Blowup of $\mathbb{A}^2$ at the origin

$$Y = Bl_0\,\mathbb{A}^2 = \{((x,y),[\xi:\eta]) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid x\eta - y\xi = 0\} \subset \mathbb{A}^2 \times \mathbb{P}^1$$

$$\operatorname{rk}\begin{pmatrix} x & y \\ \xi & \eta \end{pmatrix} \leq 1$$

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ \mathbb{A}^2 & & \mathbb{P}^1 \end{tikzcd}$$

What are the fibres of π and p?

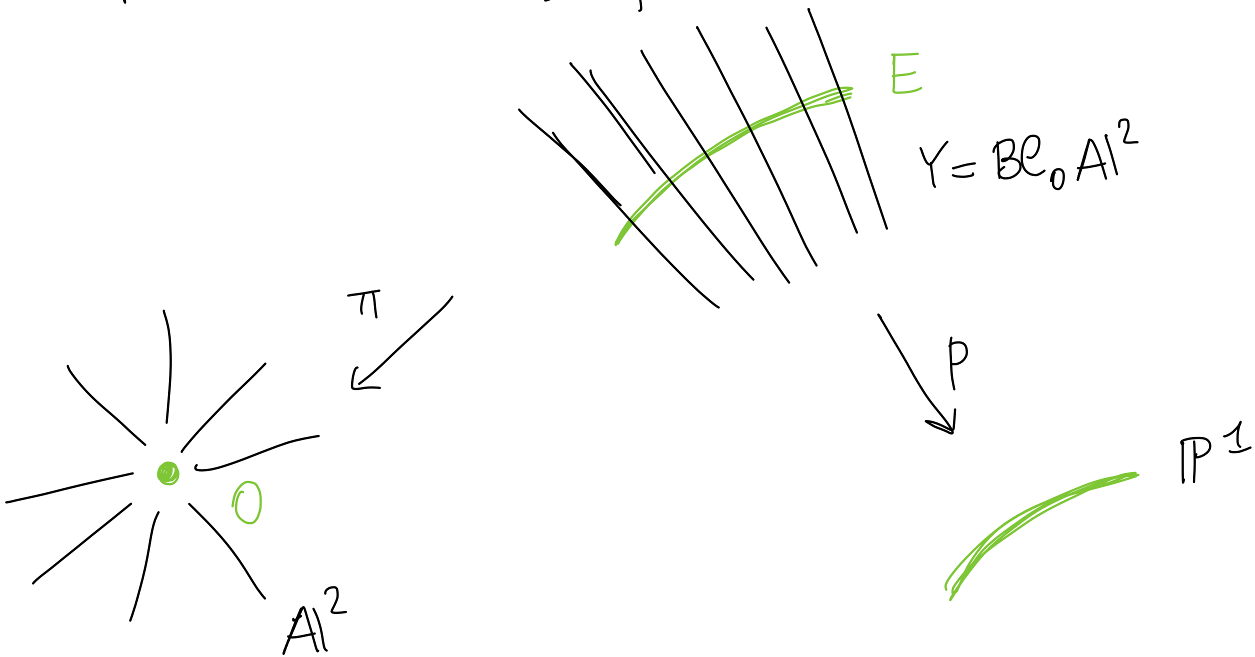
$(x_0,y_0) \in \mathbb{A}^2 \setminus \{(0,0)\}$ $\quad \pi^{-1}(x_0,y_0) = \{((x_0,y_0),[x_0:y_0])\}$ is a point

$\pi^{-1}(0,0) = \{(0,0)\} \times \mathbb{P}^1$ is isomorphic to $\mathbb{P}^1$

$\pi|_{\pi^{-1}(\mathbb{A}^2 \setminus \{(0,0)\})} : \pi^{-1}(\mathbb{A}^2 \setminus \{(0,0)\}) \longrightarrow \mathbb{A}^2 \setminus \{(0,0)\}$ is an isom.

