Castelnuovo's Contractibility Criterion

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<u>Theorem</u>: Let S be a smooth projective surface and let $E \subseteq S$ be a proj curve isomorph. to $\mathbb{P}^1$ s.t. $E^2 = -1$. Then E is an exceptional curve on S

<u>Proof</u>: Assume $S \subseteq \mathbb{P}^d$ and let Π be a hyperplane in $\mathbb{P}^d$. Let $H := S \cap \Pi \subseteq S$ a hyperplane section on S.

assume this is a curve on S

H is a very ample divisor on S.

Let $H \subseteq S$ be a hyperplane section such that

$$H^1(S, \mathcal{O}_S(H)) = 0$$

For every hyp. sect. H_0 there exists $m_0 \in \mathbb{N}$ s.t. $(\forall n \geq m_0)$ $H^1(S, \mathcal{O}_S(nH)) = 0$ (Serre's vanishing & finiteness adapted)

Define $k = H.E$

$H' = H + kE$

Rule: Let X be a smooth proj. surface

$C \subseteq X$ is a smooth projective curve

D is a divisor on X

$$D.C = \deg_C \mathcal{O}_X(D)|_C$$

Now, $\mathcal{O}_S(H')|_E$ is a coherent sheaf of $\mathcal{O}_S$-modules on E. It is also invertible. Since $E \cong \mathbb{P}^1$, invert. sheafs on E are determined by their degree.

$$\deg_E \mathcal{O}_S(H)|_E = H \cdot E = k \Rightarrow \mathcal{O}_S(H)|_E \cong \mathcal{O}_E(k)$$
$$\deg_E \mathcal{O}_S(E)|_E = E \cdot E = -1 \Rightarrow \mathcal{O}_S(E)|_E \cong \mathcal{O}_E(-1)$$
$$\deg_E \mathcal{O}_S(H')|_E = (H + k \cdot E) \cdot E = 0 \Rightarrow \mathcal{O}_S(H')|_E \cong \mathcal{O}_E$$

choose $s \in H^0(S, \mathcal{O}_S(E))$ which defines E by $\quad \mathrm{div}_0(s) = E$

$$0 \longrightarrow \underset{\substack{\| \\ \mathcal{O}_S(-E)}}{\mathcal{I}_E} \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_E \longrightarrow 0$$ s.e.s. of sheaves

(ideal sheaf of E)

For every $1 \le i \le k$ consider $H + iE$ with the associated sheaf $\mathcal{O}_S(H+iE)$ invertible and in part. flat (as an $\mathcal{O}_S$-module)

$$0 \to \underbrace{\mathcal{O}_S(-E) \otimes_{\mathcal{O}_S} \mathcal{O}_S(H+iE)}_{\cong} \longrightarrow \underbrace{\mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{O}_S(H+iE)}_{\cong} \to \underbrace{\mathcal{O}_E \otimes \mathcal{O}_S(H+iE)}_{\cong} \to 0$$

$$0 \to \mathcal{O}_S(H + (i-1)E) \to \mathcal{O}_S(H+iE) \to \mathcal{O}_S(H+iE)|_E \to 0$$

$$\cong \mathcal{O}_E(k-i)$$

Rule, using:

$$(H+iE) \cdot E = k-i$$

Long exact sequence in cohomology induced by s.e.s.

$$0 \to H^0(S, \mathcal{O}_S(H+(i-1)E)) \to H^0(S, \mathcal{O}_S(H+iE)) \to H^0(E, \mathcal{O}_E(k-i)) \longrightarrow$$
$$H^1(S, \mathcal{O}_S(H+(i-1)E)) \to H^1(S, \mathcal{O}_S(H+iE)) \to \underbrace{H^1(E, \mathcal{O}_E(k-i))}_{\cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-i)) = 0} \to \cdots$$

$$0 \to H^0(S, \mathcal{O}_S(H+(i-1)E)) \to H^0(S, \mathcal{O}_S(H+iE)) \to H^0(E, \mathcal{O}_E(k-i)) \longrightarrow$$
$$H^1(S, \mathcal{O}_S(H+\underline{(i-1)}E)) \xrightarrow{*} H^1(S, \mathcal{O}_S(H+\underline{iE})) \to 0$$

