Riemann - Roch Theorem for Surfaces

Goal: Prove Riemann-Roch theorem for surfaces.

Today:

- Self-intersection of surfaces.
- Adjunction Formula
- Riemann - Roch Theorem.

Recall: Last time: D a divisor on the surface X.

We defined: Self-intersection $D \cdot D = D^2$.

↳ With this idea: Define a numerical invariant of a surface

X a surface $/ k$

$\Omega_{X/k}$ sheaf of differentials of X/k.

$\omega_X = \Lambda^2 \Omega_{X/k}$ the canonical sheaf.

K canonical divisor. Well-def up to linear equivalence.

$K^2 :=$ Self-intersection of the canonical divisor. Depends only on X.

An anticanonical divisor is $-K$ for K canonical.

Canonical Divisor for $\mathbb{P}^n$:

Take a rational n-form on an open affine subset of $\mathbb{P}^n$: Compute what it looks like on the other charts.

Coords $(x_0 : \dots : x_n)$ on $\mathbb{P}^n$

On chart $U_0 = \{x_0 \neq 0\}$.

$$\omega = d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge \dots \wedge d\tilde{x}_n \qquad \text{for} \quad \tilde{x}_i = \frac{x_i}{x_0} \quad i>0$$

On chart $U_n = \{x_n \neq 0\}$

- $y_i = \dfrac{x_i}{x_n}$ $\quad i=0,\dots,n-1$

$$d\tilde{x}_i = d\left(\frac{x_i}{x_0}\right) = d\left(\frac{y_i}{y_0}\right) \quad \text{for } i=1,\dots,n-1$$

$$d\tilde{x}_n = d\left(\frac{x_n}{x_0}\right) = d\left(\frac{1}{y_0}\right)$$

$$\Rightarrow \quad \omega = d\left(\frac{y_1}{y_0}\right) \wedge d\left(\frac{y_2}{y_0}\right) \wedge \dots \wedge d\left(\frac{1}{y_0}\right)$$

$$= -\frac{1}{y_0^{n+1}} dy_1 \wedge \dots \wedge dy_{n-1} \wedge dy_0 \qquad d\left(\frac{y_1}{y_0}\right) = \frac{dy_1}{y_0} - \frac{y_1}{y_0^2} dy_0$$

Pole of order $n+1$ on hyperplane H given by $y_0 = 0$.

So the canonical divisor: $K_{\mathbb{P}^n} \sim -(n+1)H$

Remark: $\mathbb{P}^n \setminus (U_0 \cup U_n)$ is codimension 2 subvariety, no divisor can be contained in it.
$\Rightarrow$ Enough to consider 2 charts.

By Def: $\omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n})$

$$\Rightarrow \omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

Example: $X = \mathbb{P}^2$. Compute K^2.

- Start with a rational 2-form on one open affine subset of $\mathbb{P}^2$.
- Compute its expression in other charts.

$(x:y:z)$ coords. of $\mathbb{P}^2$.

In chart $U_z = \{z \neq 0\}$ take $x := \frac{x}{z}$, $y := \frac{y}{z}$ and take the 2-form

$$dx \wedge dy \quad \longrightarrow \quad \text{No zeros, no poles.}$$

In $U_y := \{y \neq 0\}$ take $u := \frac{x}{y}$, $v := \frac{z}{y}$ so that

$$dx \wedge dy = d\left(\frac{u}{v}\right) \wedge d\left(\frac{1}{v}\right) = \left(\frac{du}{v} - \frac{u}{v^2}\,dv\right) \wedge \left(-\frac{1}{v^2}\,dv\right)$$
$$= -\frac{1}{v^3}\,du \wedge dv.$$

Has a pole of order 3 on hyperplane $v=0$.

$\Rightarrow \quad K = -3H$, $\quad H$ hyperplane where we have poles, given by $z=0$.

So $K = -3H \Rightarrow K^2 = 9H^2 = 9$

$\downarrow$

Any two lines in $\mathbb{P}^2$ are linearly eq.
Two distinct lines meet in 1 point.

Example: X non-singular quadric surface in $\mathbb{P}^3$.
$X \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Remark: $\mathrm{Cl}\, X \simeq \mathbb{Z} \oplus \mathbb{Z}$. $(\simeq \mathrm{Pic}(X))$.