$(0,0)$ is replaced via π by a curve isom. to $\mathbb{P}^1$

$\forall [\xi_0 : \eta_0] \in \mathbb{P}^1 \quad p^{-1}([\xi_0 : \eta_0]) = \mathrm{Span}(\xi_0, \eta_0) \times \{[\xi_0 : \eta_0]\}$ is isomorphic to $\mathbb{A}^1 \Rightarrow p$ is a line bundle



$E = \pi^{-1}(0) \subset Y$ is called EXCEPTIONAL DIVISOR, isom. to $\mathbb{P}^1$

Recap about line bundles and divisors

X normal variety, $D \hookrightarrow X$ effective Cartier divisor

Let $\{U_\alpha\}_\alpha$ be an affine cover which trivialises $\mathcal{O}_X(D)$; in other words $\forall \alpha$ the divisor $D \cap U_\alpha \hookrightarrow U_\alpha$ is defined by an equation $f_\alpha \in \mathcal{O}_X(U_\alpha)$. So we are taking isomorphisms

$$\begin{array}{ccc} \mathcal{O}_{U_\alpha} & \xrightarrow{\sim} & \mathcal{O}_X(-D)|_{U_\alpha} \\ 1 & \longmapsto & f_\alpha \end{array} \qquad \begin{array}{ccc} \mathcal{O}_{U_\alpha} & \xrightarrow{\sim} & \mathcal{O}_X(D)|_{U_\alpha} \\ 1 & \longmapsto & f_\alpha^{-1} \end{array}$$

What happens on double intersections? On $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the divisor D is defined by f_α or by f_β; so there exists $g_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})^*$ s.t. $f_\alpha = g_{\alpha\beta} f_\beta$. [Notice that f_α, $g_{\alpha\beta}$ are not uniquely determined by D, but the cohomology class of the cocycle $\{g_{\alpha\beta}\}_{\alpha,\beta}$ in $\check{H}^1(\mathcal{U}, \mathcal{O}_X^*)$ is well defined.]

On $U_{\alpha\beta}$ I have two isomorphisms $\mathcal{O}_{U_{\alpha\beta}} \xrightarrow{\sim} \mathcal{O}_X(-D)|_{U_{\alpha\beta}}$: one comes from U_α, one comes from U_β.

$$U_\alpha \hookleftarrow U_{\alpha\beta} \hookrightarrow U_\beta$$

$$\begin{tikzcd}
1 \in \mathcal{O}_{U_{\alpha\beta}} \arrow[rr] \arrow[dr] & & \mathcal{O}_{U_{\alpha\beta}} \ni 1 \arrow[dl] \\
& f_\alpha \in \mathcal{O}_X(-D)|_{U_{\alpha\beta}} \ni f_\beta &
\end{tikzcd}$$

$$1 \mapsto f_\alpha, \qquad 1 \mapsto f_\beta$$

transition for the line bundle $\mathcal{O}_X(-D)$

so $\mathcal{O}_{U_{\alpha\beta}} \to \mathcal{O}_{U_{\alpha\beta}}$ is given by $1 \mapsto g_{\alpha\beta}$

$$\begin{tikzcd}
1 \in \mathcal{O}_{U_{\alpha\beta}} \arrow[rr] \arrow[dr] & & \mathcal{O}_{U_{\alpha\beta}} \ni 1 \arrow[dl] \\
& f_\alpha^{-1} \in \mathcal{O}_X(D)|_{U_{\alpha\beta}} \ni f_\beta^{-1} &
\end{tikzcd}$$

$$1 \mapsto f_\alpha^{-1}, \qquad 1 \mapsto f_\beta^{-1}$$

transition for the line bundle $\mathcal{O}_X(D)$

so $\mathcal{O}_{U_{\alpha\beta}} \to \mathcal{O}_{U_{\alpha\beta}}$ is given by $1 \mapsto g_{\alpha\beta}^{-1}$

Example $\mathbb{P}^1_k$ with homogeneous coordinates $[\xi:\eta]$, divisor $D = n\cdot[1:0]$ for $n\in\mathbb{Z}, n\geq 0$. Consider the affine charts $U_\xi = \{\xi\neq 0\}\simeq \mathbb{A}^1_s$ and $U_\eta=\{\eta\neq 0\}\simeq\mathbb{A}^1_t$ with coordinates $s=\frac{\eta}{\xi}$, $t=\frac{\xi}{\eta}$, respectively.

The ideal of $D\cap U_\xi \hookrightarrow U_\xi$ is (s^n).