__Lemma__: $H^1(S, \mathcal{O}_S(H+iE)) = 0$ for all $0 \leq i \leq k$

Proof: By induction on i

__Base__: $H^1(S, \mathcal{O}_S(H)) = 0$ by assumption

__Step__: Conclude by looking* at long exact sequence.

$\xrightarrow{0} H^0(S, \mathcal{O}_S(H+(i-1)E)) \to H^0(S, \mathcal{O}_S(H+iE)) \to H^0(E, \mathcal{O}_E(k-i)) \to 0$

<u>Choose</u>: $s_0, \dots, s_n$ basis of $H_0(\mathcal{O}_S(H))$

For all $1 \le i \le k$ choose

$$\{a_{i,0}, a_{i,1}, \dots, a_{i,k-i}\} \subseteq H^0(S, \mathcal{O}_S(H+iE)) \text{ s.t.}$$

Restriction map $\Big\downarrow r_i$ maps them to $\mathcal{O}_{\mathbb{P}^1}(k-i)(\mathbb{P}^1)$

$\cong$

a Basis of $H^0(E, \mathcal{O}_E(k-i))$

homog. polynomials of deg $k-i$ in 2 variables.
A basis is given by homog. monomials of deg $k-i$

$$\underbrace{x^{k-i}, x^{k-i-1}y, \dots, y^{k-i}}_{k-i+1, \text{ hence } a_{i,0}, \dots, a_{i,k-i}}$$

$\{ s^k s_0, \ldots, s^k s_n,$

Reminder: $s \in H^0(S, \mathcal{O}_S(E))$

$s^{k-1} a_{1,0}, \ldots, s^{k-1} a_{1,k-1},$

$s_0, \ldots, s_n \in H^0(S, \mathcal{O}_S(H))$ basis

$\vdots$

$a_{ij} \in H^0(S, \mathcal{O}_S(H + iE)), 0 \leq j \leq k-i$ restrict to a basis

$s a_{k-1,0}, s a_{k-1,1},$

$a_{k,0} \in H^0(S, \mathcal{O}_S(H'))$ a nonzero constant

$\longrightarrow H^0(E, \mathcal{O}_S(H')|_E) \cong H^0(E, \mathcal{O}_E) \cong \mathbb{C}$

$a_{k,0} \}$ forms a basis in $H^0(S, \mathcal{O}_S(H'))$

$$\left.\begin{array}{l} a_{k-i,j} \in H^0(S, \mathcal{O}_S(H + (k-i)E)) \\ s^i \in H^0(S, \mathcal{O}_S(iE)) \end{array}\right\} \Rightarrow s^i a_{k-i,j} \in H^0(S, \mathcal{O}_S(\underset{H+kE}{\overset{\shortparallel}{H'}}))$$

$\varphi : S \to \mathbb{P}^{N-1}$ defined by this linear system

The map $[s_0 : \ldots : s_n] : S \to \mathbb{P}^n$ is an embedding

(H is very ample divisor)

Look at $\varphi|_{S \setminus E}$: it is an embedding

φ maps E to the point $p := [0 : \ldots : 0 : 1] \in \mathbb{P}^{N-1}$

Denote by $\mathcal{E} : S \longrightarrow S' := \varphi(S)$ the restr. of φ

<u>Remains</u>: Prove that S' is smooth!

It suffices to prove smoothness at p

$S_\supseteq$

Consider the open set $U \supseteq E$ given by

$$a_{k,0} \neq 0 \wedge [a_{k-1,0} \neq 0 \vee a_{k-1,1} \neq 0]$$

Choose global sections of $\mathcal{O}_S(-E)$ given by

$$x = \frac{a_{k-1,0}}{a_{k,0}} \qquad y = \frac{a_{k-1,1}}{a_{k,0}}$$

These define rational maps by

$h_2 : U \to \mathbb{P}^1$ given by $[x : y]$ section of $\mathcal{O}_S$

$h_1 : U \to \mathbb{A}^2$ given by $(sx : sy)$

$h := (h_1, h_2) : U \to \mathbb{A}^2 \times \mathbb{P}^1 \supseteq \widehat{\mathbb{A}^2}$ the blowup of $\mathbb{A}^2$ at 0

coordinates (u, v, U, V)

