X normal (e.g. smooth)	X arbitrary scheme
• (Weil) divisor	• invertible sheaves
• linear equivalence $\sim$	• isomorphism $\simeq$
• sum of divisors $D_1 + D_2$	• tensor product $L_1 \otimes_{\mathcal{O}_X} L_2$
• divisor class group $Cl(X) = \{\text{divisors}\}/\sim$	• Picard group $\mathrm{Pic}(X) = \{\text{invertible sheaves}\}/\simeq$

<u>Def</u> A divisor $D = \sum a_i D_i$ is called EFFECTIVE if $a_i \geq 0 \; \forall i$.

We write $D \geq 0$

X smooth. $K(X)$ = field of rational functions on X

If D is a divisor on X, then consider $\mathcal{O}_X(D)$ sheaf of $\mathcal{O}_X$-modules
$\forall U \subseteq X$ open

non-zero rat. function

$$\mathcal{O}_X(D)(U) = \{ f \in K(X)^* \mid (\mathrm{div}(f) + D)|_U \geq 0 \} \cup \{0\}$$

- $\mathcal{O}_X(D)$ invertible sheaf
- $D_1 \sim D_2 \iff \mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$
- $\forall L$ inv. sheaf, $\exists D$ divisor on X s.t. $L \simeq \mathcal{O}_X(D)$
- D_1, D_2 divisors $\Rightarrow \mathcal{O}_X(D_1 + D_2) \simeq \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)$

$Cl(X) \longrightarrow Pic(X)$

linear eq. class of $D \longmapsto$ isom. class of $\mathcal{O}_X(D)$

isomorphism of abelian groups

Example $X = \mathbb{P}^n$, $S = \mathbb{C}[x_0, \dots, x_n]$

$H = \{x_0 = 0\}$ hyperplane, prime divisor $\qquad \mathcal{O}_{\mathbb{P}^n}(H) = \mathcal{O}_{\mathbb{P}^n}(1)$

$f \in S_d$, $f = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$, f_i irr. distinct $\qquad \mathcal{O}_{\mathbb{P}^n}(V(f)) = \mathcal{O}_{\mathbb{P}^n}(d)$

$V(f) = \sum_i \alpha_i V(f_i)$ divisor

$V(f) \sim d \cdot H$ proof: $\frac{f}{x_0^d}$ is a rational function on $\mathbb{P}^n$

$\mathrm{div}\left(\frac{f}{x_0^d}\right) = V(f) - d \cdot H$

$$\begin{aligned} Cl(\mathbb{P}^n) &\xrightarrow{\sim} \mathbb{Z} \\ [H] &\longmapsto 1 \\ [V(f)] &\longmapsto d \end{aligned} \qquad \begin{aligned} \mathbb{Z} &\xrightarrow{\sim} \mathrm{Pic}(\mathbb{P}^n) \\ d &\longmapsto [\mathcal{O}_{\mathbb{P}^n}(d)] \end{aligned}$$

__Example__ X smooth, D effective divisor on X

$D = \sum_i a_i D_i \qquad a_i \geq 0$, D_i prime distinct.

$\operatorname{Supp} D = \bigcup_i D_i$ closed subset of X

Equip $\operatorname{Supp} D$ with a scheme structure D.

(E.g. $X = \mathbb{A}^2$, $D = 1 \cdot V(x) + 3 \cdot V(y)$, consider the closed subscheme induced by the ideal (xy^3))

What is $\mathcal{O}_X(-D)$? $U \subseteq X$ open, $f \in K(X)^*$

$$f \in \mathcal{O}_X(-D)(U) \overset{\text{def}}{\Longleftrightarrow} (\operatorname{div}(f) - D)|_U \geq 0$$

$$\operatorname{div}(f) = \sum_{E \text{ prime divisors}} \operatorname{ord}_E(f) \cdot E$$

$$\operatorname{div}(f) - D = \sum_{\substack{E \text{ p.d.} \\ E \neq D_i}} \operatorname{ord}_E(f) \cdot E + \sum_i (\operatorname{ord}_{D_i}(f) - a_i) \cdot D_i$$

