# Plan

17.05

1) Introduce a new equivalence relations between Divisors and the corresponding abelian groups

i) Linear $\rightsquigarrow$ $\mathrm{Pic}(X)$

ii) Algebraic $\rightsquigarrow$ $NS(X)$

iii) Numerical $\rightsquigarrow$ $\mathrm{Num}(X)$

2) Prove the Hodge Index Theorem ;

Let $H$ be an ample divisor on a nonsingular projective surface $X$.
Let $D \in \mathrm{Div}(X)$ and suppose that

i) $D \not\equiv_{num} 0$

ii) $H \cdot D = 0$

Then $D^2 < 0$ .

# Equivalence relations between divisors

$X$ = smooth quasi-projective variety over $\mathbb{K} = \overline{\mathbb{K}}$.

$K(X)$ : its field of rational functions

Recap:

- Prime divisor: a subvariety $D \subset X$ of codimension 1
- (Weil) divisor: $\sum\limits_{\substack{D \subset X \\ \text{prime divisor}}} a_D \cdot D$ , $a_D \in \mathbb{Z}$
- $\mathrm{Div}(X)$ : free abelian group of divisors, Prime divisors as basis
- Group hom:

$$\begin{aligned} \mathrm{div} : K(X)^* &\longrightarrow \mathrm{Div}(X) \\ f &\longmapsto \sum_{\substack{D \subset X \\ \text{prime}}} \mathrm{ord}_D(f) \cdot D \end{aligned}$$

- Principal divisors: $\mathrm{Im}\, \mathrm{div}$

## i) Linear equivalence $\sim$

- Divisors $D_1, D_2$ on $X$ are linearly equivalent, $D_1 \sim D_2$, if $D_1 - D_2$ is a principal divisor i.e $\exists\, f \in K^*(K)$:

$$\mathrm{div}(f) = D_1 - D_2$$

- Divisor class group:

$$\mathrm{Div}(X)/\sim \;=:\; \mathcal{Cl}(X) \cong \mathrm{Pic}(X)$$

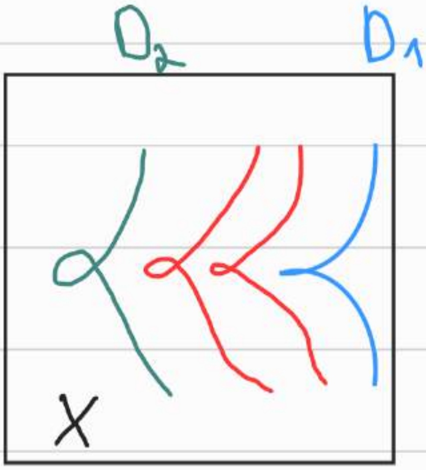
- Example: $X = \mathbb{P}^n$, $H := \{x_0 = 0\}$, $f, g \in S_d$

$\rightsquigarrow$ • $V(f) \sim V(g) \sim d \cdot H$

• $\mathcal{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$

## ii) Algebraic equivalence $\underset{alg}{\sim}$

Idea:




We can deform one divisor into the other

- $T$ irreducible variety, fix $t \in T$

$\rightsquigarrow \quad i_t : X \hookrightarrow T \times X$
$\qquad\qquad x \longmapsto (t, x)$

- a divisor $D$ on $T \times X$ s.t. $\operatorname{Supp} D \not\supset X \times \{t\}$ $\forall t \in T$ $\rightsquigarrow$ $i_t^*(D)$ : pullback divisor on $X$.