The ideal of $D\cap U_\eta \hookrightarrow U_\eta$ is (1).

$U_\xi\cap U_\eta = k[t,t^{-1}] = k[s,s^{-1}]$ with $st=1$.

$$U_\xi \hookleftarrow U_\xi\cap U_\eta \hookrightarrow U_\eta$$

$$\begin{tikzcd}
1 \in k[s^\pm] \arrow[rr, "1\mapsto s^n"] \arrow[dr] & & k[s^\pm] \ni 1 \arrow[dl] \\
& s^n \in \mathcal{O}_{\mathbb{P}^1}(-D)|_{U_\xi\cap U_\eta} \ni 1 &
\end{tikzcd}$$

transition for the line bundle $\mathcal{O}_X(-D)\simeq\mathcal{O}_{\mathbb{P}^1}($

$$\begin{tikzcd}
1 \in k[s^\pm] \arrow[rr, "1\mapsto s^{-n}"] \arrow[dr] & & k[s^\pm] \ni 1 \arrow[dl] \\
& s^{-n} \in \mathcal{O}_{\mathbb{P}^1}(D)|_{U_\xi\cap U_\eta} \ni 1 &
\end{tikzcd}$$

transition for the line bundle $\mathcal{O}_X(D)\simeq\mathcal{O}_{\mathbb{P}^1}(\ )$

Now let's go back to the blowup $Y = Bl_0 \mathbb{A}^2 \xrightarrow{\pi} \mathbb{A}^2_{x,y}$.

Y has two charts: $Y = U_\xi \cup U_\eta$

- $U_\xi = \{\xi \neq 0\} \cap Y$ on $\mathbb{P}^1_{[\xi:\eta]}$ use affine coordinate $s = \frac{\eta}{\xi}$

$$U_\xi = \{(x, y, [1:s]) \mid xs - y = 0\} \simeq \{(x,y,s) \in \mathbb{A}^3 \mid y = xs\} \simeq \mathbb{A}^2_{x,s}$$

$$(x, y, s) \longmapsto (x, s)$$

$$(x, xs, s) \qquad (x, s)$$

How do we write π in this chart?

$$\pi|_{U_\xi} : U_\xi = \mathbb{A}^2_{x,s} \longrightarrow \mathbb{A}^2_{x,y}$$

$$(x, s) \longmapsto (x, xs)$$

What is the exceptional divisor E in this chart?

$$E \cap U_\xi = (\pi|_{U_\xi})^{-1}(0,0) = \{(x,s) \in \mathbb{A}^2 \mid x=0, xs=0\} = V(x)$$

The ideal of $E \cap U_\xi \hookrightarrow U_\xi$ is generated by x.

- $U_\eta = \{\eta \neq 0\} \cap Y$.   Use the affine coordinate $t = \dfrac{\xi}{\eta}$

$$U_\eta = \{(x,y,[t:1]) \mid x - ty = 0\} = \{(x,y,t) \in \mathbb{A}^3 \mid x = yt\} \simeq \mathbb{A}^2_{y,t}$$

$$(x,y,t) \longmapsto (y,t)$$
$$(ty,y,t) \longleftarrow (y,t)$$

$$\pi|_{U_\eta} : U_\eta = \mathbb{A}^2_{y,t} \longrightarrow \mathbb{A}^2_{x,y}$$
$$(y,t) \longmapsto (yt, y)$$

Exceptional divisor in this chart: the ideal of $E \cap U_\eta \hookrightarrow U_\eta$ is generated by y.

Intersection of the two charts:

$$U_\xi \hookleftarrow U_\xi \cap U_\eta \hookrightarrow U_\eta$$

$$k[x,s] \hookrightarrow k[x,s^{\pm}] \xrightarrow{\sim} k[y,t^{\pm}] \hookleftarrow k[y,t]$$

$st = 1$,
$x = yt$, $y = xs$

What are $\mathcal{O}_Y(-E)$ and $\mathcal{O}_Y(E)$?

