INTERSECTION PRODUCT ON THE PICARD GROUP OF A SURFACE

- X will be nonsingular projective surface over $k = \bar{k}$
- A curve will mean an effective divisor
- GOAL: Define and calculate the intersection number

$$C.D \in \mathbb{Z}$$

for any two divisors $C, D \in \operatorname{Div} X$

Intersection Multiplicity:
C, D curves with no common irr. component.
If $P \in C \cap D$,

$$(C.D)_P := \dim_k \left(\mathcal{O}_{P,X} / (f,g) \right)$$

where f, g are local equations for C, D resp.

C, D meet transversally at $P \in C \cap D$ if $(C.D)_P = 1$.

D is transversal to C if

$$(C.D)_P = 1, \quad \forall P \in C \cap D.$$

Thm 1.1 (HA77, p.357)

There is a unique pairing

$$\begin{aligned} \mathrm{Div}\, X \times \mathrm{Div}\, X &\longrightarrow \mathbb{Z} \\ (C, D) &\longmapsto C.D \end{aligned}$$

s.t.

(1) if C, D are nonsingular curves meeting transversally, then
$$C.D = \#(C \cap D)$$
(2) it is symmetric: $C.D = D.C$

(3) it is additive: $(C_1 + C_2).D = C_1.D + C_2.D$

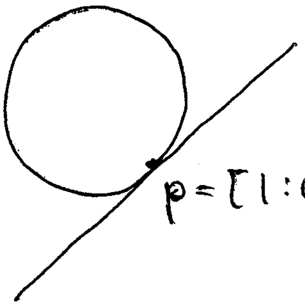
(4) it depends only on linear equivalence class: $C_1 \sim C_2 \Rightarrow C_1.D = C_2.D$

Ex: Intersection multiplicity

$X = \mathbb{P}^2$, $\mathbb{C} = k$

$\{X_0 X_2 - X_1^2 = 0\} = C$

$\{X_2 = 0\} = D$



$p = [1:0:0]$

$(C.D)_p$?

$p \in \mathbb{P}^2 \setminus \{X_0 = 0\} \xrightarrow{\simeq} \mathbb{A}^2$

$[X_0 : X_1 : X_2] \longmapsto \left(\frac{X_1}{X_0}, \frac{X_2}{X_0}\right) = (x, y)$

C has the local equation $y - x^2$ around p

D has the local equation y around p

$$(C.D)_p = \dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{p,\mathbb{P}^2}}{(f,g)}\right)$$

$$= \dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{(0,0),\mathbb{A}^2}}{(y, y-x^2)}\right)$$

$$\mathcal{O}_{(0,0),\mathbb{A}^2}/(y, y-x^2) \cong \frac{\mathbb{C}[x,y]_{(x,y)}}{(y, x^2)}.$$

Then we have the iso. of vecter spaces

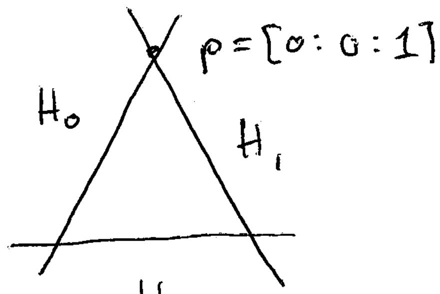
$$\frac{\mathbb{C}[x,y]_{(x,y)}}{(y, x^2)} \xrightarrow{\simeq} \mathbb{C}^2$$

$$1 \longmapsto (1,0)$$

$$x \longmapsto (0,1).$$

$$\dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{(0,0),\mathbb{A}^2}}{(y, y-x^2)}\right) = 2.$$

Ex: $X = \mathbb{P}^2$, $k = \mathbb{C}$



$H_0 = \{x_0 = 0\}$

$H_1 = \{x_1 = 0\}$

$H_2 = \{x_2 = 0\}$

$\underset{C}{\overset{H_0}{\shortparallel}}, \underset{D}{\overset{H_1}{\shortparallel}}$ $\quad H_2$

$C, D \in \operatorname{Div} X$, C, D are curves and nonsingular.

$C.D$?