$$(\operatorname{div}(f) - D)\big|_U = \sum_{\substack{E \text{ p.d.} \\ E \neq D_i \ \forall i \\ E \cap U \neq \emptyset}} \operatorname{ord}_E(f) \cdot E + \sum_{\substack{i \\ \text{s.t.} \\ D_i \cap U \neq \emptyset}} (\operatorname{ord}_{D_i}(f) - a_i) D_i$$

$$f \in \mathcal{O}_X(-D)(U) \iff \begin{cases} \forall E \ \ldots \quad \operatorname{ord}_E(f) \geq 0 \\ \forall i \text{ s.t. } D_i \cap U \neq \emptyset, \quad \operatorname{ord}_{D_i}(f) \geq a_i \end{cases} \Rightarrow f \in \mathcal{O}_X(U)$$

$\iff f \in I_{D/X}(U)$ ideal of the closed embedding $D \hookrightarrow X$

$$\Rightarrow \mathcal{O}_X(-D) = I_{D/X}$$

$$\begin{array}{c} 0 \longrightarrow I_{D/X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0 \\ \| \\ \mathcal{O}_X(-D) \end{array}$$

<u>Example</u> $X = \mathbb{P}^n$ $S = \mathbb{C}[x_0, \ldots, x_n]$

$f \in S_d \setminus \{0\}$ $D = V(f)$ with the correct scheme structure

$$0 \to S(-d) \xrightarrow{f} S \longrightarrow S/(f) \to 0$$

s.e.s. in the category of $\mathbb{Z}$-graded S-modules

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_D \to 0$$

s.e.s. in $\mathrm{Coh}(\mathbb{P}^n)$

<u>Example</u> X smooth projective curve over $k=\bar{k}$

prime divisors on X $=$ points on X

If $D = \sum_i n_i \cdot p_i$, then deg $D := \sum_i n_i \in \mathbb{Z}$

- $D_1 \sim D_2 \implies \deg D_1 = \deg D_2$

$\deg : Cl(X) \simeq \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$ group homomorph.

- $X = \mathbb{P}^1 \implies$ deg isomorphism

- $X \neq \mathbb{P}^1 \implies$ deg surjective, but not injective
(i.e. $g(X) \geq 1$)

The kernel is called JACOBIAN of X

X smooth variety, L inv. sheaf on X, $s \in H^0(X, L)$ section.

If $x \in X$, the number $s(x)$ is NOT well defined because it depends on the trivialisation of L in a nbd of x.
But it makes sense to say if $s(x) = 0$ or $s(x) \neq 0$.

If s is not identically zero, can take
$V(s) := \{x \in X \mid s(x) = 0\}$
is an effective divisor on X such that $\mathcal{O}_X(V(s)) \simeq L$.

<u>Example</u> $X = \mathbb{P}^n$, $L = \mathcal{O}(d)$, $d \geq 1$, $s \in H^0(X, L) = \mathbb{C}[x_0, \ldots, x_n]_d$
$\forall x \in X$ it makes sense to say if $s(x) = 0$ or $s(x) \neq 0$.
$V(s)$ is the hypersurface with equation s.

X smooth proj. variety, L inv. sheaf on X.

The COMPLETE LINEAR SYSTEM of L is

$$|L| := \{ D \in \mathrm{Div}(X) \mid D \geq 0, \quad \mathcal{O}_X(D) \simeq L \}.$$

$$\begin{aligned} \mathbb{P}(H^0(X,L)) &\longrightarrow |L| \\ [s] &\longmapsto V(s) \end{aligned}$$

is a bijection

this equips $|L|$ with the structure of a projective space

A LINEAR SYSTEM is a (linear) projective subspace of $\mathbb{P}(H^0(X,L)) \simeq |L|$.

Abuse of notation: if $D \in \mathrm{Div}(X)$ with $D \geq 0$, then

$$|D| := |\mathcal{O}_X(D)| = \{ D' \in \mathrm{Div}(X) \mid D' \geq 0, \ D' \sim D \}$$

X smooth proj. var over k, L inv. sheaf on X.