$$U_\xi \hookleftarrow U_\xi \cap U_\eta \hookrightarrow U_\eta$$

$$\begin{tikzcd}
1 \in \arrow[dr, mapsto] & \mathcal{O}_{U_\xi\cap U_\eta} \arrow[rr, "1\mapsto t=s^{-1}"] \arrow[dr] & & \mathcal{O}_{U_\xi\cap U_\eta} \arrow[dl] & \ni 1 \arrow[dl, mapsto] \\
& x \in & \mathcal{O}_Y(-E)|_{U_\xi\cap U_\eta} & \ni y &
\end{tikzcd}$$

because the ideal of $E\cap U_\xi \hookrightarrow U_\xi$ is generated by x

transition for $\mathcal{O}_Y(-E)$

$$\begin{tikzcd}
1 \in \arrow[dr, mapsto] & \mathcal{O}_{U_\xi\cap U_\eta} \arrow[rr, "1\mapsto s=t^{-1}"] \arrow[dr] & & \mathcal{O}_{U_\xi\cap U_\eta} \arrow[dl] & \ni 1 \arrow[dl, mapsto] \\
& x^{-1} \in & \mathcal{O}_Y(E)|_{U_\xi\cap U_\eta} & \ni y^{-1} &
\end{tikzcd}$$

transition for $\mathcal{O}_Y(E)$

Recall $\mathcal{O}_{U_\xi\cap U_\eta} = k[x,s^{\pm}] = k[y,t^{\pm}]$ while U_ξ, U_η are the charts of $Y = Bl_0\mathbb{A}^2 \longrightarrow \mathbb{A}^2$

What is the normal bundle $N_{E/Y}$?

Recall: $N_{E/Y} = \mathcal{O}_Y(E)|_E$.

Consider the charts on $E = \mathbb{P}^1_{[\xi:\eta]}$: $E \cap U_\xi = \mathbb{A}^1_s$, $E \cap U_\eta = \mathbb{A}^1_t$

$$E \cap U_\xi \hookleftarrow E \cap U_\xi \cap U_\eta \hookrightarrow E \cap U_\eta$$

$$\mathcal{O}_{E \cap U_\xi \cap U_\eta} \xrightarrow{1 \mapsto s = t^{-1}} \mathcal{O}_{E \cap U_\xi \cap U_\eta}$$ is the transition of $N_{E/Y}$

By comparing with page 5 we get $N_{E/Y} = \mathcal{O}_{\mathbb{P}^1}(-1)$

What is ω_Y ?

$$U_\xi = \mathbb{A}^2_{x,s} \implies \begin{array}{ccc} \mathcal{O}_{U_\xi} = k[x,s] & \xrightarrow{\sim} & \omega_Y|_{U_\xi} \\ 1 & \longmapsto & dx \wedge ds \end{array}$$

$$U_\eta = \mathbb{A}^2_{y,t} \implies \begin{array}{ccc} \mathcal{O}_{U_\eta} = k[y,t] & \xrightarrow{\sim} & \omega_Y|_{U_\eta} \\ 1 & \longmapsto & dy \wedge dt \end{array}$$

On the double intersection what is the relation between $dx \wedge ds$ and $dy \wedge dt$?

$$\left.\begin{array}{l} s = t^{-1} \implies ds = -t^{-2}dt \\ x = ty \implies dx = t\,dy + s\,dt \end{array}\right\} \implies dx \wedge ds = -t^{-1}\, dy \wedge dt$$

$$\begin{tikzcd} U_\xi & U_\xi \cap U_\eta \arrow[l, hook'] \arrow[r, hook] & U_\eta \end{tikzcd}$$

$$\begin{tikzcd} 1 \in \mathcal{O}_{U_\xi \cap U_\eta} \arrow[rr, "1 \mapsto -t^{-1}"] \arrow[dr] & & \mathcal{O}_{U_\xi \cap U_\eta} \ni 1 \arrow[dl] \\ & dx \wedge ds \in \omega_Y|_{U_\xi \cap U_\eta} \ni dy \wedge dt & \end{tikzcd}$$

