Rational and birational maps between surfaces

Emilio Montes de Oca

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Motivation

- **General goal**: Classify varieties.

- Today we introduce one way of doing this: birational classification.

- This classification is coarser than usual isomorphism, but nevertheless many nice properties are preserved.

- Today we will see that birational maps for surfaces have a simple structure: they are composites of a finite number of blow-ups..

Rational maps

Definition (Rational map)

Let X and Y be varieties with X irreducible. A *rational map* $\phi : X \dashrightarrow Y$ is a morphism from an open set $U \subseteq X$ to Y which cannot be extended to any larger open subset.

- **Remark:** A rational map from a smooth complete curve to another one is in fact a morphism.

Thus assume $X = S$ is a surface.

- **Remark:** $S \setminus U = F$ is finite.

Rational maps

A couple of extra definitions:

- The *image* of ϕ is $\phi(S) = \overline{\phi(U)}$.

- If $C \subseteq S$ is an irreducible curve, the *image* of C is $\phi(C) = \overline{\phi(C \setminus F)}$.

- We also have $Pic(S) \cong Pic(S \setminus F)$. Thus we can also speak of the *inverse image* of ϕ of a divisor D in Y (or of an invertible sheaf or a linear system), denoted by $\phi^*(D)$.

Example

Let $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be defined by $\phi(x, y, z) = (x, y)$.

Here $U = \mathbb{P}^2 \setminus \{(0, 0, 1)\}$ and $|F| = 1$, $\phi(\mathbb{P}^2) = \mathbb{P}^1$.(is U indeed maximal?)

(This map is projection from $(0, 0, 1)$.)

We also have $\phi(\overline{x + y + z = 0}) = \overline{\mathbb{P}^1} = \mathbb{P}^1$, and similarly $\phi(x + y = 0) = (x, -x) = \overline{(1, -1)} = (1, -1) \in \mathbb{P}^1$.

We also have $\phi^*((a, b)) = bx - ay + 0z$.

Linear systems

Let S be a smooth surface. Recall from previous sessions:

- If $D \in Div(S)$ is effective, $|D|$ denotes the set of effective divisors linearly equivalent to D.

- We have a bijection $|D| \cong \mathbb{P}(H^0(O_S(D)))$ (via $[s] \mapsto \mathbb{V}(s)$).

- A linear system on S is a linear subspace of $|D|$. $|D|$ is called complete linear system. The dimension of the system is its dimension as a projective space.

Linear systems

Assume S projective.

- If $V = span(s_0, \dots, s_n) \leq H^0(X, L)$ for $s_0, \dots, s_n \in H^0(S, L)$, write $B = \{x \in S | s_i(x) = 0 \text{ for all } i\}$.

- We thus have a morphism $\phi : X \setminus B \to \mathbb{P}^n$, $x \mapsto [s_0(x) : \dots : s_n(x)]$.

- Write $v = \mathbb{P}(V)$.

Linear systems

- The base locus of v is $B_sV = \{x \in S | s(x) = 0 \text{ for all } s \in V\}$.

- For each point $x \in S$, $ev_x : V \to K$, $s \mapsto s(x)$ is not well defined, but $[ev_x] \in \mathbb{P}(V^*)$ is well defined (as long as there exists at least one section s in which $s(x) \neq 0$).

- the map induced by the v is $\phi_v : X \setminus B \to P(V^*)$, $x \mapsto [ev_x]$.

Linear systems

- The fixed part of the system v is the biggest divisor F contained in the corresponding divisor of zeroes to every element of V.

- A base point of v is a point $p \in S$ such that every divisor of zeroes of V contains p.

Linear systems

- Linear systems induce rational maps.

- $\phi(S)$ is not contained in hyperplanes.

- Reciprocally, if $\phi : S \to \mathbb{P}^n$ is any rational map whose image is not in an hyperplane, then ϕ defines a system of divisors via

$$\phi^*(\{\text{hyperplanes in } \mathbb{P}^n\}).$$

Linear systems

We have thus the equivalence:

$A = \{$rational maps $\phi : S \dashrightarrow \mathbb{P}^n$
such that $\phi(S)$ is not contained in an hyperplane$\}$

and

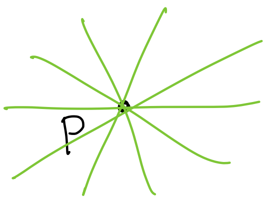
$B = \{$linear systems on S of dimension n with no fixed part$\}$.