In fact $h(U) \subseteq \widehat{\mathbb{A}^2}$

since it is given by $uV - vU = 0$ blowup

$h = (sx, sy ; x : y)$ and this fulfills

<u>Property 1</u> : h induces an isomorphism

$h|_E \; E \longrightarrow$ the exceptional divisor of $\widehat{\mathbb{A}^2}$

<u>Proof</u>: on E we have $h(u) = (0:0; x(u) : y(u))$

which is an isomorphism

linearly independent

<u>Property 2</u>: For all $q \in E \subseteq \mathcal{U}$, h is étale in a neighborhood of q

in our case: locally biholomorphic

there is a system of local coordinates of $h(q)$ which pulls back to a local system of coordinates at q on S

Take $(u, v; U : V)$ natural coordinates on $\mathbb{A}^2 \times \mathbb{P}^1$

Reminder: $\widehat{\mathbb{A}^2}$ is given by $uV - vU = 0$

Let's suppose that $x(q) = 0$ and $y(q) = 1$

$h(q)$ has coords: $(0, 0; 0 : 1) \in \widehat{\mathbb{A}^2}$ $\quad h = (sx, sy; x : y)$

Choose local coordinates at $h(q)$:

$$v \quad \text{and} \quad u/v$$

$h^*(v) = v \circ h = s \cdot y$ vanishes with order 1 on E

$h^*(u/v) = x/y$ when restricted to E a local coordinate of E at q

$s \cdot y$ and x/y are local coordinates on S at q which we were looking for.

$$
\begin{tikzcd}
S \arrow[r, "h"] \arrow[d, "\varepsilon"'] & \widehat{\mathbb{A}}^2 \arrow[d, "\eta"] \\
\varepsilon(S) \arrow[r, "\bar{h}"] & \mathbb{A}^2
\end{tikzcd}
$$

To prove the theorem we'll show:

There is a neighborhood U of E and V of the exc. curve in $\widehat{\mathbb{A}}^2$

such that

$$
\begin{tikzcd}
U \arrow[r, two heads, "h|_U"] \arrow[d, "\varepsilon"'] & V \arrow[d, "\eta"] \\
\varepsilon(U) \arrow[r, "\bar{h}|_{\varepsilon(U)}"] & \eta(V)
\end{tikzcd}
$$

$h|_U : U \longrightarrow V$ is an isomorphism in analytic sense (biholomorphism)

$$\begin{tikzcd}
U \arrow[r, "h|_U"] \arrow[d, phantom, sloped, "\subseteq"', "\text{open subset}"] & V \arrow[d, phantom, sloped, "\subseteq", "\text{open subset}"'] \\
S \arrow[r, "h"] \arrow[d, "\varepsilon"'] & \widehat{\mathbb{A}}^2 \arrow[d, "\eta"] \\
\varepsilon(S) \arrow[r, "\bar{h}"] & \mathbb{A}^2
\end{tikzcd}$$

We now prove

if $h|_U$ is biholomorphism

$\varepsilon(S)$ is smooth at p

To do this, we show

$$\varepsilon(U) \xrightarrow{\bar{h}|_{\varepsilon(U)}} \eta(V)$$

is biholomorphism

which implies

$\varepsilon(U)$ is smooth.

$$\begin{tikzcd}
U \arrow[r, "h|_U"] \arrow[d, "\varepsilon"'] & V \subseteq \widehat{\mathbb{A}}^2 \arrow[d, "\eta"] \\
\varepsilon(U) \arrow[r, "\bar{h}|_{\varepsilon(U)}"] & \eta(V) \\
 & \mathbb{A}^2 \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Blue = with assumption of $h|_U$ is biholomorphism

exceptional curve of $\widehat{\mathbb{A}}^2$ contracted to point

$$\varepsilon \circ h|_U^{-1} : V \longrightarrow \varepsilon(U)$$

Lemma (Reverse Universal Property)

Let $\varepsilon : \hat{S} \to S$ be a blowup

Let $f : \hat{S} \to X$ be a map contracting the exceptional divisor to a point.