$s_0, \ldots, s_n \in H^0(X, L)$

$B = \{x \in X \mid s_0(x) = 0, \ldots, s_n(x) = 0\}$ closed subset of X

$$\begin{aligned} \phi : X \setminus B &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \cdots : s_n(x)] \end{aligned}$$

morphism

The tuple $(s_0(x), \ldots, s_n(x)) \in k^{n+1}$ depends on the choice of a trivialisation of L in a nbd of x

But the element $[s_0(x) : \cdots : s_n(x)] \in \mathbb{P}^n$ is well defined

What is the preimage of a hyperplane?

In 2 slides we will see the definition of B and ϕ which doesn't depend on $s_0, \ldots, s_n$, but depends only on $\mathrm{Span}(s_0, \ldots, s_n) \subseteq H^0(X, L)$

<u>Example</u> $X = \mathbb{P}^2$ with homog. coord x_0, x_1, x_2. $L = \mathcal{O}(2)$

$H^0(X, L) = \mathbb{C}[x_0, x_1, x_2]_2$

Consider the sections x_0^2, $x_0 x_1$, x_2^2

The base locus is $\{[x_0 : x_1 : x_2] \in \mathbb{P}^2 \mid x_0^2 = 0, x_0 x_1 = 0, x_2^2 = 0\} =$
$= \{[0:1:0]\}$. So

$$\phi : \mathbb{P}^2 \setminus \{[0:1:0]\} \longrightarrow \mathbb{P}^2$$

$$[x_0 : x_1 : x_2] \longmapsto [x_0^2 : x_0 x_1 : x_2^2]$$

X smooth proj. var, L inv. sheaf on X
$V \subseteq H^0(X,L)$ linear subspace $\rightsquigarrow$
$\mathfrak{V} := \mathbb{P}(V) \subseteq \mathbb{P}(H^0(X,L))$ proj. subspace, i.e. LINEAR SYSTEM

BASE LOCUS $\quad \mathrm{Bs}\,\mathfrak{V} := \{x \in X \mid \forall s \in V,\ s(x) = 0\} \overset{\text{closed}}{\subseteq} X$

MAP INDUCED BY $\mathfrak{V}$ $\quad \phi_{\mathfrak{V}} : X \setminus \mathrm{Bs}\,\mathfrak{V} \longrightarrow \mathbb{P}(V^\vee)$

$$x \longmapsto \text{evaluation at } x$$

- $\mathfrak{V}$ is called BASE POINT FREE if $\mathrm{Bs}\,\mathfrak{V} = \emptyset$.
- L is called BASE POINT FREE if $|L|$ is base point free
- L is called VERY AMPLE if L is base point free and $\phi_{|L|} : X \longrightarrow \mathbb{P}(H^0(X,L)^\vee)$ is a closed embedding
- L is called AMPLE if it exists $m \in \mathbb{Z}, m \geq 1$ s.t. $L^{\otimes m}$ is very ample.

<u>Example</u> $X = \mathbb{P}^1$ with homog. coordinates x_0, x_1. $L = \mathcal{O}(2)$

- $V = \mathrm{Span}(x_0^2, x_1^2) \subseteq \mathbb{C}[x_0, x_1]_2 = H^0(\mathbb{P}^1, \mathcal{O}(2))$

$$\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$
$$[x_0 : x_1] \longmapsto [x_0^2 : x_1^2]$$

is not injective; almost every point has 2 preimages

- $H^0(\mathbb{P}^1, \mathcal{O}(2))$ has basis x_0^2, x_0x_1, x_1^2

$$\phi_{|\mathcal{O}(2)|} : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$
$$[x_0 : x_1] \longmapsto [x_0^2 : x_0x_1 : x_1^2]$$

is a closed embedding. So $\mathcal{O}(2)$ is very ample.

the image is $V(y_1^2 - y_0y_2)$.

where y_0, y_1, y_2 are the coords on $\mathbb{P}^2$

Example $d \geq 1$

A basis of $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is given by the monomials of degree d in the variables $x_0, \dots, x_n$.

$$\phi_{|\mathcal{O}(d)|} : \mathbb{P}^n \longrightarrow \mathbb{P}^N \qquad N = h^0(\mathbb{P}^n, \mathcal{O}(d)) - 1$$

is a closed embedding, called VERONESE EMBEDDING

So $\mathcal{O}_{\mathbb{P}^n}(d)$ is very ample $\forall d \geq 1$.

