19.04.2021

$X = \operatorname{Spec} A$ ($k = \bar{k}$ field, A reduced finitely generated k-algebra
X is the unique affine variety s.t. $\mathcal{O}_X(X) = A$

M A-module $\rightsquigarrow$ $\widetilde{M}$ sheaf on X

$\forall f \in A$, $X_f = X \setminus V(f)$ (open in X) $\quad \widetilde{M}(X_f) = M_f$ $\leftarrow$ localisation of M at $\{1, f, f^2, f^3, \dots\}$

<u>Def</u> X scheme (q. proj. var.) $\mathcal{F}$ sheaf on X of $\mathcal{O}_X$-modules ($\forall U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module)
is called

QUASI-COHERENT if $\exists \{U_i\}$ open affine cover, $U_i = \operatorname{Spec} A_i$,
COHERENT $\quad$ M_i finite A_i-modules s.t.

$$\mathcal{F}|_{U_i} \simeq \widetilde{M_i}$$

$QCoh(X)$, $Coh(X)$ category of q.coh / coh sheaves on X.

Thm X affine, $X = \operatorname{Spec} A$

$$\mathrm{Mod}_A \xrightarrow{\sim} \mathrm{QCoh}(X) \qquad \text{equivalence of abelian cat.}$$
$$M \longmapsto \widetilde{M}$$

$$\mathrm{FMod}_A \simeq \mathrm{Coh}(X)$$

Thm (Serre) X scheme (with very mild assumptions).

X affine $\iff$ $\forall F \in \mathrm{QCoh}(X)$ $\forall i > 0$ $\quad H^i(X, F) = 0$

Thm k field. X ~~projective~~ proper over k. $F \in \mathrm{Coh}(X)$

$\Rightarrow \forall i \geq 0$, $H^i(X, F)$ finite dimensional k-vector space.

the dimension is denoted $h^i(F)$.

Euler characteristic

$$\chi(F) := \sum_{i \geq 0} (-1)^i h^i(F)$$

X scheme

<u>Def</u> A LOCALLY FREE SHEAF of rank n on X is a sheaf F of $\mathcal{O}_X$-modules such that $\exists \{U_i\}$ open cover s.t.
(affine)
$F|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus n}$ ($\Rightarrow$ coherent)

<u>Def</u> An INVERTIBLE SHEAF on X is a locally free sheaf of rank 1.

<u>Example</u> $X = \mathbb{P}^n_k$, $d \in \mathbb{Z}$, $\mathcal{O}_{\mathbb{P}^n_k}(d)$ invertible sheaf

<u>Caution</u> $\exists X$ affine and L invertible sheaf on X s.t. $L \not\simeq \mathcal{O}_X$

$A = \mathbb{Z}[\sqrt{-5}]$, $X = \operatorname{Spec} A$, $I = (2, 1+\sqrt{-5})$ $\quad \widetilde{I}$ is invertible

but $\widetilde{I} \not\simeq \widetilde{A} = \mathcal{O}_X$

<u>Example</u> $\mathcal{O}_{\mathbb{P}^n_A}(d) \quad d \in \mathbb{Z}$

homog. polynomials of degree d

$$H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) = \begin{cases} A[x_0, \dots, x_n]_d & d \geq 0 \\ 0 & d < 0 \end{cases}$$

$$H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) = 0 \qquad 0 < i < n, \quad i > n$$

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) \qquad (\rightsquigarrow \text{Serre duality})$$

Example $Z \hookrightarrow X$ closed embedding

$\mathcal{O}_X$ sheaf of regular functions on X

$\mathcal{O}_Z$ " " " " " Z

$$\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$$
$$f \longmapsto f|_Z$$

I = ideal sheaf of $Z \hookrightarrow X$

= { functions on X which vanishes on Z }

$$\boxed{0 \to I \to \mathcal{O}_X \to \mathcal{O}_Z \to 0}$$

s.e.s. of coherent sheaves

If X (and consequently Z) is affine:

$X = \operatorname{Spec} A$

$Z = \operatorname{Spec}(A/J) = V(J)$ $\qquad J \subseteq A$ ideal

$$0 \to I_{Z/X} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \qquad \text{s.e.s. in } \operatorname{Coh}(X)$$

$$0 \to J \to A \to A/J \to 0 \qquad \text{s.e.s. } \operatorname{Mod}_A$$

E.g. $X = \mathbb{A}^2$, $Z = \{(0,0)\}$

$$0 \to (x,y) \to k[x,y] \to k[x,y]/(x,y) \simeq k \to 0$$

it is not an invertible sheaf

Example $d \in \mathbb{Z}$, $d \geq 0$, $f \in S_d \setminus \{0\}$

$S = k[x_0, \dots, x_n]$ with standard grading

$$0 \to S \xrightarrow{\cdot f} S \longrightarrow S/(f) \longrightarrow 0$$ s.e.s. in Mod_S