<u>Def</u>: An algebraic family of divisors is a map $f : T \longrightarrow \operatorname{Div}(X)$ s.t there exist $D \in \operatorname{Div}(T \times X)$ and $\operatorname{Supp} D \not\supset X \times \{t\}$ $\forall t \in T$ with:

i) $i_t^*(D) \in \operatorname{Div}(X)$ $\quad \forall t \in T$

ii) $i_t^*(D) = f(t)$

**Def**: Two divisors $D_1, D_2$ on $X$ are allgebraically equivalent, $D_1 \underset{alg}{\sim} D_2$, if there exists an algebraic family of divisors $f: T \longrightarrow \mathrm{Div}(X)$ with two points $t_1, t_2 \in T$ such that

$$f(t_1) = D_1 \quad \text{and} \quad f(t_2) = D_2$$

**Ex.**: show $\underset{alg}{\sim}$ is an equivalence relation and closed under addition in $\mathrm{Div}(X)$.

**Def**: The group $\mathrm{Div}(X) / \underset{alg}{\sim}$ is called the Neron-Severi group and is denoted by $NS(X)$.

**Fact**: Let $X$ be a nonsingular projective variety. Then $NS(X)$ is a finitely generated abelian group.

- Linear equivalence $\Longrightarrow$ Algebraic equivelence

Proof:

- Suffices to show for $D \sim 0_{div} \Leftrightarrow D = div(g)$, $g \in K^*(X)$

Let $g \in K^*(X)$ : $D = div(g)$

Consider $\mathbb{A}^2$ with coordinates $u, v$ and define $T := \mathbb{A}^2 \setminus \{(0,0)\}$.

projections: $T \xleftarrow{\pi_T} T \times X \xrightarrow{\pi_X} X$

$\rightsquigarrow$ Pullbacks
$\tilde{u} := \pi_T^*(u)$
$\tilde{v} := \pi_T^*(v)$
$\tilde{g} := \pi_X^*(g)$
are functions on $T \times X$

Define $D := div(\tilde{u} + \tilde{g}\tilde{v})$ divisor on $T \times X$

$(u,v) \in T$ induces a family of algebraic divisors

$$f : T \longrightarrow Div(X)$$
$$(u,v) \longmapsto div(u + v \cdot g)$$

In Particular, $f(0,1) = div(g)$
$f(1,0) = 0_{div}$

$\rightsquigarrow$ $0_{div} \underset{alg}{\sim} div(g) = D$

· Algebraic equivalence $\not\Rightarrow$ Linear equivalence

$X$ smooth proj. curve with genus $\geq 1$, $p, q \in X$ points with $p \neq q$.

Claim: $p \underset{alg}{\sim} q$ but $p \not\sim q$.

Proof (By contradiction):

A divisor on $X$ is of the form $\sum\limits_{\substack{x_i \in X \\ \text{Point}}} a_i x_i$ $\quad a_i \in \mathbb{Z}$

The embedding $\Delta : X \hookrightarrow X \times X$, $\quad x \longmapsto (x, x)$ $\quad$ $\Delta$ is a divisor on $X \times X$

~~Any divisor $D \in Div(X)$ induces a trivial~~ and induces

the algebraic family of divisors:

$$f : X \longrightarrow Div(X)$$

$$f(p) = p \;, \; f(q) = q$$

$\rightsquigarrow$ $p \underset{alg}{\sim} q$

Assume however that $p \sim q$, i.e

$\exists\, f \in K^*(X)$ with $div(f) = p - q$.

$\rightsquigarrow$ · $f$ has a simple zero in $p$

· $f$ has a simple pole in $q$

· $f$ doesn't have any other zeros

$\rightsquigarrow \quad f : U \longrightarrow \mathbb{A}^1 \quad$ and $\quad U = X \setminus \{q\}$

Recall that $f$ admits an extension

$$\tilde{f} : X \longrightarrow \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$$

Such that

- $$[f] = [(\underbrace{\tilde{f}^{-1}(\mathbb{A}^1)}_{= X \setminus \tilde{f}^{-1}(\infty)}, \tilde{f}|_{\tilde{f}^{-1}(\mathbb{A}^1)})]$$

In particular,

- $\tilde{f}^{-1}(0) = \{p\}$
- $\tilde{f}^{-1}(\infty) = \{q\}$
- every point in $\mathbb{P}^1$ has exactly one pre-image in $X$

$\rightsquigarrow \quad \tilde{f} : X \xrightarrow{\sim} \mathbb{P}^1$ isomorphism.