$p_1 : X \longrightarrow \mathbb{P}^1$, $p_2 : X \longrightarrow \mathbb{P}^1$ projections on factors

Induces: $p_1^* : \mathrm{Cl}\, \mathbb{P}^1 \longrightarrow \mathrm{Cl}\, X$, $p_2^* : \mathrm{Cl}\, \mathbb{P}^1 \longrightarrow \mathrm{Cl}\, X$.

Identify $\mathrm{Cl}(\mathbb{P}^1) \simeq \mathbb{Z}$, $1 \longmapsto$ class of a point.

$$\mathbb{Z} \xrightarrow{p_2^*} \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X \setminus \{pt\} \times \mathbb{P}^1) \simeq \mathrm{Cl}(\mathbb{A}^1 \times \mathbb{P}^1) \simeq \mathrm{Cl}(\mathbb{P}^1) \text{ isom.}$$

$$\mathbb{Z} \xrightarrow{p_1^*} \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(\mathbb{A}^1 \times \mathbb{P}^1) \longrightarrow 0$$

$$\Rightarrow \mathrm{Cl}(X) \simeq \mathrm{Im}\, p_1^* \oplus \mathrm{Im}\, p_2^* \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

- If D is any divisor on X, let (a,b) the ordered pair of integers in $\mathbb{Z} \oplus \mathbb{Z}$ corresponding to the class of D under this isomorphism.

Then D is of type (a,b) on X.

Since $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$: Rational 2-form on X $\longleftrightarrow$ Rational 1-forms on both copies of $\mathbb{P}^1$

zeros on 2-form $\longleftrightarrow$ Pre-images of zeros on 1-forms on $\mathbb{P}^1$.

So to compute K_X : Compute first $K_{\mathbb{P}^1}$

As before : Take coords $(x:y)$ on $\mathbb{P}^1$.

For the chart $U_y = \{y \neq 0\}$ take $x = \frac{x}{y}$ and the 1-form : dx

$\Rightarrow$ On $U_x = \{x \neq 0\}$: $u = \frac{y}{x}$ so the 1-form becomes

$$dx = d\left(\frac{1}{u}\right) = -\frac{1}{u^2}\,du \longrightarrow \text{Pole of order } 2.$$

$\Rightarrow \quad K_{\mathbb{P}^1} \sim -2H$

↳ Hyperplane in $\mathbb{P}^1$: Point

By fact that will be proven below (on Canonical sheaf of product of two smooth varieties):

K_X is of type $(-2,-2)$

Recall: C, D divisors on X,

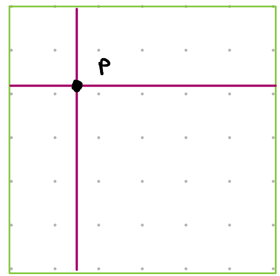
C of type (a,b), D of type (a',b')

then $C \cdot D = ab' + a'b$

$K_X^2 = (-2)(-2) + (-2)(-2) = 8.$

$\mathbb{P}^1 \times \mathbb{P}^1$

$l \sim (1,0)$



$m \sim (0,1)$

$(l \cdot m)_p = 1.$

Riemann-Roch for curves:

Def: Let X be a curve. The <u>genus</u> of X is

$$g := \dim H^1(X, \mathcal{O}_X) = h^1(\mathcal{O}_X)$$

Remark: By Serre-Duality,

$$g = \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \omega_X) = h^0(\omega_X)$$

Notation: D a divisor on a curve X,

$$\ell(D) := \dim H^0(X, \mathcal{O}_X(D)).$$

Theorem (Riemann-Roch): Let D a divisor on a curve X of genus g. Then

$$\ell(D) - \ell(K-D) = \deg D + 1 - g$$

Example: On a curve X of genus g, the canonical divisor K has degree $2g-2$.