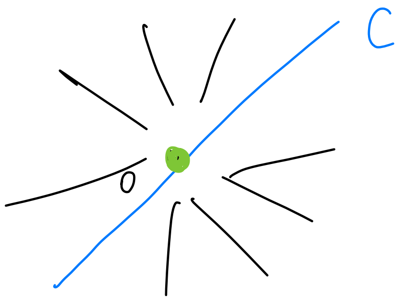
transition for ω_Y

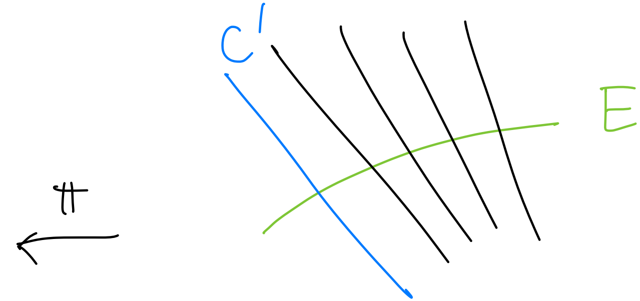
$$\implies \omega_Y \simeq \mathcal{O}_Y(E)$$

Recap $\pi : Y = Bl_0 \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ blowup of $\mathbb{A}^2$ at the origin

- $E = \pi^{-1}(0)$ exceptional divisor
- π birational, induces an isomorphism $Y \setminus E \simeq \mathbb{A}^2 \setminus \{0\}$
- Y smooth
- $E \simeq \mathbb{P}^1$, $N_{E/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$
- $\omega_Y \simeq \mathcal{O}_Y(E)$

$C = (y = x) \subset \mathbb{A}^2_{x,y}$



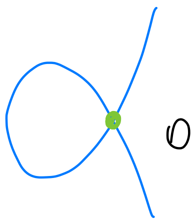


$$\pi^{-1}C = \{((x,x),[\xi:\eta]) \mid x(\xi-\eta)=0\}$$ set-theoretically

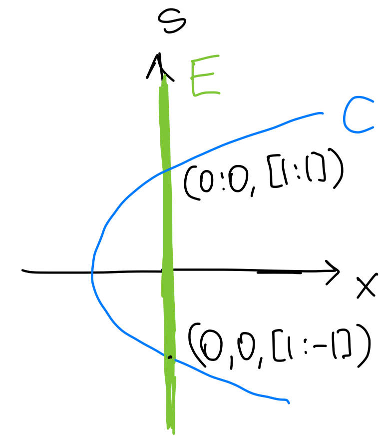
$$= \underbrace{\{((0,0),[\xi:\eta])\}}_{= E} \cup \underbrace{\{((x,x),[\xi:\xi])\}}_{= C'}$$

as divisors on Y
or schemetheoretically $\quad \pi^{*}C = E + C'$

$C = V(x^2 - y^2 + x^3) \subset \mathbb{A}^2_{x,y}$



$\xleftarrow{\ \pi\ }$

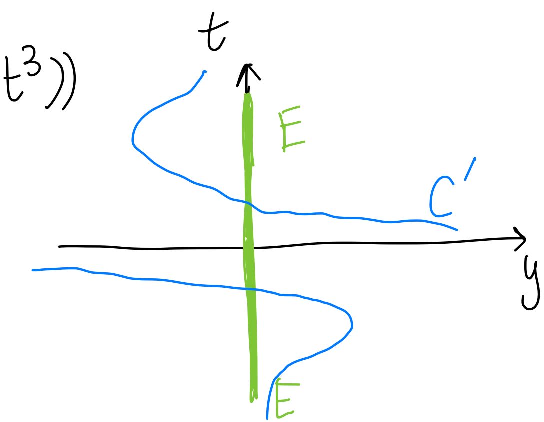


$U_\xi \cap \pi^{-1} C \underset{\uparrow \; y = xs}{=} V(x^2 - (xs)^2 + x^3) = V(x^2(1 - s^2 + x))$

$U_\xi = \mathbb{A}^2_{x,s}$

$U_\eta \cap \pi^{-1} C \underset{\uparrow \; x = yt}{=} V((yt)^2 - y^2 + (yt)^3) = V(y^2(t^2 - 1 + yt^3))$