Claim: C, D meet transversally

$\rightsquigarrow C.D = \#(C \cap D) = 1$

P is only intersection point

$p \in \mathbb{P}^2 \setminus H_2 \xrightarrow{\simeq} \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) = (x, y)$

C has the local equation y around p

D ——— " ——— x around p

$$(C.D)_p = \dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{p,X}}{(x,y)}\right) = \dim_{\mathbb{C}}\left(\frac{\mathbb{C}[x,y]_{(x,y)}}{(x,y)}\right)$$

$$= 1$$

where $\mathcal{O}_{p,X} \cong \mathcal{O}_{(0,0), \mathbb{A}^2}$.

Remark: (4) implies that this pairing induce a pairing on

$$\mathrm{Div}\, X/_{\sim} \cong \mathrm{Pic}(X)$$

Before proof:

Lemma 1.2 (HA77, p. 358).
Let $C_1, \dots, C_r$ be irr. curves on X, and let D be a very ample divisor. Then almost all curves $D' \in |D|$ are irr. nonsingular and meet each of the C_i transversally.

Lemma 1.3 (HA77, p. 358)
Let C be an irr. nonsingular curve on X and let D be any curve meeting C transversally. Then

$$\#(C \cap D) = \deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_C).$$

Pf: If time.

Pf of Thm 1.1:

Uniqueness:

Let $H \in \operatorname{Div} X$ be ample.

Given $C, D \in \operatorname{Div} X$ we want to find $n \in \mathbb{N}$ s.t $C + nH$, $D + nH$, nH are all very ample.

The existence of such $n \in \mathbb{N}$ follows from these general statements

•) H ample iff. $\forall \mathcal{M} \in \operatorname{Coh}(X)$ $\exists k \in \mathbb{N}$ s.t

$\mathcal{M} \otimes H^k$ is generated by global sections

*) H very ample, $\mathcal{M}$ generated by global sections, both invertible, then

$\mathcal{M} \otimes H$ is very ample.

Find $l \in \mathbb{N}$ s.t lH is very ample, then

$n = l + k$.

Lemma 1.2 gives

$C' \in |C + nH|$

$D' \in |D + nH|$ transversal to C'

$E' \in |nH|$ —— " —— D'

$F' \in |nH|$ —— " —— C' and E'

and

$C \sim C' - E'$ and $D \sim D' - F'$

From property (1)-(4) of the pairing:

$$C.D = \#(C' \cap D') - \#(C' \cap F') - \#(E' \cap D') + \#(E' \cap F').$$

This shews uniqueness.

Existence

Let $B \subseteq \mathrm{Div}\, X$ denote subgroup of very ample divisors.

Define

$$C.D = \#(C' \cap D'), \quad C, D \in B$$

$C' \in |C|$ nonsingular

$D' \in |D|$ nonsingular and transversal to C'

Well-defined?

$D'' \in |D|$, $D'' \neq D'$, nonsingular and transversal to C'

From lemma 1.3

$$\#(C' \cap D') = \deg_C(\mathcal{O}_X(D') \otimes \mathcal{O}_{C'})$$
$$= \deg_C(\mathcal{O}_X(D'') \otimes \mathcal{O}_{C'}) = \#(C' \cap D'')$$

By symmetry, we have shown well-definedness.

This pairing satisfies (1)-(4). (9)

Let $C, D \in \operatorname{Div} X$

$C \sim C' - E'$, $D \sim D' - F'$

where $C', D', E', F' \in B$.

We define

$$C.D = C'.D' - C'.F' - E'.D' + E'.F'$$

Well-defined?

$C \sim C'' - E''$ with $C'', E'' \in B$

$\rightsquigarrow \quad C' - E' \sim C'' - E''$

$\rightsquigarrow \quad C'.D' - E'.D' = C''.D' - E''.D'$

Similar for $-F'$.

So
$$\begin{aligned} \operatorname{Div} X \times \operatorname{Div} X &\to \mathbb{Z} \\ (C, D) &\mapsto C.D \end{aligned}$$

is well-defined.

$$\mathrm{Div}\,X \times \mathrm{Div}\,X \to \mathbb{Z}$$

satisfies (2) - (4) by construction.