By Riemann-Roch with $D = K$.

$$\ell(K) - \ell(0) = \deg K + 1 - g$$

Now $\ell(K) = \dim H^0(X, \mathcal{O}_X(K)) = \dim H^0(X, \omega_X) = g$

$\ell(0) = 1$ since $H^0(X, \mathcal{O}_X(0)) \cong \mathbb{K}$.

$$\Rightarrow \quad g - 1 = \deg K + 1 - g \quad \Leftrightarrow \quad \deg K = 2g-2.$$

Adjunction Formula:

Prop: C a non-singular irreducible curve. For D a divisor on X,

$$C \cdot D = \deg_C \mathcal{O}_X(D)|_C$$

Prop (Adjunction Formula): Let C a non-singular irreducible curve of genus g on a surface X. Let K be the canonical divisor on X, then

$$2g-2 = C\cdot(C+K).$$

Proof: Let $i : C \longrightarrow X$.
By Adjunction:

$$\omega_C \simeq (\mathcal{O}_X(C) \otimes_{\mathcal{O}_X} \omega_X)|_C \qquad \text{where } \omega_X = \mathcal{O}_X(K_X)$$
$$\simeq \mathcal{O}_X(K_X + C)|_C$$

Now taking degree:

$$\deg \omega_C = 2g-2 \quad \text{from Ex above.}$$

$$\deg_C \mathcal{O}_X(K_X+C)|_C = C\cdot(C+K_X) \quad \text{from Previous Prop.}$$

$$\Rightarrow \quad 2g-2 = C\cdot(K_X+C).$$

$\blacksquare$

Example: C a smooth irreducible curve of degree d in $\mathbb{P}^2$.

We have seen $K_{\mathbb{P}^2} \sim -3H$

Hence $C \sim dH$ and $C + K_{\mathbb{P}^2} \sim (d-3)H$

By adjunction formula: $2g - 2 = (d-3) \cdot d \cdot H^2 = (d-3) \cdot d$.

$$\Rightarrow \quad g = \frac{1}{2}(d-1)(d-2).$$

Which is the genus-degree formula for plane curves.

Example: Let C be a curve of type (a,b) on the quadric surface in $\mathbb{P}^3$.

Since $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ is of type $(-2,-2)$ then

$$C + K_{\mathbb{P}^1 \times \mathbb{P}^1} \text{ is of type } (a-2,\ b-2)$$

By Adjunction Formula:

$$2g - 2 = a(b-2) + (a-2)b$$

$$\Rightarrow \quad g = ab - a - b + 1.$$

Prop 1: Let $f : X \longrightarrow Y$ a morphism, let $g : Y' \longrightarrow Y$ also a morphism of schemes, and let: $f' : X' := X \times_Y Y' \longrightarrow Y'$ be obtained by base extension.

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Then,

$$\Omega_{X'/Y'} \cong g'^{*}(\Omega_{X/Y}).$$

where $g' : X' \longrightarrow X$ the projection.

Proof: Follows from properties of modules of relative differential forms.

Prop 2: Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ morphism of schemes. There is an exact sequence of sheaves on X,

$$f^{*}\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Lemma: Let X, Y be non-singular varieties over $\mathbb{K}$. Then

$$\omega_{X \times Y} \simeq p_1^* \omega_X \otimes p_2^* \omega_Y .$$

for $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ the projections.

Proof: First of all we prove

$$\Omega_{X \times Y / \mathbb{K}} \simeq p_1^* \Omega_{X/\mathbb{K}} \oplus p_2^* \Omega_{Y/\mathbb{K}} .$$

By Prop 1 writing $\mathbb{K}$ for Y and Y for Y' we get,

$f : X \longrightarrow \mathbb{K}$ and $g : Y \longrightarrow \mathbb{K}$ morphisms

and $f' : X \times_{\mathbb{K}} Y \longrightarrow Y$ by base extension, then we obtain

$$\Omega_{X \times Y / Y} \simeq p_1^* (\Omega_{X/\mathbb{K}})$$

and also $$\Omega_{X \times Y / X} \simeq p_2^* (\Omega_{Y/\mathbb{K}})$$

Now by Prop 2 writing $\mathbb{K}$ for Z, X for Y and $X \times Y$ for X we have

$$f = p_1 : X \times_{\mathbb{K}} Y \longrightarrow X \quad , \quad g : X \longrightarrow \mathbb{K}$$

morphisms of schemes. Then

$$(*) \quad \Omega_{X\times Y/Y} \simeq p_1^*(\Omega_{X/\mathbb{K}}) \longrightarrow \Omega_{X\times Y/\mathbb{K}} \longrightarrow \Omega_{X\times Y/X} \simeq p_2^*(\Omega_{Y/\mathbb{K}}) \longrightarrow 0$$

is exact.