$U_\eta = \mathbb{A}^2_{y,t}$



$C = V(x^2 - y^2 + x^3) \subset \mathbb{A}^2$

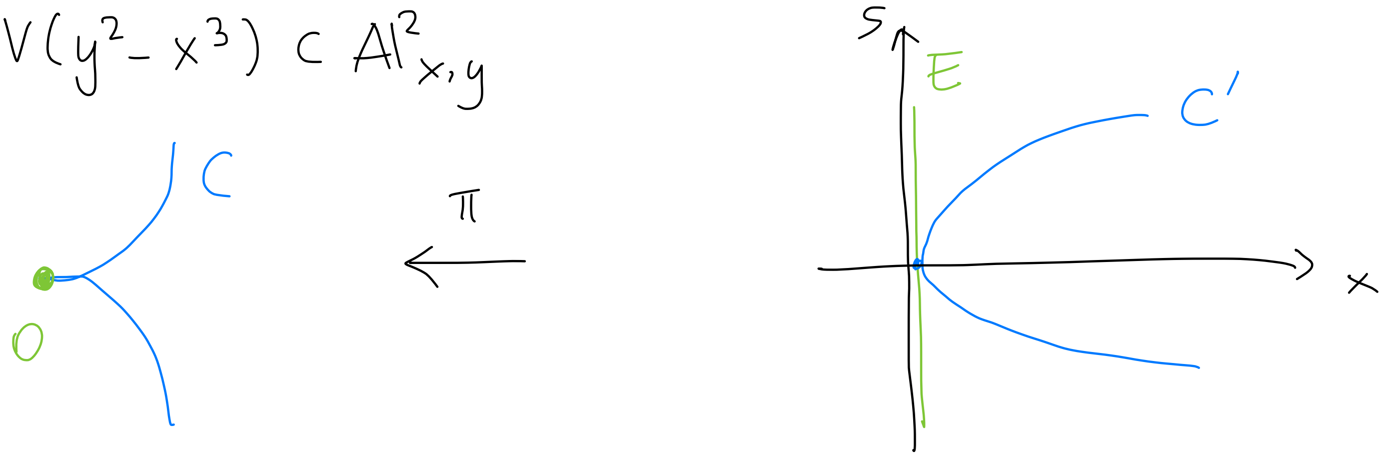
$\pi^{-1} C = C' \cup E$ set-theoretically

$C' = \{ \xi^2 - \eta^2 + x\xi^2 = 0 \} \subset Y = Bl_0 \mathbb{A}^2$ strict transform of C

$\pi^* C = C' + 2E$ scheme-theoretically or as divisors in $Y = Bl_0 \mathbb{A}^2$

Notice also that C' is smooth

$$C = V(y^2 - x^3) \subset \mathbb{A}^2_{x,y}$$



$$\underset{\mathbb{A}^2_{x,s}}{\underset{\shortparallel}{U_\xi}} \cap \pi^{-1} C \underset{\underset{y = xs}{\uparrow}}{=} V\left((xs)^2 - x^3\right) = V\big(\underbrace{x^2}_{E}\,(\underbrace{s^2 - x}_{C'})\big)$$

set-theoretically $\quad \pi^{-1} C = E \cup C'$

scheme-theoretically $\quad \pi^* C = C' + 2E$

or as divisors

$C = V(f) \subseteq \mathbb{A}^2_{x,y}$ curve

$f = f_0 + f_1(x,y) + f_2(x,y) + \cdots$, $f_i(x,y)$ homog. of degree i

multiplicity of C at 0 : $\mu_0(C) = \min\{i \mid f_i(x,y) \neq 0\}$

- $\mu_0(C) = 0 \iff 0 \notin C$
- $\mu_0(C) = 1 \iff 0$ is a smooth point of C
- $\mu_0(C) \geq 2 \iff 0$ is a singular point of C

$C' =$ Zariski closure in $Y = Bl_0 \mathbb{A}^2$ of $\pi^{-1}(C \setminus \{(0,0)\})$ "strict transform of C"

$$\pi^* C = C' + \mu_0(C) \cdot E$$

Blowup of a smooth surface at a point

X smooth proj. surface over $k=\bar{k}$, $p \in X$

$\pi : \tilde{X} = Bl_p X \longrightarrow X$ BLOW-UP of X at p

How to construct this? $\mathcal{O}_{X,p}$ is a regular local ring of dim 2, Choose x, y local parameters. ($\{\bar{x}, \bar{y}\}$ is a basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$; $\widehat{\mathcal{O}_{X,p}} \simeq k[[x,y]]$)

Choose U open affine s.t. x, y are defined over U

Now repeat what we have done by replacing $\mathbb{A}^2_{x,y}$ with U.