(1)? C, D nonsingular meeting transv.

$$C.D = C'.D' - C'.F' - E'.D' + E'.F'$$

$$= \deg(\mathcal{O}_X(D') \otimes \mathcal{O}_{C'}) - \deg(\mathcal{O}_X(F') \otimes \mathcal{O}_{C'})$$

$$- \deg(\mathcal{O}_X(D') \otimes \mathcal{O}_{E'}) + \deg(\mathcal{O}_X(F') \otimes \mathcal{O}_{E'})$$

$$= \deg(\mathcal{O}_X(\overbrace{D' - F'}^{D}) \otimes \mathcal{O}_{C'}) - \deg(\mathcal{O}_X(D' - F') \otimes \mathcal{O}_{E'})$$

Choose C', E' s.t. C', E' meets D transv. using Lemma 1.2.

$$\rightsquigarrow \deg(\mathcal{O}_X(D) \otimes \mathcal{O}_{C'}) = \#(C'.D)$$

$$= \deg(\mathcal{O}_X(C') \otimes \mathcal{O}_D)$$

$$\rightsquigarrow \deg(\mathcal{O}_X(D) \otimes \mathcal{O}_{C'}) - \deg(\mathcal{O}_X(D) \otimes \mathcal{O}_{E'})$$

$$= \deg(\mathcal{O}_X(C) \otimes \mathcal{O}_D) = \#(C \cap D).$$

Corollary

X surface. C nonsingular irr. curve on X. $D \in \operatorname{Div} X$.
Then

$$C.D = \deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_C).$$

Pf: Consult the proof of Thm 1.1.

Proposition 1.4 (HA77, p. 360)
If C, D are curves on X having no common irr. component, then

$$C.D = \sum_{p \in C \cap D} (C.D)_p$$

Pf: Claim: The RHS is independent of representaties.

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \quad \text{SES}$$

Tensoring with $\mathcal{O}_C$ preserves exactness

$$(*) \quad 0 \to \mathcal{O}_X(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{D \cap C} \to 0 \quad \text{SES}$$

From (*) we get

$$(+)\quad \chi(\mathcal{O}_{C\cap D}) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_X(-D)\otimes\mathcal{O}_C).$$

$$\chi(\mathcal{O}_{C\cap D}) = \dim_k H^0(X, \mathcal{O}_{C\cap D})$$

$$= \dim_k\Big(\bigoplus_{p\in C\cap D} \mathcal{O}_{p,X}/(f_p, g_p)\Big)$$

$$= \sum_{p\in C\cap D} (C.D)_p$$

(+) shows that RHS is independent of representative.

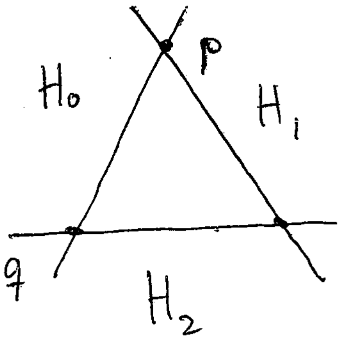
Choose $C \sim C' - E'$, $D \sim D' - F'$ as in "Uniqueness" part.

$$\sum_{p\in C\cap D} (C.D)_p = \sum_{p\in (C'-E')\cap(D'-E')} (C'-E'.D'-F')_p =$$

$$\sum_{p\in C'\cap D'} (C'.D')_p + \sum_{p\in E'\cap F'} (E'.F')_p - \sum_{p\in D'\cap E'} (D'.E')_p - \sum_{p\in C'\cap F'} (C'.F')_p = (C.D$$

Ex (Revisited)

$X = \mathbb{P}^2$, $k = \mathbb{C}$ $\quad C = 2H_0$, $D = H_1 + 3H_2$.



Both curves having no common irr. component.

$C \cap D = \{p, q\} = \{[0:0:1], [0:1:0]\}$

Use prop 1.4 to calculate $C.D$.