Also,

$$(**) \quad p_2^*(\Omega_{Y/\mathbb{K}}) \longrightarrow \Omega_{X\times Y/\mathbb{K}} \longrightarrow p_1^*(\Omega_{X/\mathbb{K}}) \longrightarrow 0$$

is exact.

Both sequences imply that $(*)$ splits so

$$\Omega_{X\times Y/\mathbb{K}} \simeq p_1^*(\Omega_{X/\mathbb{K}}) \oplus p_2^*(\Omega_{Y/\mathbb{K}})$$

Now let $m = \dim X$, $n = \dim Y$.

By def. of the canonical sheaf:

$$\begin{aligned}
\omega_{X\times Y} &= \Lambda^{n+m} \Omega_{X\times Y} \\
&\simeq \Lambda^{n+m} (p_1^*(\Omega_X) \oplus p_2^*(\Omega_Y)) \\
&\simeq \Lambda^m (p_1^*(\Omega_X)) \otimes \Lambda^n (p_2^*(\Omega_Y)) \\
&\simeq p_1^*(\Lambda^m \Omega_X) \otimes p_2^*(\Lambda^n(\Omega_Y)) \\
&\simeq p_1^*\omega_X \otimes p_2^*\omega_Y .
\end{aligned}$$

$\blacksquare$

Prop: let $f: X \longrightarrow Y$ a flat morphism, D an effective divisor on Y, then $f^{-1}D$ is an effective divisor on X and

$$\mathcal{O}_X(f^{-1}D) = f^*(\mathcal{O}_Y(D))$$

$$\Rightarrow \quad H^0(X, \mathcal{O}_X(f^{-1}D)) \longleftarrow H^0(Y, \mathcal{O}_Y(D))$$

$$\text{section def. } f^{-1}D = f^* s_0 \longleftarrow\!\shortmid \; s_0 = \text{Section defining } D$$

Remark: Prop holds also if f is an embedding of a divisor that has no common components with D.

Example (Product of two curves):

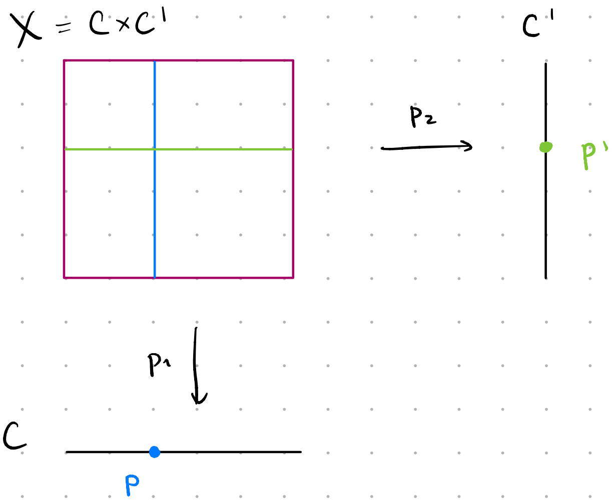
Let X be a product of two non-singular projective curves C, C' of genus g, g' resp.

$$X = C \times C'$$

Let $p_1 : X \longrightarrow C$, $p_2 : X \longrightarrow C'$ the projections.
By the Lemma,

$$K_X = p_1^* K_C + p_2^* K_{C'}$$

$$K_X^2 = (p_1^* K_C)^2 + 2\, p_1^* K_C \cdot p_2^* K_{C'} + (p_2^* K_{C'})^2$$



$\forall\, L \in \mathrm{Pic}(C)$, $p_1^* L \in \mathrm{Pic}(X)$

$L' \in \mathrm{Pic}(C')$, $p_2^* L \in \mathrm{Pic}(X)$

$\forall\, p \in C$, $p_1^{-1}(p) = \{p\} \times C'$

$$p_1^* \mathcal{O}_C(p) = \mathcal{O}_X(\{p\} \times C')$$

$\forall\, p' \in C'$,

$$p_2^* \mathcal{O}_{C'}(p') = \mathcal{O}_X(C \times \{p'\})$$

(1) $p_1^* \mathcal{O}_C(p) \cdot p_2^* \mathcal{O}_{C'}(p') = (\{p\} \times C') \cdot (C \times \{p\}) = 1.$