Is this a sequence in the category of graded S-modules?

$$0 \to S(-d) \xrightarrow{\cdot f} S \longrightarrow S/(f) \longrightarrow 0$$ s.e.s. in cat. of $\mathbb{Z}$-graded S-modules

shift of degrees

$\mathbb{P}^n_k \supset V(f) = Z$

$$0 \longrightarrow \underset{\substack{\parallel \\ I_{Z/\mathbb{P}^n}}}{\mathcal{O}_{\mathbb{P}^n}(-d)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$ s.e.s. in $\mathrm{Coh}(\mathbb{P}^n_k)$

X scheme.

$\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a new sheaf:

$\forall U \subseteq X$ open $\quad \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\right)(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$

coherent.

i.e: $\quad U = \mathrm{Spec}\, A \subseteq X$

$\mathcal{F}|_U = \widetilde{M}, \quad M \in \mathrm{Mod}_A$

$\mathcal{G}|_U = \widetilde{N}, \quad N \in \mathrm{Mod}_A$

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\big|_U = \left(\mathrm{Hom}_A(M, N)\right)^{\sim}$

$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})\big|_U = (M \otimes_A N)^{\sim}$

X scheme.

DUAL INVERTIBLE SHEAF

- L inv. sheaf on X $\rightsquigarrow$ $L^\vee := \mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$ inv. sheaf on X

It is enough to check locally, so can assume X affine and L trivial:

$X = \operatorname{Spec} A$, $L = \tilde{A}$ $\Rightarrow$ $L^\vee = (\operatorname{Hom}_A(A,A))^\sim \simeq \tilde{A}$

- L, M inv. sheaves on X $\rightsquigarrow$ $L \otimes_{\mathcal{O}_X} M$ inv. sheaf on X

Do it locally.

- L inv. sheaf $\Rightarrow$ $L^\vee \otimes_{\mathcal{O}_X} L \simeq \mathcal{O}_X$

$\operatorname{Pic}(X) = \{\text{invertible sheaves}\}/\text{isomorphism}$, $\otimes$ operation, $\mathcal{O}_X = 1$

PICARD GROUP

abelian group

$X = \mathbb{P}^1 \times \mathbb{P}^1$

$x_0, x_1 \quad y_0, y_1$

$f(x_0, x_1, y_0, y_1)$ bihomog. of bidegree (d, e) $\quad d \geq 1$, $e \geq 1$

$C = \{f = 0\}$ irreducible curve

$$\mathbb{Z} \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X \setminus C) \longrightarrow 0$$

$$\mathrm{Pic}(X) \cong \mathbb{Z}^2$$

$\Rightarrow \mathrm{Pic}(X \setminus C) \neq 0$

$X \setminus C$ affine!

DVR discrete valuation ring

A domain. A is a DVR if one (and hence every) of the following eq. conditions holds:

1) A normal, local, $\dim A = 1$ (integrally closed in the fraction field)

0) local PID

2) $\exists t \in A \setminus \{0\}$, $\forall a \in A \setminus \{0\}$, $\exists!$ (unique $n \in \mathbb{N}$, $u \in A^*$ s.t. $a = ut^n$

3) If $K = \operatorname{Frac} A$, then exists a discrete valuation $v : K^* \to \mathbb{Z}$ s.t. $A = \{a \in K^* \mid v(a) \geq 0\} \cup \{0\}$

- group homomor. $v(ab) = v(a) + v(b)$
- surjective
- $a, b \in K^*$, $a+b \neq 0 \Rightarrow v(a+b) \geq \min\{v(a), v(b)\}$

$2 \Rightarrow 3$: if $a = ut^n$ where $u \in A^*$, $n \in \mathbb{N}$

$$\nu(a) := n$$

In this way you have defined ν on $A \setminus \{0\}$

$$\nu\left(\frac{a}{b}\right) = \nu(a) - \nu(b)$$

$3 \Rightarrow 2$: ν surjective $\Rightarrow$ choose $t \in A$ s.t. $\nu(t) = 1$.