$X = \mathbb{P}^2$ $\quad |\mathcal{O}_{\mathbb{P}^2}(1)|$ complete linear system of lines

$$\phi_{|\mathcal{O}_{\mathbb{P}^2}(1)|} : \begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ [x_0 : x_1 : x_2] & \longmapsto & [x_0 : x_1 : x_2] \end{array}$$ is the identity

Fix $p \in \mathbb{P}^2$. Let $\mathfrak{V} \subset |\mathcal{O}_{\mathbb{P}^2}(1)|$ be the linear system given by the lines passing through p.

Assume $p = [1:0:0]$. Then x_1, x_2 are a basis.



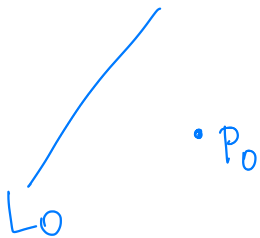
$$\phi_{\mathfrak{V}} : \begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \\ [x_0 : x_1 : x_2] & \longmapsto & [x_1 : x_2] \end{array}$$ is a projection with centre p

The base locus of $\mathfrak{V}$ is $\mathrm{Bs}\,\mathfrak{V} = \{p\}$. So $\phi_{\mathfrak{V}}$ is defined over $\mathbb{P}^2 \setminus \{p\}$.

$X = \mathbb{P}^2$. $|\mathcal{O}_{\mathbb{P}^2}(2)|$ = complete linear system of conics

$\phi_{|\mathcal{O}_{\mathbb{P}^2}(2)|} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ is the 2nd Veronese embedding.

We consider the point $P_0 = [1:0:0]$ and the line $L_0 = (x_0 = 0)$



We consider two linear subsystems of $|\mathcal{O}_{\mathbb{P}^2}(2)|$:

$\mathcal{V} = \{ L_0 + \ell \mid \ell \text{ line} \}$

$\mathcal{W} = \{ L_0 + \ell \mid \ell \text{ line passing through } P_0 \}$

The base loci are: $Bs\,\mathcal{V} = \underbrace{L_0}_{\text{fixed part}}$, $Bs = \underbrace{L_0}_{\text{fixed part}} \cup \{P_0\}$

x_0^2, x_0x_1, x_0x_2 is a basis of $\mathcal{V}$. So

$$\phi_{\mathcal{V}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$
$$[x_0:x_1:x_2] \longmapsto [x_0^2 : x_0x_1 : x_0x_2]$$

is the identity, it coincides with the rational map induced by $|\mathcal{O}_{\mathbb{P}^2}(1)|$.

$x_0x_1,\ x_0x_2$ is a basis of W

$$\phi_W : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$

$$[x_0 : x_1 : x_2] \longmapsto [x_0x_1 ; x_0x_2]$$

coincides with the rational map $[x_0 : x_1 : x_2] \mapsto [x_1 : x_2]$ which is induced by the linear subsystem of $|\mathcal{O}_{\mathbb{P}^2}(1)|$ considered 2 slides ago.

The last 2 examples should convince you that:

L line bundle on X, $V \subseteq |L|$ linear system.

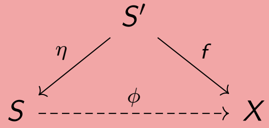
Let $D \in \mathrm{Div}^+ X$ be the fixed part of V. So that there exists a linear system $\overline{V} \subseteq |L \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)|$ such that $V = D + \overline{V}$.

Then $\phi_V = \phi_{\overline{V}}$ and $\overline{V}$ is without fixed part.

Theorem (Elimination of indeterminacy)

Let $\phi : S \dashrightarrow X$ be a rational map from a surface S to a projective variety X.

There is a surface S', a morphism $\eta : S' \to S$ given by composition of finitely many blow-ups and a morphism $f : S' \to X$ such that



commutes.

Rational maps and linear systems

Proof outline:

We may assume WLOG $X = \mathbb{P}^m$ and that $\phi(S)$ lies in no hyperplane of $\mathbb{P}^m$.

Let $P \subseteq |D|$ be the corresponding linear system of dimension m (under the last correspondence).

If P has no base point, ϕ is already a morphism & take $S' = S$, $\eta = id$.

Rational maps and linear systems

Suppose x is a base point. Consider the blow-up $\epsilon : S_1 \to S$ at x.

Pull back the system: $\epsilon^*(P) \subseteq |\epsilon^* D|$. The exceptional curve E appears on the fixed part of this system, with multiplicity $k \geq 1$.

Take out fixed components of the system: $P_1 = |\epsilon^* D - kE|$.

This defines a rational map $\phi_1 : S_1 \dashrightarrow \mathbb{P}^m$, $\phi_1 = \phi \circ \epsilon$.

Rational maps and linear systems

We thus produce a sequence $\epsilon_n : S_n \to S_{n-1}$ of blow-ups and systems $P_n \subseteq |D_n|$ on S_n with no fixed part and $D_n = \epsilon^* D_{n-1} - k_n E_n$.

