$k = \bar{k}$ X quasi-projective variety $X = U \cap Z$ Zariski

$U \subseteq \mathbb{P}^n$ open

$Z \subseteq \mathbb{P}^n$ closed

X CURVE if $\dim X = 1$, X irreducible

X SURFACE if $\dim X = 2$, " "

<u>Goal</u> Study birational classification of surfaces.

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k field, can work with separated schemes of finite type over k

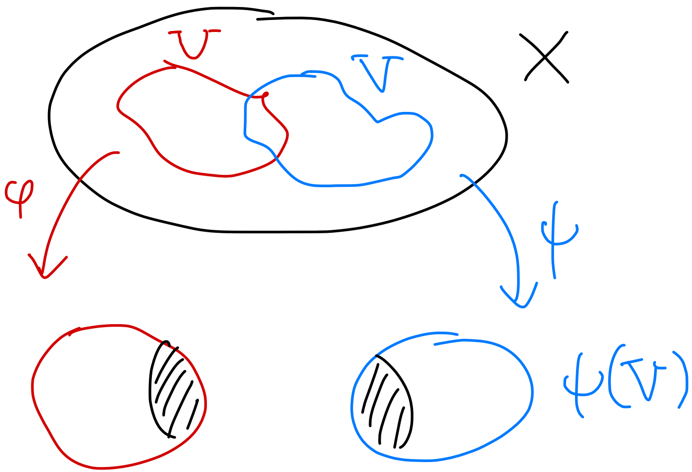
If $k = \bar{k}$

q.proj. varieties in the classical sense	$\overset{1:1}{\longleftrightarrow}$	reduced q. proj. schemes over k
affine	$\longleftrightarrow$	affine
proj	$\longleftrightarrow$	proj.

X top space.

Def A CHART on X is (U,φ) where $U \subseteq X$ open, $\varphi : U \to \varphi(U)$ is a homeo, $\varphi(U)$ is open in $\mathbb{R}^n$. ($\mathbb{C}^n$)

Two charts are compatible if



$$\varphi(U\cap V) \longrightarrow \psi(U\cap V)$$

$$x \longmapsto \psi(\varphi^{-1}(x))$$

is C^∞ (holomorphic)

$\varphi(U)$

(COMPLEX)

An ATLAS is a collection of charts (U,φ) which are pairwise compatible, $\bigcup U = X$.

COMPLEX

A DIFFERENTIABLE/SMOOTH MANIFOLD is X with an atlas.

complex manifold of dim n $\longrightarrow$ smooth manifold of dim 2n $\longrightarrow$ topological manifolds of dim 2n

biholomorph. diffeomorph. homeomorph.

Freedman
Donaldson
Milnor
"exotic"

X q.proj. var., $x \in X$ regular

$\mathcal{O}_{X,x}$ local ring of germs of functions at x

$\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ maximal ideal

$\mathfrak{m}_x/\mathfrak{m}_x^2$ k-vector space

$\Theta_{X,x} = \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$ TANGENT SPACE of X at x

$X = V(f_1, \dots, f_r) \subset \mathbb{A}^n$ Jacobian matrix $J_x = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{1 \le i \le r \\ 1 \le j \le n}}$

$\Theta_{X,x} = \ker J_x \subseteq k^n$

- $\dim \Theta_{X,x} \geq \dim_x X$

<u>Def</u> $x \in X$ is SMOOTH if $\dim \Theta_{X,x} = \dim_x X$.

$X/\mathbb{C}$, $x \in X$ smooth $\Longrightarrow$ IMPLICIT FUNCTION THEOREM

a nbd of x in X can be identified with an open subset of $\mathbb{C}^n$, where $n = \dim_x X$.

in the analytic topology

$X \subset \mathbb{P}^N$ smooth q.proj. over $\mathbb{C}$.

$\mathbb{P}^N$ Zariski topology

metric / analytic / Euclidean topology $\quad \mathbb{P}^N = \mathbb{C}^{N+1} \setminus \{0\} / \mathbb{C}^*$

$X^{an} = X$ with the analytic topology has a natural complex manifold structure.

from smooth q.proj. varieties over $\mathbb{C}$ X to complex manifold. X^{an} ANALYTIFICATION

X projective $\Rightarrow$ X^{an} compact.