$p \in \mathbb{P}^2 \setminus H_2 \xrightarrow{\simeq} \mathbb{A}^2$

$[x_0 : x_1 : x_2] \longmapsto \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) = (x, y)$

$q \in \mathbb{P}^2 \setminus H_1 \xrightarrow{\simeq} \mathbb{A}^2$

$[x_0 : x_1 : x_2] \longmapsto \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) = (\tilde{x}, \tilde{y})$

C has the local equation y^2 around p

C ———— " ———— $\tilde{y}^2$ around q

D ———— " ———— x around p

D ———— " ———— $\tilde{x}^3$ around q

$$(C.D)_p = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}[x,y]_{(x,y)}}{(x,y^2)} \right) = 2$$

$$(C.D)_q = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}[\tilde{x},\tilde{y}]_{(\tilde{x},\tilde{y})}}{(\tilde{x}^3,\tilde{y}^2)} \right) = 6$$

$$\leadsto \quad C.D = (C.D)_p + (C.D)_q = 8.$$

Ex (HA77, p. 361, 1.4.2)

Let $X = \mathbb{P}^2$. Recall $\operatorname{Pic} \mathbb{P}^2 \cong \mathbb{Z}$.

Then the linear equivalence class containing all the lines is a generator, denoted h.

$$h.h = \sum_{p \in H_0 \cap H_1} (H_0 . H_1) = 1.$$

This determines the intersection product by linearity.

Ex (HA77, p. 361, 1.4.3)

Let X be the nonsingular quadratic surface in $\mathbb{P}^3$. $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

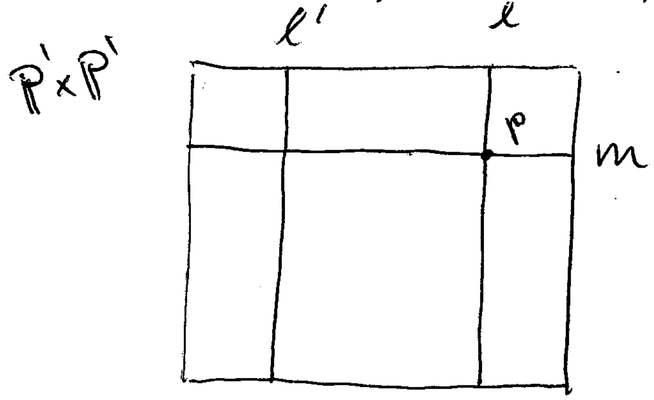
Then $\operatorname{Pic} X \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Let l be of type $(1,0)$

and m —— " —— $(0,1)$.

Want to argue that

$$l^2 = 0, \quad m^2 = 0, \quad l.m = 1.$$



$l' \sim l$.

$l^2 = l.l' = 0$ and similarly for m.

$l.m = (l.m)_p = 1$.

This determines the intersection product on X

Ex (HA77, p. 360, 1.4.1)

X surface, $D \in \mathrm{Div}\, X$

Was is the self-intersection number

$D.D =: D^2$

Cannot use prop. 1.4

If D is a nonsingular irr. curve on X apply corollary

$$\rightsquigarrow D^2 = \deg_D(\mathcal{O}_X(D) \otimes \mathcal{O}_D)$$

$$= \deg_D(N_{C/X}).$$

Proposition (HA77, p. 366, exercise 1.1)
Let X be surface, $C, D \in \operatorname{Div} X$.
Then

$$C.D = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(C)^{-1}) - \chi(\mathcal{O}_X(D)^{-1}) + \chi(\mathcal{O}_X(C)^{-1} \otimes \mathcal{O}_X(D)^{-1})$$

Pf of Lemma 1.3

$$0 \to \mathcal{O}_X(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{C \cap D} \to 0 \quad \text{SES}$$

$\mathcal{O}_X(-D) \otimes \mathcal{O}_C = i^*(\mathcal{O}_X(-D))$, $\quad i : C \hookrightarrow X$.

$i^*(\mathcal{O}_X(-D))$ ideal sheaf of $C \cap D \hookrightarrow C$.

$\rightsquigarrow i^*(\mathcal{O}_X(-D)) = \mathcal{O}_C(-(C \cap D))$

$$\begin{aligned}
\rightsquigarrow \mathcal{O}_C(C \cap D) &= i^*(\mathcal{O}_X(-D))^\vee \\
&= i^*(\mathcal{O}_X(-D)^\vee) \\
&= i^*(\mathcal{O}_X(D)) \\
&= \mathcal{O}_X(D) \otimes \mathcal{O}_C.
\end{aligned}$$

$\deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_C) = \deg_C(C \cap D) = \#(C \cap D)$,

since C, D meet transversally.