Recall: for D, D' divisors on S, $\epsilon^* D . \epsilon^* D' = D.D'$, $E.(\epsilon^* D) = 0$, $E^2 = -1$.

Observe:

$$D_n^2 = \epsilon^* D_{n-1} . \epsilon^* D_{n-1} + k_n E_n . k_n E_n - \epsilon^* D_{n-1} . k_n E_n - k_n E_n . \epsilon^* D_{n-1}$$

i.e.

$$\text{i.e., } D_n^2 = D_{n-1}^2 - k_n^2 < D_{n-1}^2.$$

$D_n^2 \geq 0$ as P_n has no fixed part. Hence the process ends.

Remarks:

- Proof is constructive.
- D^2 bounds above the number of blow-ups.

Birational maps

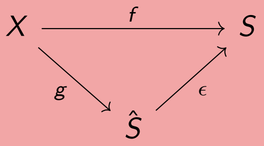
Definition (Birational)

A rational map $f : X \dashrightarrow Y$ is a *birational map* if there is another rational map $Y \dashrightarrow X$ inverse to f. In this case, we say X and Y are birational.

If in addition f is a morphism, we call it a birational morphism.

Theorem (Universal property of Blow-Up)

Let $f : X \to S$ be a birational morphism of surfaces. Suppose f^{-1} is undefined at $p \in S$. Then f factors as



where g is a birtaional morphism and ϵ the blow-up at p.

Blow-ups

Lemma

Let S be an irreducible (but possibly singular) surface. Let S' be a smooth surface and $f : S \to S'$ a birational morphism.

If f^{-1} is undefined at p, then $f^{-1}(p)$ is a curve on S.

Blow-ups

Proof outline:

- Assume WLOG that S is affine, i.e., we have an embedding $S \hookrightarrow \mathbb{A}^n$.

- $j \circ f^{-1} : S' \to \mathbb{A}^n$ is defined by rational functions $g_1, \ldots, g_n$. Say g_1 is undefined at x, i.e., $g_1 \notin \mathcal{O}_{S',p}$.

- Write $g_1 = u/v$, with $u, v \in \mathcal{O}_{S',p}$ coprime and $v(p) = 0$.

- The pullback of v defines a curve $C = \mathbb{V}(f^*v) \subseteq S$.

- If $S \subseteq \mathbb{A}^n$ has coordinates x_i, then $f^*u = x_1 f^*v$. But this means:

$$f^*(u) = f^*(v) = 0 \text{ in } C,$$

or

$$C = f^{-1}(u = v = 0).$$

Blow-ups

- u, v coprime imply that $u = v = 0$ is finite.

- By shrinking S' if necessary, we can assume $u = v = 0$ is simply P.

Thus $C = f^{-1}(p)$.

Blow-ups

Lemma

Let $\phi : S \dashrightarrow S'$ be a rational map of surfaces such that ϕ^{-1} is undefined at $p \in S'$. Then there is a curve C on S such that $\phi(C) = \{p\}$.

Proof outline:

- ϕ corresponds to a morphism $f : U \to S'$ for some $U \subseteq S$ open.

- Let $\Gamma = \{(u, f(u)) \in U \times S' | u \in U\}$ be the graph of f.

- Let $S_1 = \overline{\Gamma} \subseteq S \times S'$. S_1 is an irreducible surface, possibly with singularities.

Blow-ups

- We also have the diagram:

$$\begin{array}{ccc} & S_1 & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ S & \overset{\phi}{\dashrightarrow} & S' \end{array}$$

with the projections birational morphisms.

- Use last lemma: if ϕ^{-1} is undefined at $p \in S'$, then π_2^{-1} is undefined at p.

- But then there is a $C_1 \subseteq S_1$ irreducible curve with $\pi_2(C_1) = p$.

- $\pi_1(C_1)$ is thus a curve in S with image $\{p\}$.

Blow-ups

We now prove the universal property of blow-up:

Theorem (Universal property of Blow-Up)

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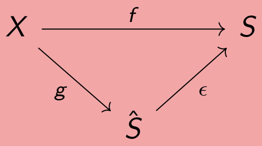
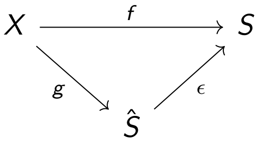


where g is a birtaional morphism and ϵ the blow-up at p.

Blow-ups

Proof outline:

- Our goal is to prove the factorization:



- Take $g = \epsilon^{-1} \circ f$, which is birational. Denote $s = g^{-1}$.