(2) $\forall\, p, q \in C,\quad p_1^* \mathcal{O}_C(p) \cdot p_1^* \mathcal{O}_C(q) = 0$

- Let $L \in \mathrm{Pic}(C)$.

$$L = \mathcal{O}_C\Big(\sum_i a_i p_i\Big) \qquad a_i \in \mathbb{Z}$$

$$p_1^* L = \mathcal{O}_X\Big(\sum_i a_i (\{p_i\} \times C')\Big)$$

$$(p_1^* L)^2 = \sum_{i,j} a_i a_j (\{p_i\} \times C') \cdot (\{p_j\} \times C') \overset{(2)}{=} 0$$

Similarly, $\quad 0 = (p_2^* L')^2 \quad$ for $L' \in \mathrm{Pic}(C')$.

- let $L \in \mathrm{Pic}(C)$, $L' \in \mathrm{Pic}(C')$

$$L = \mathcal{O}_C\left(\sum_i a_i p_i\right) \qquad L' = \mathcal{O}_{C'}\left(\sum_j b_j q_j\right)$$

$$(p_1^* L) \cdot (p_2^* L') = \sum_{i,j} a_i b_j \left(\{p_i\} \times C'\right) \cdot \left(C \times \{q_j\}\right)$$

$$\overset{(1.)}{=} \sum_{i,j} a_i b_j = \sum_i a_i \cdot \sum_j b_j$$

$$= (\deg_C L) \cdot (\deg_{C'} L')$$

Thus,

$$K_X^2 = (p_1^* \cancel{K_C})^2 + \underbrace{2\, p_1^* K_C \cdot p_2^* K_{C'}}_{2(2g-2)(2g'-2)} + (p_2^* \cancel{K_{C'}})^2$$

Riemann – Roch for curves

$$\Rightarrow \quad K_X^2 = 8(g-1)(g'-1).$$

Example: Let X be a surface of degree d in $\mathbb{P}^3$.

$X \sim dH$ for H a hypersurface.

First we compute: $K_{\mathbb{P}^3}$.

Same as before: $dx \wedge dy \wedge dz$ in chart $U_w = \{w \neq 0\}$ for

$$x = \frac{X}{W}, \quad y = \frac{Y}{W}, \quad z = \frac{Z}{W}.$$

In chart $y \neq 0$: $u = \frac{x}{y}$, $v = \frac{z}{y}$, $r = \frac{w}{y}$.

$$dx \wedge dy \wedge dz = -\frac{1}{r^4}\, du \wedge dr \wedge dv$$

$$\Rightarrow \quad K_{\mathbb{P}^3} = -4H.$$

By Adjunction: $K_X = (K_{\mathbb{P}^3} + dH)|_X = (d-4)H|_X$

So $K_X^2 = (d-4)^2 \, H|_X^{\,2}$

Compute $H|_X{}^2$:

H', H'' general hyperplanes in $\mathbb{P}^3$.

Now,

$$H|_X{}^2 = \mathcal{O}_X(1)^2 = \mathcal{O}_{\mathbb{P}^3}(H')|_X \cdot \mathcal{O}_{\mathbb{P}^3}(H'')|_X$$

$\mapsto$ on X

$$= (X \cap H') \cdot (X \cap H'')$$
$$= \# (X \cap H') \cap (X \cap H'')$$
$$= \# (X \cap H' \cap H'')$$
$$= \underbrace{(X \cap H'')}_{\text{Curve of degree } d \text{ in } H''} \cdot \underbrace{(H' \cap H'')}_{\text{Line in } H''} = d \cdot 1 = d.$$

$$\Rightarrow \quad K_X{}^2 = (d-4)^2 \cdot d.$$

Example: Let X be the smooth intersection of 2 quadrics in $\mathbb{P}^4$

$$X = Y_1 \cap Y_2$$

Exact Sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{Y_1} \longrightarrow 0$$