(Every alg. var. with the Zariski top is compact)

$\longmapsto$

- classical algebraic varieties over $k = \bar{k}$

- schemes of finite typ over k

- complex manifold.

X top. space.

<u>Def</u> A PRESHEAF (of abelian groups) on X is the datum F of

- an abelian group $F(U)$ $\forall U \subseteq X$ open
- $\forall V \subseteq U \subseteq X$ open, group hom. $\rho^U_V : F(U) \longrightarrow F(V)$, $s \longmapsto s|_V$

such that

- $U \subseteq X$ open $U \subseteq U$ $\rho^U_U = id_{F(U)}$
- $W \subseteq V \subseteq U \subseteq X$ open

$$\begin{tikzcd} F(U) \arrow[r, "\rho^U_V"] \arrow[dr, "\rho^U_W"'] & F(V) \arrow[d, "\rho^V_W"] \\ & F(W) \end{tikzcd}$$

commutes

<u>Example</u> C^0_X sheaf of continuous functions with values in $\mathbb{R}$ (in $\mathbb{C}$).

$\forall U \subseteq X$ open, $C^0_X(U) = \{U \longrightarrow \mathbb{R}$ continuous$\}$

<u>Example</u> G abelian group

$\forall U \subseteq X$ open $\quad F(U) = G$

$$\rho^U_V = \mathrm{id}_G$$

F is constant presheaf

not a sheaf

<u>Def</u> A presheaf F is a SHEAF if

$\forall U \subseteq X$ open, $\forall \{U_i\}_i$ open cover of U

$s_i \in F(U_i)$

$\forall i, j \quad s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$

$\Rightarrow \exists!\, s \in F(U)$ s.t. $\forall i,\ s|_{U_i} = s_i$.

local sections which are pairwise compatible glue to a unique global section

<u>Examples</u>

- X top. space, C^0_X sheaf of continuous functions $X \to \mathbb{R}$ / $X \to \mathbb{C}$

- X smooth manifold C^∞_X sheaf of C^∞ functions $X \to \mathbb{R}$
  sheaf of $\mathbb{R}$-algebras

- X complex manifold $\mathcal{O}_X$ sheaf of holomorphic functions $X \to \mathbb{C}$
  sheaf of $\mathbb{C}$-algebras

- X q. proj. variety over $k = \bar{k}$ $\mathcal{O}_X$ sheaf of regular functions

  $\forall U \subseteq X$ Zariski open $\mathcal{O}_X(U) = \{ U \to k \text{ regular} \}$
  sheaf of k-algebras

<u>Example</u> X top space, G abelian group

$\underline{G}_X$ constant sheaf on X with constant group G

$\forall U \subseteq X$ open

$$\underline{G}_X(U) = \{U \to G \text{ locally constant}\}$$

$$= \{U \to G \text{ continuous}\}$$

where G is equipped with the discrete topology.

<u>Example</u> $X = \mathbb{R}$, $G = \mathbb{Z}$

$$\underline{G}_X(\mathbb{R}) = \{\mathbb{R} \to \mathbb{Z} \text{ loc. const.}\} \simeq \mathbb{Z}$$

$\mathbb{R}$ connected

$$\underline{G}_X(\mathbb{R} \setminus \{0\}) = \{\mathbb{R} \setminus \{0\} \xrightarrow[\text{loc. const}]{} \mathbb{Z}\} \simeq \mathbb{Z} \oplus \mathbb{Z}$$

__Example__ X complex manifold (e.g. $X \subseteq \mathbb{C}^n$ open)

$\mathcal{O}_X^*$ sheaf of nowhere vanishing holomorphic functions

$$\mathcal{O}_X^*(U) = \{ U \to \mathbb{C}^* \text{ holomorphic} \}$$

the abelian group structure is given by the product in $\mathbb{C}^*$

$Sh(X) = \{$ sheaves of abelian groups on $X \}$ category

HOMORPHISM OF SHEAVES

$F, G \in Sh(X)$, $f: F \to G$ is the datum of a group hom. $f_U : F(U) \to G(U)$ $\forall U \subseteq X$ suchthat it is compatible with the restrictions: $V \subseteq U$