$\rightsquigarrow$ genus $X = 0$ $\lightning$

we conclude that $p \not\sim q$

<u>Remark</u>: $X$ nonsingular projective curve

$D_1, D_2 \in \mathrm{Div}(X)$. Then

$$D_1 \underset{alg}{\sim} D_2 \iff \deg D_1 = \deg D_2$$

We saw that for all points $x, x' \in X$:

$$x \underset{alg}{\sim} x'$$

Let $D \in \mathrm{Div}(X)$: $D = \sum_{x_i \in X} a_i \cdot x_i$ , $\deg D := \sum_i a_i$

and $x_0 \in X$ a point. Then:

- $x_i \underset{alg}{\sim} x \quad \forall i$

$\rightsquigarrow \quad a_i \cdot x_i \underset{alg}{\sim} a_i \cdot x_0 \quad \forall i$

$\rightsquigarrow \quad D \underset{alg}{\sim} \left(\sum_i a_i\right) \cdot x_0 = \deg D \cdot x_0$

$$\Rightarrow \quad D_1 \underset{alg}{\sim} D_2 \iff \deg D_1 = \deg D_2$$

<u>Remark</u>: $X = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ then

$$\sim \iff \underset{alg}{\sim}$$

iii) Numerical equivalence $\underset{num}{\sim}$

$X$ = smooth projective surface

<u>Def</u>: Two divisors $D_1, D_2 \in Div(X)$ are numerically equivalent, $D_1 \underset{num}{\sim} D_2$, if

$$D_1 . C = D_2 . C \quad \forall\, C \in Div(X)$$

or equivalently if $\quad D_1 - D_2 \underset{num}{\sim} 0_{div}$

<u>Def</u>: The divisor group of numerical equivalence classes is given by

$$Num(X) := Div(X) / \underset{num}{\sim}$$

<u>Fact</u>:

$$Num(X) \cong NS(X) / NS(X)^{tor} \cong \mathbb{Z}^{\rho}$$

$NS(X)^{tor} \subseteq NS(X)$ the torsion subgroup

$\rho$ : Picard rank (number)

<u>Ex</u>: Algebraic equivalence $\Rightarrow$ Numerical equivalence (Hartshorne 5.1.7)

Summary:

$X$ = smooth projective surface

with divisor group $\mathrm{Div}(X)$

$$\sim \implies \underset{\text{alg}}{\sim} \implies \underset{\text{num}}{\sim}$$

yields

$$\mathrm{Pic}(X) \twoheadrightarrow NS(X) \twoheadrightarrow \mathrm{Num}(X)$$

# II Hodge Index Theorem

$X$ = smooth projective surface over $\mathbb{K} = \overline{\mathbb{K}}$

Recap: ample divisors

- Ample divisor: a divisor $H$ such that $H^{\otimes m}$ $(\mathcal{O}_X(H)^{\otimes m})$ is very ample for some $m \geq 1$.

- Very ample divisor $H$ on $X$ gives a closed embedding $i_H : X \hookrightarrow \mathbb{P}^n$

Fact 1: $E, H \in \mathrm{Div}(X)$ : $H$ ample $E$ effective and $E \not\equiv_{num} 0$, Then

$$H.E > 0$$

- Replace $H$ with $mH$ $m >> 0$ to get $i : X \hookrightarrow \mathbb{P}^n$

- General element $C \in |H| := \{0 \leq D \in \mathrm{Div}(X) : D \sim H\}$ having no common irreducible components with $E$

  - $C = X \cap$ hyperplane
  - $E \cap C = E \cap$ hyperplane $\neq \cancel{\emptyset}$