Tensoring by $- \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_{Y_2}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2)|_{Y_2} \longrightarrow \mathcal{O}_{Y_2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Exactness of the sequence implies:

$$\mathcal{O}_{Y_2}(X) = \mathcal{O}_{\mathbb{P}^4}(2)\Big|_{Y_2}$$

By Adjunction:

$$\omega_{Y_i} \simeq \left(\mathcal{O}_{\mathbb{P}^4}(2) \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_{\mathbb{P}^4}(-5) \right)\Big|_{Y_i}$$

$$\simeq \mathcal{O}_{\mathbb{P}^4}(-3)\Big|_{Y_i}$$

Again by adjunction:

$$\omega_X \simeq (\mathcal{O}_{Y_i}(X) \otimes_{\mathcal{O}_{Y_i}} \omega_{Y_i})|_X$$

$$\simeq (\mathcal{O}_{\mathbb{P}^4}(2) \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_{\mathbb{P}^4}(-3))|_X$$

$$\simeq \mathcal{O}_{\mathbb{P}^4}(-1)|_X = \mathcal{O}_X(-1).$$

Compute K_X^2:

Take $H_1, H_2 \subset \mathbb{P}^4$ general hyperplanes. $\mathcal{O}_{\mathbb{P}^4}(H_i) = \mathcal{O}_{\mathbb{P}^4}(1)$

Let $C_i := H_i \cap X$ C_i is a smooth irreducible curve.

H_1, H_2 general $\Rightarrow$ C_1, C_2 curves meeting transversally on X.

$$\mathcal{O}_{\mathbb{P}^4}(H_i) = \mathcal{O}_{\mathbb{P}^4}(1) \Rightarrow \mathcal{O}_{\mathbb{P}^4}(1)|_X = \mathcal{O}_X(H_i \cap X) = \mathcal{O}_X(C_i)$$

$$\Rightarrow K_X^2 = \mathcal{O}_X(1) \cdot \mathcal{O}_X(1) = C_1 \cdot C_2 = \#(C_1 \cap C_2)$$

Compute $\#(C_1 \cap C_2)$:

Let $\Pi = H_1 \cap H_2 \simeq \mathbb{P}^2$.

$$\left.\begin{array}{l} Y_1 \cap \Pi \\ Y_2 \cap \Pi \end{array}\right\} \text{ Plane conics in } \Pi$$

$$\Downarrow \text{ Bezout's thm}$$

They intersect in 4 pts.

Now,

$$\begin{aligned}(Y_1 \cap \Pi) \cap (Y_2 \cap \Pi) &= Y_1 \cap Y_2 \cap H_1 \cap H_2 \\ &= X \cap H_1 \cap H_2 \\ &= (X \cap H_1) \cap (X \cap H_2) \\ &= C_1 \cap C_2 .\end{aligned}$$

Finally we get : $K_X^2 = \#(C_1 \cap C_2) = 4$.

Riemann - Roch for surfaces

For D any divisor on the surface X, let

$$l(D) := \dim H^0(X, \mathcal{O}_X(D))$$

Thus $l(D) = \dim |D| + 1$, for $|D|$ the complete linear system of D.

Def: The <u>superabundance</u> $s(D)$ of D is

$$s(D) = \dim H^1(X, \mathcal{O}_X(D))$$

Side Note on Terminology: Before invention of cohomology, Riemann-Roch Formula was written only with $l(D)$, $l(D-K)$.

Superabundance $\Rightarrow$ Amount by which it failed to hold.

Def: The <u>arithmetic genus</u> p_a of X is dim $X = r$

$$p_a^X = (-\chi(\mathcal{O}_X) + 1)(-1)^r$$

where $\chi(\mathcal{O}_X)$ is Euler-characteristic of $\mathcal{O}_X$ given by

$$\chi(\mathcal{O}_X) = \sum_i (-1)^i \dim H^i(X, \mathcal{O}_X)$$

Recall: Euler-characteristic is additive on short exact sequences,

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

e.s. of coherent sheaves on X, then

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$$

Theorem (Riemann-Roch for surfaces):

If D is any divisor on the surface X, then

$$\ell(D) - s(D) + \ell(K-D) = \frac{1}{2} D\cdot(D-K) + 1 + p_a^X$$

Proof: By Serre Duality, we have

$$\begin{aligned} \ell(K-D) &= \dim H^0(X, \mathcal{O}_X(K-D)) \\ &= \dim H^0(X, \mathcal{O}_X(D)^\vee \otimes \mathcal{O}_X(K)) \quad \text{where by def. } \omega_X = \mathcal{O}_X(K) \\ &= \dim H^0(X, \mathcal{O}_X(D)^\vee \otimes \omega_X) \\ &= \dim H^2(X, \mathcal{O}_X(D)). \end{aligned}$$