$\forall a \in A \setminus \{0\}$ $\quad a = \underbrace{\frac{a}{t^{\nu(a)}}}_{\text{invertible}} \cdot t^{\nu(a)}$

<u>Example</u> 1) $A = k[t]_{(t)}$ t-adic valuation

2) $k[[t]]$, 3) $\mathbb{Z}_{(p)}$ p-adic valuation

$k[t]$

$A = k[t]_{(t)}$ locolisation of $k[t]$ at the prime ideal (t)

$K = \text{Frac } A = k(t)$

t-adic valuation $\quad v : K^* \longrightarrow \mathbb{Z}$

$$t^n \cdot \frac{a(t)}{b(t)} \longmapsto n$$

the order of vanishing of a rational function at 0

$a, b \in k[t]$

$a(0) \neq 0$

$b(0) \neq 0$

$n \in \mathbb{Z}$

$v\left(\frac{1}{1+t}\right) = 0$ invertible in a nbd of 0

$v\left(\frac{t}{t^2+1}\right) = 1$

$k = \bar{k}$, $\lambda \in k$

$v_\lambda : k(t) \longrightarrow \mathbb{Z}$

order of vanishing at the point λ

X normal (e.g. smooth)

Def A PRIME DIVISOR is an irreducible subvariety $D \subset X$ s.t. $\dim D = \dim X - 1$.

$K(X) =$ function field of X

FIELD $= \{$rational functions on $X\}$

$= \{ (U, f) \mid U \subseteq X \text{ open}, U \neq \emptyset, f \in \mathcal{O}_X(U) \} / \sim$

$\mathcal{O}_{X,D} = \{ (U, f) \mid U \subseteq X \text{ open}, U \cap D \neq \emptyset, f \in \mathcal{O}_X(U) \} / \sim$

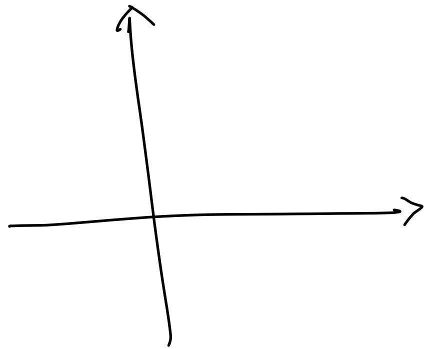
LOCAL RING

$\mathcal{O}_{X,D} \subseteq K(X)$ fraction field

X normal $\Rightarrow$ $\mathcal{O}_{X,D}$ DVR and the valuation $\mathrm{ord}_D : K(X)^* \to \mathbb{Z}$ is order of vanishing along D.

$X = \mathbb{A}^2$

$D = V(x)$ y-axis



$E = V(y)$ x-axis

$\operatorname{ord}_D(xy) = 1$ $\operatorname{ord}_E = 1$

$\operatorname{ord}_D\left(\frac{x}{x+y}\right) = 1$ $\operatorname{ord}_E = 0$

$\operatorname{ord}_D\left(\frac{1}{y^2+y^3x}\right) = 0$

$\operatorname{ord}_E = -2$

X normal.

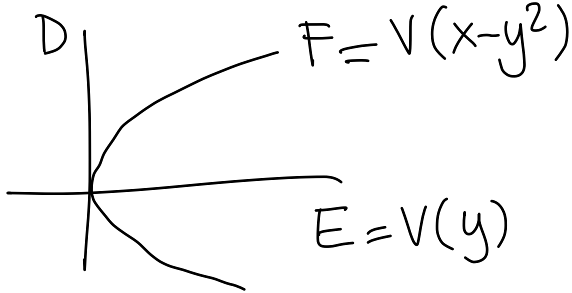
Def A (WEIL) DIVISOR on X is a finite formal sum

$$\sum_{\substack{D \subset X \\ \text{prime divisor}}} a_D \cdot D \qquad a_D \in \mathbb{Z}$$

$$\mathrm{Div}(X) := \{\text{divisors on } X\}$$

free abelian group on prime divisors on X

Example $X = \mathbb{A}^2$



$$2 \cdot D + E - 5 \cdot F \in \mathrm{Div}(X)$$

X normal, $K(X)$ = function field of X

$\mathrm{div}: K(X)^* \longrightarrow \mathrm{Div}(X)$ is a group homomorphism

$$f \longmapsto \sum_{\substack{D \subset X \\ \text{prime divisor}}} \underset{\substack{\cap \\ \mathbb{Z}}}{\mathrm{ord}_D(f)} \cdot D$$

$\mathrm{im}\,\mathrm{div} = \{\text{principal divisors on } X\} \subseteq \mathrm{Div}(X)$

Def $D_1, D_2 \in \mathrm{Div}(X)$ are LINEARLY EQUIVALENT if $D_1 - D_2$ is principal. (we write $D_1 \sim D_2$) $\equiv$, $\sim_{lin}$, $\equiv_{lin}$

Def DIVISOR CLASS GROUP $\mathrm{Cl}(X) := \mathrm{Div}(X)/\sim$.