    (E, C share no irred. components)

$$\rightsquigarrow \quad C.E > 0$$

<u>Remark</u>: $H$ ample. For every effective divisor $E$, the number $H.E$ plays a similar role to the degree of a divisor on a curve

<u>Fact 2</u>: $X$ = nonsingular projective variety over $\mathbb{K} = \overline{\mathbb{K}}$, $D \in \mathrm{Div}(X)$. Then each non-zero global section $s \in H^0(X, \mathcal{O}_X(D))$ induces an effective divisor $E$ on $X$ s.t.

$$E \sim D$$

(Hartshorne 2.7.1)

<u>Lemma</u>: Let $X$ be a nonsingular projective surface, $H$ an ample divisor on $X$. Then there exists an $n \in \mathbb{Z}$ such that for any $D \in \mathrm{Div}(X)$

$$D.H > n \Rightarrow H^2(X, \mathcal{O}_X(D)) = 0$$

Proof: (By contraposition)

Recall

- $\Omega_X$ cotangent bundle
- $\omega_X := \Lambda^2 \Omega_X$ Canonical bundle
- $K_X$ canonical divisor, $\mathcal{O}_X(K_X) \simeq \omega_X$

$\dim H^0(X, \mathcal{O}_X(K_X - D)) =$

$$= \dim H^0(X, \mathcal{O}_X(D)^\vee \otimes \mathcal{O}_X(K_X))$$

$$= \dim H^0(X, \mathcal{O}_X(D)^\vee \otimes \omega_X)$$

Serre duality

$$= \dim H^2(X, \mathcal{O}_X(D))$$

Assume $\dim H^0(X, \mathcal{O}_X(K_X - D)) > 0$

Fact 2 implies

$$K_X - D \sim E \quad , \quad E \geq 0 \quad (\text{i.e effective})$$

Fact 1, and since the intersection product only depends on the linear class, imply

- $(K_X - D).H > 0$

$\Rightarrow$ • $K_X . H > D . H$

Define $n := K_X . H$ . This implies

- $\exists n \in \mathbb{Z}$ s.t.

$$\dim H^2(X, \mathcal{O}_X(D)) > 0 \Rightarrow D.H < n$$

The Lemma follows by counter position

$\square$

<u>Remark</u>: There is an analogous statement in case $X$ is a curve. There exists an integer $n := 2g - 2$, $g$ genus of $X$ such that any divisor $D$ with $\deg D > n \Rightarrow H^1(X, \mathcal{O}_X(D)) = 0$.

<u>Corollary</u>: Let $X$ be a nonsingular surface, $H$ ample divisor on $X$. Let $D$ be a divisor on $X$ such that

i) $D.H > 0$

ii) $D^2 > 0$

Then there exists an $n_0 \in \mathbb{N}$ such that

$nD$ is linearlly equivalent to an effective divisor for all $n > n_0$.

Proof: (Apply Riemann-Roch to $nD$)

Recall that for

$$l(D) := \dim H^0(X, \mathcal{O}_X(D))$$

$$s(D) := \dim H^1(X, \mathcal{O}_X(D))$$

$$\chi(\mathcal{O}_X) := \sum_i (-1)^i \dim H^i(\mathcal{O}_X, \mathcal{O}_X(D))$$ Euler char.

$$p_a^X := \chi(\mathcal{O}_X) - 1$$ arith. genus

the Riemann-Roch theorem yields

$$\ell(D) - s(D) + \ell(K_X - D) = \frac{1}{2} D.(D - K_X) + 1 + p_a^X$$

From the Lemma $\exists\, n_0 \in \mathbb{Z}$ s.t.

$$nD.H > n_0 \implies H^2(X, \mathcal{O}(nD)) = 0 \qquad *$$

$$\overset{\text{Serre duality}}{\iff} \ell(K_X - nD) = 0$$

Since $D.H > 0$ per construction, choose $n$ big enough such that $*$ holds.