So we see LHS:

$$\begin{aligned} \ell(D) - s(D) + \ell(K-D) &= \dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) + \dim H^2(X, \mathcal{O}_X(D)) \\ &= \chi(\mathcal{O}_X(D)). \end{aligned}$$

Hence we have to show: $\chi(\mathcal{O}_X(D)) = \frac{1}{2} D\cdot(D-K) + 1 + p_a^X$.

Both sides $\Rightarrow$ Depend only on the linear eq. class of D.

We can write: $D = C - E$

Non-singular curves.

By choosing linearly equivalent non-singular $D' \sim D$.

Let: Ideal sheaf of C in X $= \mathcal{O}_X(-C)$

" E in X $= \mathcal{O}_X(-E)$

We have exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0$$

Tensoring with $\mathcal{O}_X(C)$:

$$0 \longrightarrow \mathcal{O}_X(C-E) \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_X(C) \otimes \mathcal{O}_E \longrightarrow 0 \qquad (1)$$

Similarly with

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

Tensoring with $\mathcal{O}_X(C)$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_X(C) \otimes \mathcal{O}_C \longrightarrow 0 \qquad (2)$$

Since χ is additive on s.e.s :

(1) $\Rightarrow \quad \chi(\mathcal{O}_X(C-E)) + \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_E) = \chi(\mathcal{O}_X(C))$

(2) $\Rightarrow \quad \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C) = \chi(\mathcal{O}_X(C)).$

Combining these eqs:

$$\chi(\mathcal{O}_X(C-E)) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C) - \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_E) \quad (3)$$

By def. $\chi(\mathcal{O}_X) = 1 + p_a.$

By Riemann-Roch for curves:

$$\dim H^0(X, \mathcal{O}_X(C) \otimes \mathcal{O}_C) - \dim H^0(X, (\mathcal{O}_X(C) \otimes \mathcal{O}_C)^\vee \otimes \omega_X) = \deg_C(\mathcal{O}_X(C)|_C) + 1 - g_C$$

But

$$\dim H^0(X, (\mathcal{O}_X(C) \otimes \mathcal{O}_C)^\vee \otimes \omega_X) = \dim H^1(X, \mathcal{O}_X(C) \otimes \mathcal{O}_C)$$

So $\quad \dim H^0(X, \mathcal{O}_X(C)\otimes \mathcal{O}_C) - \dim H^0(X, (\mathcal{O}_X(C)\otimes \mathcal{O}_C)^\vee \otimes \omega_X) = \chi(\mathcal{O}_X(C)\otimes \mathcal{O}_C)$

$$\Rightarrow \quad \chi(\mathcal{O}_X(C)\otimes \mathcal{O}_C) = \deg_C (\mathcal{O}_X(C)\otimes \mathcal{O}_C)|_C + 1 - g_C .$$

Now use: $\quad C\cdot D = \deg_C (\mathcal{O}_X(D)\otimes \mathcal{O}_C)|_C$

To get $\quad \chi(\mathcal{O}_X(C)\otimes \mathcal{O}_C) = C^2 + 1 - g_C \qquad (4)$

Similarly for $\mathcal{O}_X(C)\otimes \mathcal{O}_E$

$$\chi(\mathcal{O}_X(C)\otimes \mathcal{O}_E) = C\cdot E + 1 - g_E \qquad (5)$$

Finally by Adjunction Formula:

$$g_C = \frac{1}{2} C\cdot(C+K) + 1 \quad , \quad g_E = \frac{1}{2} E\cdot(E+K) + 1 . \qquad (6)$$

Combining (3) – (6):

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(C-E))$$

$$= \chi(\mathcal{O}_X) + C^2 \cancel{+1} - g_C - (C\cdot E \cancel{+1} - g_E)$$

$$= p_a^X + 1 + C^2 - \left(\frac{1}{2} C\cdot(C+K) + 1\right) - C\cdot E + \left(\frac{1}{2} E\cdot(E+K) + 1\right)$$

$$= \frac{1}{2}(C-E)\cdot(C-E-K) + 1 + p_a^X$$

$$= \frac{1}{2} D\cdot(D-K) + 1 + p_a^X$$

$\blacksquare$

Remark: Another formula considered to be part of Riemann-Roch Thm:

$$12(1+p_a) = K^2 + \underbrace{C_2}_{\text{Second Chern class of the tangent sheaf of } X.}$$