$$\begin{tikzcd} F(U) \arrow[r, "f_U"] \arrow[d, "\rho^U_V \text{ for } F"'] & G(U) \arrow[d, "\rho^U_V \text{ for } G"] \\ F(V) \arrow[r, "f_V"'] & G(V) \end{tikzcd}$$

commutes

$f_1, f_2 \in \mathcal{O}_X(U)$

$e^{(f_1+f_2)(z)} = e^{f_1(z)} \cdot e^{f_2(z)}$

<u>Example</u> X complex manifold

$$\underline{\mathbb{Z}}_X \longrightarrow \mathcal{O}_X$$
$$a \longmapsto 2\pi i a$$

$$\begin{array}{ccc} + & & \cdot \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^* \\ f & \longmapsto & e^f \end{array}$$

$e^{f(z)}$

$f : F \to G$ hom. of sheaves.

$\ker f$ is the sheaf given by $(\ker f)(U) = \ker(f_U : F(U) \to G(U))$

$U \mapsto \operatorname{im}(f_U : F(U) \to G(U))$ is not a sheaf

$$(\operatorname{im} f)(U) = \left\{ s \in G(U) \;\middle|\; \begin{array}{l} \exists \{U_i\} \text{ open cover}, \; s_i \in F(U_i) \text{ s.t.} \\ s|_{U_i} = f_{U_i}(s_i) \end{array} \right\}$$

im f is a sheaf

maybe s is not the image of elements of F(U), but it is locally the image of elements from F.

I can talk about short exact sequences of sheaves

$$0 \to F' \longrightarrow F \longrightarrow F'' \longrightarrow 0.$$

__Example__ $X \subseteq \mathbb{C}$ open

$$\mathbb{Z}_X \xrightarrow{\phi} \mathcal{O}_X \qquad\qquad \mathcal{O}_X \xrightarrow{E} \mathcal{O}_X^*$$

$$a \longmapsto 2\pi i \cdot a \qquad\qquad f \longmapsto e^f$$

$\ker \phi = 0$

$\ker E$: $(\ker E)(U) = \ker\left(\mathcal{O}_X(U) \xrightarrow{E_U} \mathcal{O}_X^*(U)\right) = (\operatorname{im}\phi)(U)$

E is surjective: $U \subseteq X$ arbitrary, $g \in \mathcal{O}_X^*(U)$ $\quad U \xrightarrow{g} \mathbb{C}^*$

choose $\{U_i\}$ open covering of U such that U_i is simply connected

$g|_{U_i}$ has a logarithm, $\exists\, f_i \in \mathcal{O}_X(U_i)$ s.t. $e^{f_i} = g|_{U_i}$

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{\phi} \mathcal{O}_X \xrightarrow{E} \mathcal{O}_X^* \longrightarrow 0 \quad \text{s.e.s. of sheaves.}$$

missing!

$$0 \longrightarrow \mathbb{Z}_X(X) \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X^*(X) \longrightarrow 0$$

$U \subseteq \mathbb{C}$

$g : U \to \mathbb{C}^*$ hol.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ U \arrow[r, "g"] \arrow[ur, dashed] & \mathbb{C}^* \end{tikzcd}$$

$\exists f : U \to \mathbb{C}$ hol/cont

s.t. $\exp \circ f = g$?

f is a $\underline{\underline{\underline{\text{log of } g}}}$

This exists $\iff \pi_1(g, z_0) = 0$.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ \mathbb{C}^* \arrow[r, "id"] \arrow[ur, dashed] & \mathbb{C}^* \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ U \arrow[r, hook, "emb"] \arrow[ur, dashed] & \mathbb{C}^* \end{tikzcd}$$

$g \in \mathcal{O}_X^*(X)$ $\qquad X \xrightarrow{g} \mathbb{C}^*$

X connected
$z_0 \in X$

complex geometry

g has a logarithm $\iff$ $\forall \gamma$ loop in X

$$\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$$

$(\exists f : X \to \mathbb{C}$ hol.

s.t. $\exp \circ f = g)$

$\ker \partial =$ im Ex

$$\mathcal{O}_X^*(X) \xrightarrow{\partial} \mathrm{Hom}(\pi_1(X, z_0), \mathbb{Z}) = H^1(X, \mathbb{Z})$$

$$g \longmapsto \begin{pmatrix} \pi_1(X, z_0) \longrightarrow \mathbb{Z} \\ [\gamma] \longmapsto \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)}\,dz \end{pmatrix}$$

$X \subseteq \mathbb{C}$ open subset, connected

$$0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \xrightarrow{E} \mathcal{O}_X^* \to 0 \quad \text{s.e.s. of sheaves}$$

$$0 \to \underline{\mathbb{Z}}_X(X) \to \mathcal{O}_X(X) \xrightarrow{E_X} \mathcal{O}_X^*(X) \xrightarrow{\partial} H^1(X, \mathbb{Z})$$

$$\underline{\mathbb{Z}}_X(X) = \mathbb{Z}$$

X top. space.