Example $\mathbb{P}^m \times \mathbb{P}^n$

$$
\begin{tikzcd}
 & \mathbb{P}^m \times \mathbb{P}^n \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
\mathbb{P}^m & & \mathbb{P}^n
\end{tikzcd}
$$

$\forall a, b \in \mathbb{Z}$

$$\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a,b) = pr_1^* \mathcal{O}_{\mathbb{P}^m}(a) \otimes_{\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}} pr_2^* \mathcal{O}_{\mathbb{P}^n}(b)$$

$\phi_{|\mathcal{O}(1,1)|}$ is a closed embedding, called SEGRE EMBEDDING

$\mathcal{O}(a,b)$ (very) ample $\Longleftrightarrow$ $a \geq 1, b \geq 1$

$X \xrightarrow{f} Y$

F locally free sheaf on Y of rank n $\Rightarrow$

f^*F " " " X " "

Locally: $Y = \operatorname{Spec} A$ $\quad A \to B$
$X = \operatorname{Spec} B$

$F = (A^{\oplus n})^{\sim}$ $\quad f^*F = (B \otimes_A A^{\oplus n})^{\sim} = (B^{\oplus n})^{\sim}$

$(X, \mathcal{O}_X)$ top. space with a sheaf of $\mathbb{K}$-algebras

E.g.: variety / scheme / complex manifold / smooth manifold
regular functions / holomorph. functions / C^∞ functions

$\pi : E \to X$ **VECTOR BUNDLE** of rank n:

$\exists \{U_i\}$ open cover of X s.t.

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{K}^n \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

$\forall x \in X$ $\pi^{-1}(x)$ has a structure of $\mathbb{K}$-vector space of dim n

If $U \subseteq X$ open, a **SECTION** of $\pi : E \to X$ on U is
$s : U \to E$ s.t. $\pi \circ s$ is the inclusion $U \hookrightarrow X$.

$\pi : E \to X$ vector bundle of rank n

$\forall\, U \subseteq X$ open, $\mathcal{E}(U) := \{\text{sections of } E \xrightarrow{\pi} X \text{ over } U\}$
is an $\mathcal{O}_X(U)$-module

$\mathcal{E}$ sheaf of $\mathcal{O}_X$-modules

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{K}^n \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

$\forall\, U \subseteq U_i$ open
$\Rightarrow \mathcal{E}(U) \simeq \{U \to \mathbb{K}^n\} \simeq \mathcal{O}_X(U)^n$

$\Rightarrow \mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus n}$

$\Rightarrow \mathcal{E}$ is a locally free sheaf of rank n.

vector bundles of rank n on X $=$ locally free sheaf of $\mathcal{O}_X$-modules of rank n

$E \longmapsto$ sheaf of sections

$\underline{\mathrm{Spec}}_X \, \mathrm{Sym}^{\bullet}_{\mathcal{O}_X} \mathcal{E}^\vee \longleftarrow\!\!| \; \mathcal{E}$

line bundle (vector bundle of rank 1) $=$ invertible sheaf (locally free sheaf of rank 1)

X smooth manifold of dimension n

T_X tangent bundle	locally free sheaf of rank n
$\mathcal{A}_X^1$ cotangent bundle	" " " " n
$\mathcal{A}_X^i = \Lambda^i \mathcal{A}_X^1$ bundle of i-forms	" " " " $\binom{n}{i}$

of C_X^∞-modules

This can be done in algebraic geometry:

X smooth variety over k of dimension n

T_X tangent sheaf / bundle, rank n

Ω^1_X cotangent sheaf / bundle, rank n

$\Omega^i_X = \Lambda^i \Omega^1_X$ sheaf of i-forms, rank $\binom{n}{i}$

$\omega_X := \Lambda^n \Omega^1_X$ CANONICAL BUNDLE, rank 1 $\Rightarrow$ line bundle

<u>Def</u> A CANONICAL DIVISOR of X is any divisor K_X on X such that $\omega_X \simeq \mathcal{O}_X(K_X)$.
(It is well defined only up to linear equivalence)

Example $X = \mathbb{P}^n$

$$\omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1)$$

Example X smooth proj. curve over $k = \bar{k}$ of genus $g = h^1(\mathcal{O}_X)$.