Prop: If D is an effective divisor on the surface X,

$$2p_a^D - 2 = D\cdot(D+K)$$

Proof: By Riemann-Roch for surfaces:

$$\chi(\mathcal{O}_X(-D)) = \frac{1}{2}(-D)\cdot(-D-K_X) + 1 + p_a^X$$

Using the exact sequence: $0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$

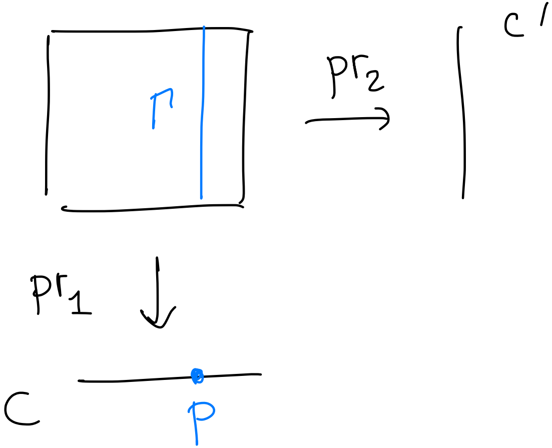
Using also: $\chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_D) = \chi(\mathcal{O}_X)$

Then:
$$\chi(\mathcal{O}_D) = \frac{1}{2}D\cdot(-D-K_X) - 1 - p_a^X + \chi(\mathcal{O}_X)$$

$$1 - p_a^D = \frac{1}{2}D\cdot(-D-K_X) - \cancel{1} - \cancel{p_a^X} + \cancel{p_a^X} + \cancel{1}.$$

$$\Rightarrow \quad 2p_a^D - 2 = D\cdot(D+K_X)$$

$\blacksquare$



$X = C \times C'$

$\Gamma = pr_1^{-1}(p) = \{p\} \times C'$

normal bundle

$N_{p/C} = \mathcal{O}_p$ trivial line bundle

$N_{\Gamma/X} = pr_1^* N_{p/C} = \mathcal{O}_\Gamma$ trivial line bundle

$\Rightarrow \Gamma^2 = \deg_\Gamma N_{\Gamma/X} = 0.$

$f: X \to Y$ morphism of schemes, flat

$D \hookrightarrow Y$ effective Cartier divisor

$\implies f^{-1}D$ effective Cartier divisor on X:

Locally: $Y = \operatorname{Spec} A$, $D = (s=0) = \operatorname{Spec} A/(s)$ $s \in A$

D Cartier iff s is a regular element of A

$A \xrightarrow{\cdot s} A$ is injective

non-zero-divisor

$f: X \to Y$
$B \xleftarrow{\varphi} A$

$f^{-1}D = \operatorname{Spec} B/Bs$

Is $\varphi(s)$ non-zero divisor in B? Is $B \xrightarrow{\cdot s} B$ injective?

If B is flat over A, yes

Riemann-Roch: X smooth projective, connected.

- $\dim X = 1$, $L \in \mathrm{Pic}(X) \implies \chi(L) = \deg L + 1 - g$

  where $g = h^1(\mathcal{O}_X)$ $\qquad = \chi(\mathcal{O}_X) + \underline{\deg L}$

- $\dim X = 2$, $L \in \mathrm{Pic}(X) \implies \chi(L) = \underline{\frac{1}{2} L\cdot(L-K_X)} + \chi(\mathcal{O}_X)$

depends only on topological properties of L.

Hirzebruch-Riemann-Roch:

E locally free sheaf on X

$$\chi(E) = \int_X \mathrm{ch}(E)\,\mathrm{td}(T_X)$$

$X \subset \mathbb{P}^n$ complete intersection of codim r

$X = V(f_1, \dots, f_r)$ $\quad \deg f_i = d_i > 0$

$\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$

$\omega_X = \mathcal{O}_X(-n-1+d_1+\dots+d_r)$ by Adjunction.