$\forall F \in Sh(X)$, $\forall i \geq 0$ integer, $H^i(X,F)$ abelian group

ith cohomology group of F

- $H^0(X,F) = F(X)$

- $0 \to F' \to F \to F'' \to 0$ s.e.s. of sheaves $\Rightarrow$

$$0 \to H^0(X,F') \to H^0(X,F) \to H^0(X,F'') \to$$
$$\to H^1(X,F') \to H^1(X,F) \to H^1(X,F'') \to$$
$$\to H^2(X,F') \to \cdots$$

long exact sequence

- X decent top space (CW complex), G abelian group

$$H^i(X, \underline{G}_X) \simeq H^i(X, G)$$

sheaf cohomology

constant sheaf

singular cohomology

- X C^∞ manifold, F sheaf of C^∞_X-modules

$\Rightarrow H^i(X, F) = 0 \quad \forall i > 0$ (partition of unity)

- $0 \to G \to F_0 \to F_1 \to F_2 \to \cdots$ exact seq. of sheaves on X

$\forall i > 0, \forall j \geq 0 \quad H^i(X, F_j) = 0$

consider the complex of ab. groups $\quad F_0(X) \to F_1(X) \to F_2(X) \to \cdots$

then the ith cohomology of this complex is $H^i(X, G)$.

Example X complex manifold, connected

$$0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$ exp. exact seq.

$$0 \to \underset{\mathbb{Z}}{\underset{\parallel}{H^0(X,\mathbb{Z})}} \to H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^*) \xrightarrow{\partial}$$

$$\to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{\text{first Chern class}}$$

$$\to H^2(X,\mathbb{Z}) \to \cdots$$

Example X C^∞ manifold.

$\mathcal{A}^p_X$ sheaf of C^∞ p-forms $\quad dx_{i_1} \wedge \dots \wedge dx_{i_p}$

C^∞_X-module

kernel of d = closed forms

im of d = exact forms

$d : \mathcal{A}^p_X \longrightarrow \mathcal{A}^{p+1}_X$

$$0 \to \underline{\mathbb{R}}_X \to \mathcal{A}^0_X \xrightarrow{d} \mathcal{A}^1_X \xrightarrow{d} \mathcal{A}^2_X \xrightarrow{d} \cdots \quad \text{exact}$$

(every closed form is locally exact

since $\mathcal{A}^p_X$ don't have higher cohomology,

$$H^i(X, \underline{\mathbb{R}}_X) = i\text{th cohomology of } \mathcal{A}^0_X(X) \to \mathcal{A}^1_X(X) \to \cdots$$

$$H^i(X, \underline{\mathbb{R}}_X) = H^i(X, \mathbb{R}) = H^i_{dR}(X)$$

de Rham theorem

<u>Example</u> X complex manifold. $\mathcal{O}_X \hookrightarrow C^\infty_X$

$\mathcal{A}^{p,q}_X$ C^∞_X-module of (p,q)-forms.

A local section is $f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ where $f \in C^\infty_X$.

Ω^p_X $\mathcal{O}_X$-module of $(p,0)$-forms

A local section of Ω^p_X is $f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ where $f \in \mathcal{O}_X$.

$\partial : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q}_X$, $\bar{\partial} : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p,q+1}_X$

$0 \to \Omega^p_X \to \mathcal{A}^{p,0}_X \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}_X \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2}_X \to \cdots$ exact sequence of sheaves

$H^q(X, \Omega^p_X) = q$th cohomology of $\mathcal{A}^{p,0}_X(X) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}_X(X) \to \cdots$

$=: H^{p,q}_{\bar{\partial}}(X)$ Dolbeaut cohomology