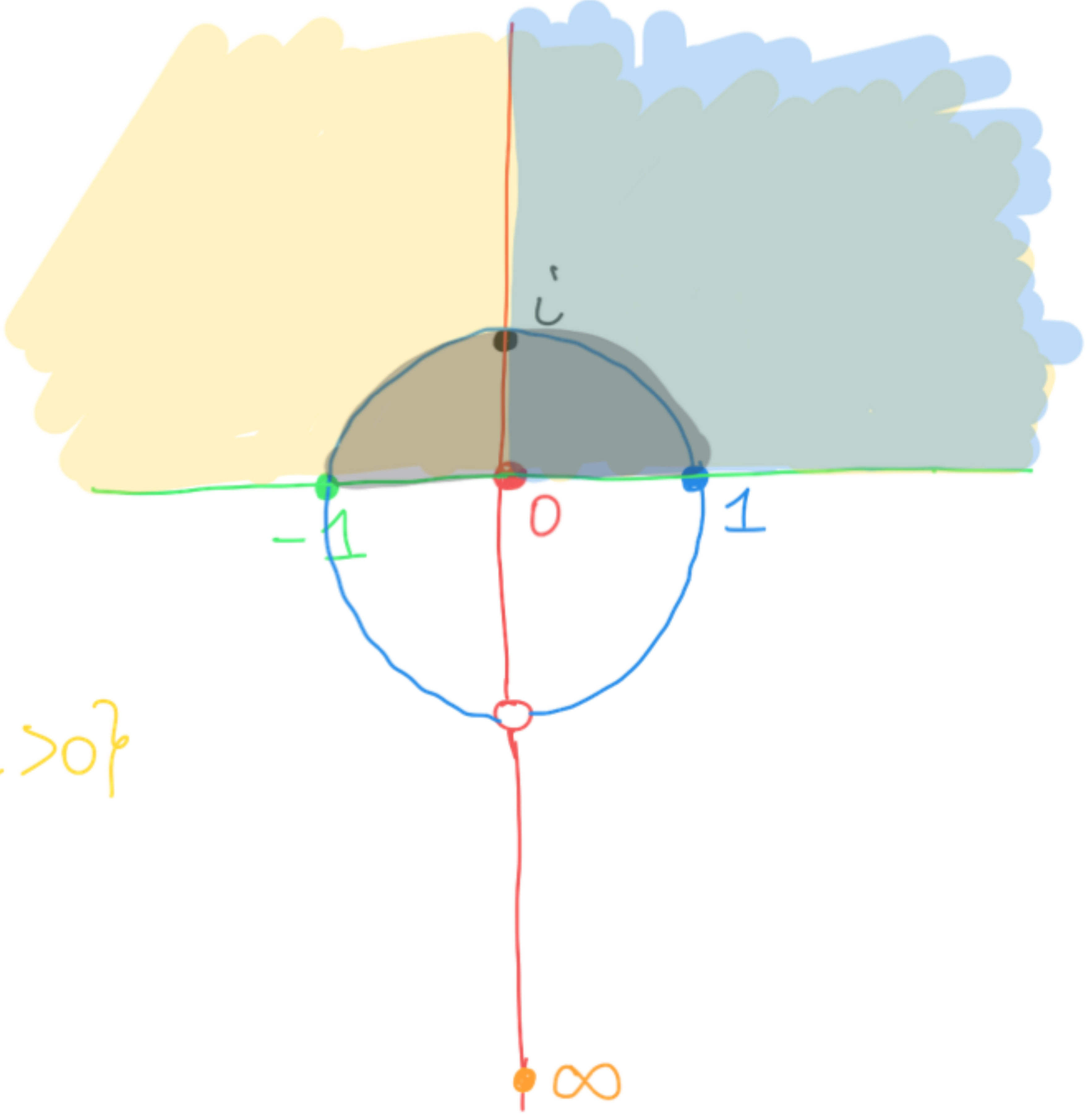
$$f^{-1} = [A^{-1}] : z \longmapsto \frac{1+iz}{1-iz} = \frac{i-z}{i+z}$$

$$-i \longmapsto \infty$$

$$\infty \longmapsto -1$$



$\xrightarrow{f}$



$f(B_1(0)) = \mathbb{H} = \{\operatorname{Im} z > 0\}$

$V = \{|z|<1, \operatorname{Re} z > 0\}$ $\quad f(V) = U$.