$Bl_p U \subseteq U \times \mathbb{P}^1_{[\xi:\eta]}$ defined by $y\xi - x\eta = 0$.

$\pi : Bl_p U \longrightarrow U$ induces an isom on $Bl_p U \setminus E \longrightarrow U \setminus \{p\}$

This can be glued to $X \setminus \{p\}$.

Prop $\tilde{X} = Bl_p X \xrightarrow{\pi} X$ blow up of p. Then:

1) $\tilde{X}$ is a smooth proj. surface

2) π is projective and birational

$E = \pi^{-1}(p)$ is isom to $\mathbb{P}^1$, $N_{E/\tilde{X}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$

$\pi|_{\tilde{X} \setminus E} : \tilde{X} - E \longrightarrow X \setminus \{p\}$ is an isom.

3) $$\begin{aligned} \mathrm{Pic}(X) \oplus \mathbb{Z} &\longrightarrow \mathrm{Pic}(\tilde{X}) \\ (L, n) &\longmapsto \pi^* L \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}}(nE) \\ (D, n) &\longmapsto \pi^* D + nE \end{aligned}$$ is an isomorphism

4) The intersection pairing on $\mathrm{Pic}(\tilde{X})$ is determined by

$E^2 = -1$, $\pi^* D \cdot \pi^* D' = D \cdot D'$, $\pi^* D \cdot E = 0$ $\forall D, D' \in \mathrm{Div}\, X$

5) $NS(X) \oplus \mathbb{Z} \simeq NS(\tilde{X})$

6) $\omega_{\tilde{X}} \simeq \pi^* \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(E)$

$K_{\tilde{X}} \sim \pi^* K_X + E$

7) $C \subseteq X$ irr. curve, $m = \mathrm{mult}_p(C)$

$C' =$ Zariski closure of $\pi^{-1}(C \setminus \{p\})$ in $\tilde{X}$

Then $\pi^* C = C' + mE$

Pf In 1) and 2) we just need to show that π is projective.

In 4) use the fact that D and D' can be replaced by divisors not passing through p.

5) follows from 3) and 4). 7) as in the case of $\mathbb{A}^2$.

Prop X normal variety, $U \subseteq X$ open subset
$D_1, \dots, D_r$ are the prime divisors of X which are contained in $X \setminus U$

$$\Rightarrow \quad \mathbb{Z}^r \longrightarrow Cl(X) \longrightarrow Cl(U) \longrightarrow 0 \qquad \text{exact.}$$
$$(a_1, \dots, a_r) \mapsto a_1 D_1 + \dots + a_r D_r$$

Pf of 3)

$$\boxed{\begin{array}{c} \tilde{X} \setminus E \\ \downarrow \\ \tilde{X} \end{array}} \simeq \boxed{\begin{array}{c} X \setminus \{p\} \\ \downarrow \\ X \end{array}} \Rightarrow Cl(X) \xrightarrow{\sim} Cl(X \setminus \{p\})$$

$$\Downarrow$$

$$\mathbb{Z} \xrightarrow{\varphi} Cl(\tilde{X}) \longrightarrow Cl(\tilde{X} \setminus E) \longrightarrow 0$$
$$1 \longmapsto E$$

I want to show that φ is injective: by contradiction $\exists n \in \mathbb{Z}$
$n \neq 0$, $nE \sim 0 \Rightarrow 0 = (nE)^2 = n^2 E^2 = -n^2$ ↯

$m + m \Rightarrow 0 \to \mathbb{Z} \xrightarrow{\varphi} Cl(\tilde{X}) \longrightarrow Cl(X) \longrightarrow 0.$

$$0 \to \mathbb{Z} \longrightarrow \mathrm{Pic}(\tilde{X}) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$$
$$1 \longmapsto E$$

splits by

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\tilde{X})$$
$$L \longmapsto \pi^* L$$

6) $\tilde{X} \setminus E \simeq X \setminus \{p\} \Rightarrow K_{\tilde{X}} - \pi^* K_X$ is supported on E

Find $n \in \mathbb{Z}$ s.t. $K_{\tilde{X}} \sim \pi^* K_X + nE$.