Then:

- $\deg \omega_X = 2g - 2$
- $h^0(\omega_X) = g$
- $g = 0 \iff X \simeq \mathbb{P}^1 \iff \deg \omega_X < 0$
- $g = 1 \iff \deg \omega_X = 0 \iff \omega_X \simeq \mathcal{O}_X$
- $g \geq 2 \iff \deg \omega_X > 0 \iff \omega_X$ ample

Important example X smooth variety. D prime divisor on X.
Assume D smooth. $i : D \hookrightarrow X$

NORMAL BUNDLE $N_{D/X}$
(it is a line bundle)

differential geometry:
$\forall x \in D \quad (N_{D/X})_x = T_{X,x} / T_{D,x}$

algebraic geometry:

$\mathcal{O}_X(D)$ line bundle on X

$$N_{D/X} := i^*(\mathcal{O}_X(D)) =: \mathcal{O}_D(D)$$
$$=: (\mathcal{O}_X(D))\big|_D$$

$$0 \to \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

apply $- \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow N_{D/X} \longrightarrow 0$$

$i : D \hookrightarrow X$

$i^* : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(D)$

$i_* : \mathrm{Coh}(D) \longrightarrow \mathrm{Coh}(X)$ fully faithful, exact

$$i_*\mathcal{O}_D \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) = i_* i^* \mathcal{O}_X(D)$$

Continuation $D \xhookrightarrow{i} X$ smooth divisor in smooth variety $n = \dim X$

In differential geometry: $i^* T_X = T_D \oplus N_{D/X}$

In complex/algebraic geometry: $0 \to T_D \to i^* T_X \to N_{D/X} \to 0$

is a sequence of locally free sheaves of $\mathcal{O}_D$-modules.

$\mathcal{H}om_{\mathcal{O}_D}(\cdot\ , \mathcal{O}_D)$:

$$0 \to (N_{D/X})^\vee \to i^*\Omega^1_X \to \Omega^1_D \to 0$$

conormal sequence

$$(N_{D/X})^\vee = I_{D/X}/I_{D/X}^2$$

$$\Rightarrow \quad \Lambda^n i^* \Omega^1_X \simeq (N_{D/X})^\vee \otimes_{\mathcal{O}_X} \Lambda^{n-1} \Omega^1_D$$

$$i^* \Lambda^n \Omega^1_X \qquad (N_{D/X})^\vee \otimes_{\mathcal{O}_X} \omega_D$$

$$i^* \omega_X$$

$$\Lambda^n(E \oplus F) \simeq \bigoplus_{i+j=n} \Lambda^i E \otimes \Lambda^j F$$

E has rank $n-1$
F has rank 1

$$\Rightarrow \quad \Lambda^n(E \oplus F) \simeq \Lambda^{n-1} E \otimes \underbrace{\Lambda^1 F}_{\parallel \atop F}$$

$D \overset{i}{\hookrightarrow} X$ smooth divisor in smooth variety

$$i^*\omega_X \simeq (N_{D/X})^\vee \underset{\mathcal{O}_D}{\otimes} \omega_D$$

$$\omega_D \simeq N_{D/X} \underset{\mathcal{O}_D}{\otimes} i^*\omega_X = i^*\mathcal{O}_X(D) \underset{\mathcal{O}_D}{\otimes} i^*\omega_X$$

$$= i^*\left(\mathcal{O}_X(D) \underset{\mathcal{O}_X}{\otimes} \omega_X\right)$$

ADJUNCTION

" $K_D \sim (D+K_X)|_D$ "

Theorem (Serre duality)

X smooth projective variety over k of dimension n

F locally free sheaf on X

ω_X canonical bundle

dual vector space

$\Rightarrow \forall i \geq 0$

$$H^i(X, F) \simeq \left(H^{n-i}(X, F^\vee \otimes_{\mathcal{O}_X} \omega_X) \right)^\vee$$

$F^\vee = \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$

dual sheaf