Calculations

$C+nH$ very ample?

$$\mathcal{O}_X(C+nH) = \mathcal{O}_X(C) \otimes \mathcal{O}_X(nH)$$

$$= \mathcal{O}_X(C) \otimes \mathcal{O}_X(H)^{n\otimes} = \mathcal{O}_X(C) \otimes \mathcal{O}_X(H)^{k\otimes} \otimes \mathcal{O}^{l\otimes}$$

By • $\mathcal{O}_X(C) \otimes \mathcal{O}_X(H)^{k\otimes}$ generated by global sections. By $*$

$$\mathcal{O}_X(C) \otimes \mathcal{O}_X(H)^{k\otimes} \otimes \mathcal{O}_X(H)^{l\otimes}$$

very ample

Additivity of $B \times B \to \mathbb{Z}$.

WLOG: C, D, D' nonsingular and D, D' transversal to C.

$$C.(D+D') = \#(C \cap D+D') = \deg_C(\mathcal{O}_X(D+D') \otimes \mathcal{O}_C)$$

$$= \deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_X(D') \otimes \mathcal{O}_C)$$

$$= \deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_C) + \deg_C(\mathcal{O}_X(D') \otimes \mathcal{O}_C)$$

$$= \#(C \cap D) + \#(C \cap D') = C.D + C.D'$$

X smooth projective surface over $k = \bar{k}$

curve on X = effective divisor on X

irreducible curve on X = prime divisor on X

<u>Prop</u> C, D curves on X without common irr. components $\Rightarrow C \cdot D \geq 0$

<u>Pf</u> $C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p \geq 0$. $\square$

<u>Prop</u> C irr. curve on X, $C^2 < 0$
$\Rightarrow |C| = \{C\}$ and $\dim_k H^0(X, \mathcal{O}_X(C)) = 1$.

<u>Pf</u> By contradiction, $\exists D \in |C| \setminus \{C\}$. Then $D \sim C$, $D \geq 0$, $D \neq C$. $0 > C^2 = C \cdot D$. By the previous proposition C and D have a common irr. component. But C is irr. So C appears in D: $D = C' + C$ where $C' \geq 0$. From $D \sim C$ we get $C' \sim 0$.

Now $C' \geq 0$, $C' \sim 0 \Rightarrow C' = 0 \Rightarrow C = D$ $\lightning$ $\square$

X surface. $p \in X$ point

$\pi : \tilde{X} \longrightarrow X$ blow up of X at p

$E = \pi^{-1}(p)$ is a smooth irreducible curve on $\tilde{X}$

$$E \simeq \mathbb{P}^1$$

$$N_{E/\tilde{X}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$$

so $E^2 = \deg_E N_{E/\tilde{X}} = -1.$

Lemma $(A,\mathfrak{m})$ local Cohen-Macaulay (e.g. regular) ring.
$f, g \in \mathfrak{m}$.
If $\dim A = 2$ and $\dim A/(f,g) = 0$, then

$$A/(g) \xrightarrow{\cdot f} A/(g) \quad \text{is injective.}$$

Pf See Theorem 17.4 (iii) in Commutative Ring Theory by Matsumura

Prop X smooth surface, C and D curves on X with no common irreducible component.

$\Rightarrow 0 \to \mathcal{O}_X(-C)|_D \to \mathcal{O}_D \to \mathcal{O}_{C\cap D} \to 0$ exact.

Pf Consider $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ and tensor by $- \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_D$, get

$$\mathcal{O}_X(-C)|_D \to \mathcal{O}_D \to \mathcal{O}_C \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_D = \mathcal{O}_{C\cap D} \to 0 \quad (*)$$

We need to prove that it is left exact. Recall that $\mathcal{O}_C$ is not a flat $\mathcal{O}_X$-module!

We examine the stalks of the sheaves in $(*)$ at each point $p \in X$.

Denote $A = \mathcal{O}_{X,p}$.

There are several cases:

- $p \in X \setminus D$

$$0 \to 0 \to 0 \to 0 \quad \text{is left exact}$$

- $p \in D \setminus C$: let g be the local equation of D at p

$$A/(g) \xrightarrow{1} A/(g) \to 0 \to 0 \quad \text{is left exact}$$

- $p \in C \cap D$: let g be the local equation of D at p
  " f " " " C "

$$A/(g) \xrightarrow{f} A/(g) \to A/(f,g) \to 0$$

is left exact because of lemma. $\square$