Riemann-Roch for $nD$ gives

$$\ell(nD) - s(nD) + \ell(K_X - D) = \frac{1}{2} nD.(nD - K_X) + 1 + p_a^X$$

- $s(nD) \geq 0$ and $\ell(K_X - D) = 0$ yields

$$\ell(nD) \geq \frac{1}{2} n^2 D^2 - \frac{1}{2} nD.K_X + 1 + p_a^X$$

Per construction $D^2 > 0$. This implies RHS becomes large for $n >> 0$. Hence

$$\ell(nD) \xrightarrow{n \to \infty} \infty$$

In particular, $H^0(X, \mathcal{O}_X(nD)) \neq 0$. From Fact 2 we conclude that $nD$ is linearlly equivalent to an effective divisor.

$\square$

__Theorem__ (Hodge Index):
Let $H$ be an ample divisor on a nonsingular projective surface $X$.
Let $D \in \operatorname{Div}(X)$ and suppose that

i) $D \not\equiv_{num} 0$

ii) $H.D = 0$

Then $D^2 < 0$.

Proof: (By contradiction and using the corollary)
Given the above setup assume $D^2 \geq 0$
$\rightsquigarrow$ 2 cases: $D^2 > 0$ $\qquad$ $D^2 = 0$

__case 1: $D^2 > 0$__

Define a divisor $H' := D + nH$, $n \in \mathbb{Z}$ to be determined later. In the proof of the intersection theorem (uniqueness part) we saw $\exists\, n \in \mathbb{N}$ s.t. $H'$ is very ample. For all schemes over a Noetherian ring and in particular for schemes over a field this implies

$H'$ is ample

We compute:

$$D.H' = D.(D+nH)$$

$$= D^2 + nD.H$$

$D.H = 0$
per construction

$$= D^2 > 0$$

per assumption

Applying the corollary on $H'$, $D$ implies

$$mD \geq 0 \quad \text{(effective divisor)}$$

for large enough $m$

From Fact 1 it follows:

$$mD.H > 0$$

$$\Rightarrow D.H > 0 \quad ↯$$

($D.H = 0$ per construction)

## Case 2: $D^2 = 0$

strategy: define $D' \in \mathrm{Div}(X)$ s.t.

i) $D'.H = 0$

ii) $D'^2 > 0$

Then use the same argument as in case 1 to construct a contradiction to $D'.H > 0$.

- Per construction $D \not\equiv_{num} 0$, i.e there exists

$$E \in \mathrm{Div}(X): \quad D.E \neq 0$$
$$\text{WLOG} > 0$$

- Define $E' := (H^2)E - (E.H)H$. Then

i) $E'.H = (H^2)E.H - (E.H)H^2$
$= 0$

ii) $E'.D = (H^2)E.D - (E.H)(H.D)$
$= (H^2)E.D > 0$

H.D = 0 per constr

self intersection of ample divisor on surface is > 0

Define a divisor $D' := nD + E'$.

$\leadsto$ • $D'.H = (nD + E').H$

$$= nD.H + E'.H$$

$$= 0$$

$D.H = 0$ per construction

$E'.H = 0$ (i)

• $D'^2 = (nD + E').(nD + E')$

$$= n^2 D^2 + 2n(D.E') + (E')^2$$

$D^2 = 0$ per assumption.

$$= 2nD.E' + (E')^2$$

Since $D.E' \neq 0$ choose $n \in \mathbb{Z}$ s.t

$$2n(D.E') + (E')^2 > 0$$

$$\Longrightarrow \quad D'^2 > 0$$

Thus $\cdot \quad D' \underset{num}{\not\equiv} 0$

$\cdot \quad D'.H = 0$

$\cdot \quad D'^2 > 0$

Using the same argument as in case 1 we also reach a contradiction

$$D'.H > 0 \quad ↯$$

We conclude that $D^2 < 0$

$\square$

<u>Remark</u> (Why is it called the Index theorem):