$f(\mathbb{R}) = i\mathbb{R} \cup \{\infty\} \setminus \{-i\}$

For every 3 points in $\mathbb{C} \cup \{\infty\}$, $\exists!$ Möbius line passing through them.

$f(i\mathbb{R}) = S^1 \setminus \{-i\}$ $\quad f(S^1) = \mathbb{R} \cup \{\infty\}$

$U = \{|z|<1, \operatorname{Im} z > 0\}$ $\quad f(U) = \{\operatorname{Re} z > 0, \operatorname{Im} > 0\}$


f
∞

f induces a biholomorphism

$$B_1(0) = \{z \in \mathbb{C} \mid |z| < 1\} \longrightarrow \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$$

11.4 S^2 Riemann sphere

$f: S^2 \longrightarrow \mathbb{C}$ holomorphic

$\Rightarrow f$ constant

<u>Pf</u> $S^2 = \mathbb{C} \cup \{\infty\}$

S^2 compact $\Rightarrow f(S^2)$ compact $\Rightarrow f(S^2) \subseteq \mathbb{C}$ bounded

$\Rightarrow f|_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$ bounded

Liouville $\Rightarrow f|_{\mathbb{C}}$ constant.

$f: S^2 \to \mathbb{C}$ continuous $\Rightarrow f$ constant. $\square$

$\{\text{Laurent series}\}$ is not a ring:

$f = \sum_{n\in\mathbb{Z}} T^n$ $\quad f\cdot f$ The constant term is

$$\sum_{n\in\mathbb{Z}} 1$$

$A = \{z\in\mathbb{C} \mid r^- < |z| < r^+\}$

$a, b \in \mathcal{O}(A)$ $\quad a = \sum_{n\in\mathbb{Z}} a_n T^n \quad b = \sum_{n\in\mathbb{Z}} b_n T^n$

$a\cdot b = \sum_{n\in\mathbb{Z}} c_n T^n \quad c_n = \sum_{\substack{m,l\in\mathbb{Z}\\ m+l=n}} a_m\cdot b_l = \sum_{m\in\mathbb{Z}} a_m b_{n-m}$

converges

$R^-(a) \leq r^- \quad R^+(a) \geq r^+$

$R^-(b) \leq r^- \quad R^+(b) \geq r^+$

6.2.a $f: \mathbb{C} \to \mathbb{C}$

$p \geq 0$, $p \in \mathbb{Z}$, $C, R \in \mathbb{R}_{>0}$

$\forall z \in \mathbb{C} \setminus B_R(0)$, $|f(z)| \leq C|z|^p$

$\Rightarrow$ f is a polynomial of degree $\leq p$

Pf Power series expansion of f in 0:

$\mathbb{C}$ is a ball $\Rightarrow$ this power series expansion is valid everywhere on $\mathbb{C}$

$\forall z \in \mathbb{C}$ $\quad f(z) = \sum_{n \geq 0} a_n z^n$.

Cauchy: $a_n = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(z)}{z^{n+1}} dz \quad \forall r > 0$

Now assume that $r \geqslant R$:

then $\partial B_r(0) \subseteq \mathbb{C} \setminus B_R(0)$

$$\forall z \in \partial B_r(0), \quad |f(z)| \leq C|z|^p = Cr^p$$

$$|a_n| = \left| \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(z)}{z^{n+1}} dz \right| \leqslant \frac{1}{2\pi} \cdot \max_{z \in \partial B_r(0)} \left| \frac{f(z)}{z^{n+1}} \right| \cdot L(\partial B_r(0))$$

$$\leqslant \frac{1}{2\pi} \frac{Cr^p}{r^{n+1}} \cdot 2\pi r = C r^{p-n}$$

$p - n < 0$

If $n > p$ then $|a_n| \leq \inf_{r \geqslant R} C r^{p-n} = 0 \Rightarrow a_n = 0$

$$\forall n > p, \ a_n = 0 \quad \Longrightarrow \quad f(z) = \sum_{n=0}^{p} a_n z^n.$$

$$n > p \implies p - n < 0$$

$$\lim_{r \to +\infty} r^{p-n} = 0$$