Blow-ups

Proof outline:

- Our goal is to prove the factorization:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & S \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle \epsilon} & \\ & C \subseteq \hat{S} & \end{array}$$

- Take $g = \epsilon^{-1} \circ f$, which is birational. Denote $s = g^{-1}$.

- Suppose g is undefined at $q \in X$. Use above lemma to find $C \subseteq \hat{C}$ with $s(C) = \{q\}$.

- This means $\epsilon(C) = f(q)$ & $E = C$.

Blow-ups

$$
\begin{array}{ccc}
q \in X & \xrightarrow{\quad f \quad} & S \supset f(q) = \epsilon(C) = p \\
& {\scriptstyle g} \searrow \qquad \nearrow {\scriptstyle \epsilon} & \\
& C = E \subseteq \hat{S} &
\end{array}
$$

- Let m_q be the maximal ideal of $O_{X,q}$ and let (x, t) be a local coordinate system at $p \in S$.

- If $g^* t \notin m_q^2$, i.e., if it vanishes on $g^{-1}(p)$ with multiplicity 1, then it defines a local equation for $g^{-1}(p)$ in $O_{X,q}$.

- But then $g^* x = ug * t$ for some $u \in O_{X,q}$, and hence if we put $y = x - u(q)t$ we have $g^* y = (u - u(q))g^* t \in m_q^2$.

- **Hence**: There is a local coordinate y on S at p with $f^* y \in m_q^2$.

Blow-ups

$$\begin{array}{ccc} q \in X & \xrightarrow{\quad f \quad} & S \supset f(q) = \epsilon(C) = p \\ & {\scriptstyle g\,(s = g^{-1})} \searrow \quad \nearrow {\scriptstyle \epsilon} & \\ & C = E \subseteq \hat{S} & \end{array}$$

- Take $e \in E$ any point where s is defined. Observe:

$$s^* f^* y = \epsilon^* y \in m_e^2,$$

and this holds for all $e \in E$ outside some finite set.

- Then $\epsilon^* y$ is a local coordinate at every point of E except one!

Blow-ups

Theorem

Let $f : S \to S_0$ be a birational morphism of surfaces. Then there is a sequence of blow-ups $\epsilon_k : S_k \to S_{k-1}$ $(k = 1, \dots, n)$ and an isomorphism $u : S \to S_n$ such that

$$f = \epsilon_1 \circ \cdots \epsilon_n \circ u.$$

Blow-ups

Proof outline:

- If f is an iso., we are done.

- Use univ. prop. of blow-up: $f = \epsilon_1 \circ f_1$, ϵ_1 blow-up of S at p and $f_1 : S \to S_1$ birational morphism.

- If we are not yet done, repeat to construct an infinite sequence of blow-ups $\epsilon_k : S_k \to S_{k-1}$ and birational morphisms $f_k : S \to S_k$ with $\epsilon_k \circ f_k = f_{k-1}$.

Blow-ups

- $\epsilon_k \circ f_k = f_{k-1}$ implies that the number of irreducible curves contracted to a point under f_k is non increasing.

- Exceptional curve is contracted by f_{k-1} but not by f_k, thus on each step the number of irreducible curves decreases strictily.

- This is a contradiction for k big enough.

Blow-ups

Corollary

Let $\phi : S \dashrightarrow S'$ be a birational map of surfaces. Then there is a surface $\hat{S}$ and a commutative diagram

$$\begin{array}{ccccc} & & \hat{S} & & \\ & \swarrow^{f} & & \searrow^{g} & \\ S & & \overset{\phi}{\dashrightarrow} & & S' \end{array}$$

where the morphisms f,g are composites of blowups and isomorphisms.

Blow-ups

Remarks:

1. Every birational morphism from S to itself is an isomorphism.

2. The blow-up $\epsilon : \hat{S} \to S$ at p has also a universal property in the *other direction*:

Every morphism $f : \hat{S} \to X$ that contracts E to a point factors through S.

Birational surfaces

Denote by $B(S)$ the set of isomorphism classes of surfaces birational to S.

Definition

For $S_1, S_2 \in B(S)$, we say that S_1 dominates S_2 if there exists a birational morphism $S_1 \to S_2$.

Definition

A surface S is minimal if its class in $B(S)$ is minimal, i.e., if every birational morphism $S \to S'$ is an isomorphism.

Theorem

Every surface dominates a minimal surface.

Proof outline:

- Take a surface S. If S is not minimal then there is a birational morphism $S \to S_1$ which is not an isomorphism.

- If S_1 is not minimal, take $S_1 \to S_2$ birational.

- At each step, the rank of the Neron-Severi group decreases strictly (follows from last lecture), and thus we eventually get a minimal surface.

Thank you!