Consider the intersection product

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$$

Notice that degeneracy is possible
i.e $C \in \mathrm{Div}(X)$:

- $C \not\sim \mathcal{O}_{div}$
- $C.D = 0 \quad \forall\, D \in \mathrm{Div}(X)$.

$$\overset{\text{Per Def.}}{\Longrightarrow} \qquad C \underset{num}{\sim} \mathcal{O}_{div}$$

Hence, the intersection product induces a nondegenerate symmetric bilinear form on $\mathrm{Num}(X)$.

Recall $\quad \mathrm{Num}(X) \simeq \mathbb{Z}^{\rho} \qquad \rho$ : Picard rank

Nondegenerate sym. bil. form:

$$\begin{tikzcd} \mathrm{Num}(X)\times\mathrm{Num}(X) \arrow[r] \arrow[d, bend right, "-\otimes_{\mathbb{Z}}\mathbb{R}"'] & \mathbb{Z} \\ \mathbb{R}^{\rho}\times\mathbb{R}^{\rho} \arrow[r] & \mathbb{R} \end{tikzcd}$$

Recall that, over $\mathbb{R}$, such a form can be diagonalized $\rightsquigarrow$ $\begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix}$

The number $(\#{+1} - (\#{-1}))$ is called the signature or index of the bilinear form.

In our case, the diagonalized form has one $+1$ corresponding to a real multiple of $H$ and all the rest are $-1$.

$$\begin{array}{cc} r\cdot H \leftarrow & \begin{bmatrix} +1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix} \\ & \end{array}$$

**Example:** Recall the quadric surface

$X \subseteq \mathbb{P}^3$

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \quad \text{coordinates } x_0, x_1, x_2, x_3$$
$$[w:x] \times [y:z] \longmapsto [wy : wz : xy : xz]$$

$$X := V(x_0 x_3 - x_1 x_2) \subseteq \mathbb{P}^3$$

with $\mathrm{Pic}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and the intersection product

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$$
$$(a,b) \times (a',b') \longmapsto ab' + ba'$$

- Recall that on $\mathbb{P}^n \times \mathbb{P}^m$ $\sim \iff \underset{alg}{\sim}$

and for $D \in \mathrm{Div}(X)$: $D \sim O_{div} \iff D \underset{num}{\sim} O_{div}$

$$\Rightarrow \quad \mathrm{Pic}(X) \simeq NS(X) \simeq \mathrm{Num}(X)$$

- Recall that for $H \in Div(X)$:

$$\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$$

We say that $H$ is of type $(a,b)$

- $a, b \geq 1 \implies H$ is (very) ample

Let

- $H \in Div(X)$ be of type $(1,1) \rightsquigarrow H$ ample

- $D \in Div(X)$ be of type $(1,-1)$.

$\rightsquigarrow$ • $D \not\sim_{num} 0_{div}$

• $D.H = (1,-1).(1, 1) = -1+1 = 0$

Hodg Index theorem $\implies D^2 < 0$

Indeed, $D^2 = (1,-1).(1,-1) = -1-1 = -2$.

$$\begin{tikzcd}
\mathrm{Pic}(X)\times\mathrm{Pic}(X) \arrow[r] \arrow[d, bend right, "-\otimes_{\mathbb{Z}}\mathbb{R}"'] & \mathbb{Z} \\
\mathbb{R}^2\times\mathbb{R}^2 \arrow[r] & \mathbb{R}
\end{tikzcd}$$

$$(a,b)\times(a',b') \longmapsto ab'+ba'$$

With respect to basis $H, D$ $\rightsquigarrow$ is represented by $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

Using the basis $\frac{1}{\sqrt{2}}H, \frac{1}{\sqrt{2}}D$ we can diagonalize to

$$\rightsquigarrow \frac{1}{\sqrt{2}}H \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$