Example $X = \mathbb{A}^2$

$$f = \frac{x\,y^3}{x^2 + x^3 y}$$

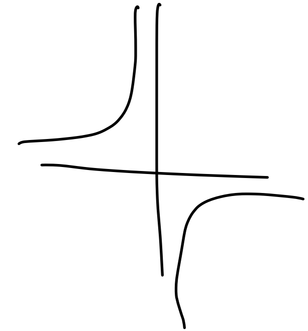
What is $\operatorname{div}(f)$?

$$f = \frac{1}{1+xy} \cdot \frac{1}{x} \cdot y^3$$

$D = V(x)$ y-axis

$E = V(y)$ x-axis

$F = V(1+xy)$



$$\operatorname{div}(f) = -F - D + 3 \cdot E$$

<u>Example</u> $X = \mathbb{P}^n$, $\quad S = k[x_0, \dots, x_n]$

$f \in S_d$ $\quad V(f)$ is a prime divisor $\iff$ f is irreducible.

I cannot talk about $\mathrm{div}(f)$ because f is <u>not</u> a rational function on $\mathbb{P}^n$.

$f, g \in S_d$ $\quad \dfrac{f}{g}$ is a rational function on $\mathbb{P}^n$

What is $\mathrm{div}\left(\frac{f}{g}\right)$?

$f = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$

$g = g_1^{\beta_1} \cdots g_r^{\beta_r}$

irreducible factorisation

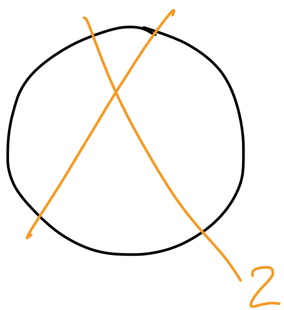
$$\mathrm{div}\left(\frac{f}{g}\right) = \sum_i \alpha_i V(f_i) - \sum_i \beta_i V(g_i)$$

principal divisor.

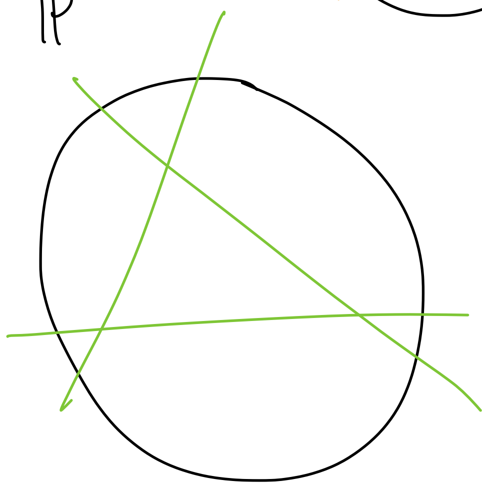
$f = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$ $\qquad$ $g = g_1^{\beta_1} \cdot \ldots \cdot g_s^{\beta_s}$ $\qquad$ $f, g \in S_d$

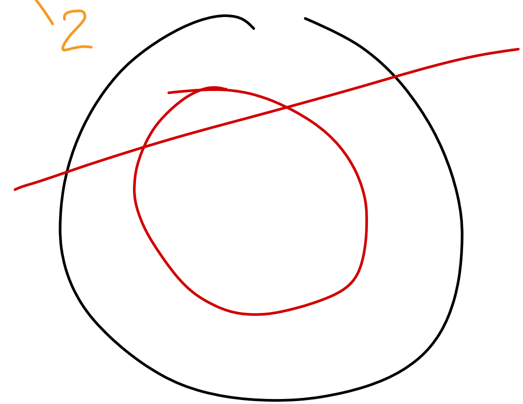
SAME DEGREE

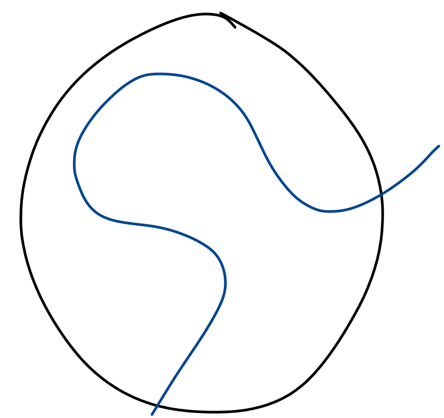
$$\sum_i \alpha_i V(f_i) \sim \sum_i \beta_i V(g_i)$$



$X = \mathbb{P}^2$







cubic