$E \cdot K_{\tilde{X}} = E(\pi^* K_X + nE) = -n$

$E \simeq \mathbb{P}^1 \Rightarrow g(E) = 0 \xRightarrow{\text{adjunction}} -2 = E \cdot (E + K_{\tilde{X}}) = -1 - n$

$\Rightarrow n = 1 \Longrightarrow K_{\tilde{X}} \sim \pi^* K_X + E$.

<u>Cor</u> $(K_{\tilde{X}})^2 = K_X^2 - 1$.

What is left to be done? To show that $\tilde{X}$ is projective and π is projective. This can be done by using the "algebraic" construction of blowup:

$p \in X$, $\mathcal{I}$ sheaf of ideals of $\{p\} \hookrightarrow X$

$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ quasi-coherent sheaf of $\mathbb{N}$-graded $\mathcal{O}_X$-algebras

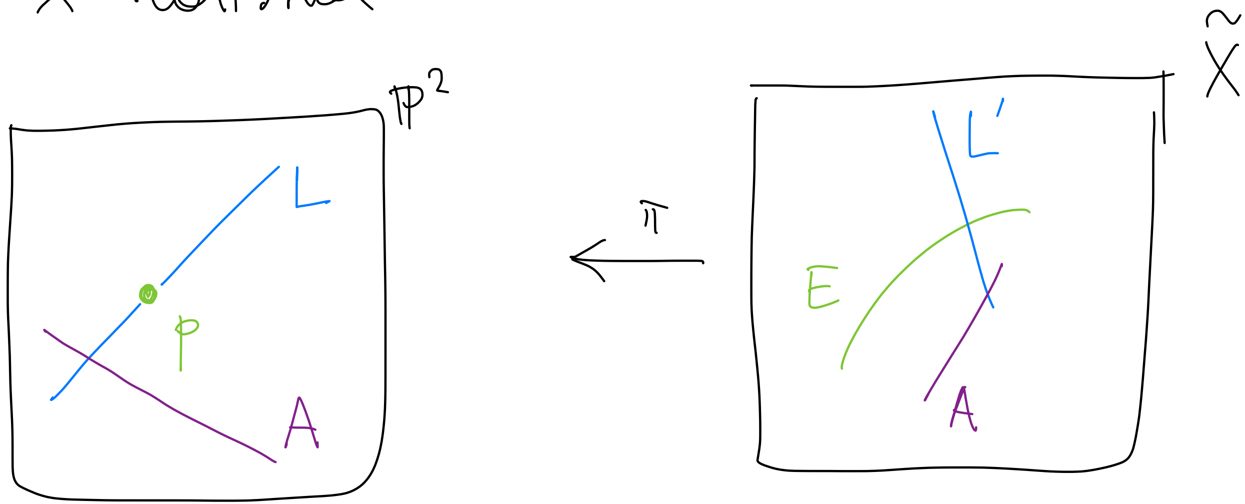
Then $\tilde{X} = \underline{\mathrm{Proj}}_X \mathcal{A} \xrightarrow{\pi} X$ is the blowup of X at p.

The exceptional divisor is the fibred product

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X} \arrow[d, "\pi"] \\ \{p\} \arrow[r, hook] & X \end{tikzcd}$$

Example $X = \mathbb{P}^2 \ni p$ $\quad \tilde{X} = Bl_p X \xrightarrow{\pi} \mathbb{P}^2$

$\tilde{X}$ rational



$h = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ $\qquad \mathrm{Pic}(\tilde{X}) \simeq \mathbb{Z}^2$ with basis h, E.

$A \sim \mathcal{O}_{\mathbb{P}^2}(1) \qquad \pi^* A = h$

$L \sim \mathcal{O}_{\mathbb{P}^1}(1) \qquad \underset{\substack{\| \\ L' + E}}{\pi^* L} = h \qquad \Longrightarrow \qquad L' = h - E.$

$$(L')^2 = (h-E)^2 = h^2 - \cancel{2hE} + E^2 = 1 - 1 = 0$$