Then

$$\begin{tikzcd} f : \hat{S} \arrow[r] \arrow[dr, "\varepsilon"'] & X \\ & S \arrow[u, "\bar{f}"'] \end{tikzcd}$$

commutes

(in analytic category)

$$
\begin{tikzcd}
U \arrow[r, "h|_U"] \arrow[d, "\varepsilon"'] & V \subseteq \hat{\mathbb{A}}^2 \arrow[d, "\eta"] \\
\varepsilon(U) \arrow[r, "\bar{h}|_{\varepsilon(U)}"] & \eta(V) \\
 & \rotatebox{90}{$\subseteq$} \\
 & \mathbb{A}^2
\end{tikzcd}
$$

$\varepsilon \circ h|_U^{-1} : V \to \varepsilon(U)$

factors through η

and a map

$\eta(V) \longrightarrow \varepsilon(U)$

which is the inverse

of $\bar{h}|_{\varepsilon(u)}$

Meaning that $\bar{h}|_{\varepsilon(u)}$ is a biholomorphism

Consequently $\varepsilon(u)$ is smooth.

Lemma: Let $f: X \to Y$ be a continuous map of Hausdorff topological spaces.
Let $K \subseteq X$ be a compact subset.
Suppose that

(i) $f|_K : K \to f(K)$ is a homeomorphism,

(ii) $(\forall k \in K)$ f is a local homeomorphism at k.

Then there is a neighborhood U of K and an open set $V \subseteq Y$ such that

$$f|_U : U \to V \text{ is a homeomorphism.}$$

In our case

$\mathbb{P}^1 \cong E \subseteq S \Rightarrow E$ is compact

$$\begin{tikzcd}
U \arrow[r, "h|_U"] \arrow[d, hook, "\text{open subs}"'] & V \arrow[d, hook, "\text{open subset}"] \\
S \arrow[r, "h"] \arrow[d, "\varepsilon"'] & \mathbb{A}^2 \arrow[d, "\eta"] \\
\varepsilon(S) \arrow[r, "\bar{h}"] & \mathbb{A}^2
\end{tikzcd}$$

<u>Property 1:</u>

$h|_E : E \longrightarrow$ except. div. of $\mathbb{A}^2$ is biholomorphism, in particular homeomorphism;

<u>Property 2:</u> $(\forall q \in E)$ h is étale in a neighborhood of q.

$\Rightarrow$ for every $q \in E$ there exist

$U_q \ni q$ open neighborhood and $V_q \ni h(q)$ open neighborhood such that $h|_{U_q} : U_q \to V_q$ is a local biholomorphism, in particular a local homeomorph.

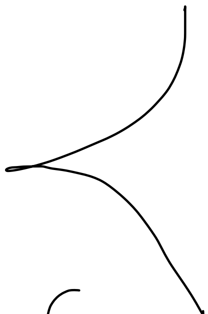
$\Rightarrow$ The assumptions of the lemma are satisfied

Lemma implies the existence of $U \supseteq E$ an open neighborhood and $V \supseteq$ except. dir. of $\widehat{\mathbb{A}^2}$ such that $h|_U : U \to V$ is homeomorphism

$$\left.\begin{array}{l} S \text{ smooth, } \widehat{\mathbb{A}^2} \text{ smooth} \\ h|_U \text{ is holomorphic} \\ h|_U \text{ is homeomorphism} \end{array}\right\} \Rightarrow h|_U \text{ is biholomorphism}$$

As explained, this finishes the proof. $\square$

$C = V(y^2 - x^3) \subset \mathbb{A}^2$



$C \xleftarrow[\text{normalisation}]{\pi} \mathbb{A}^1$

$$\pi : \mathbb{A}^1 \longrightarrow C$$
$$t \longmapsto (t^2, t^3)$$

π algebraic map

π homeomorphism both w.r.t. Zariski topology and analytic topology

π is not an isomorphism neither in alg. category nor in holomorph. category

X smooth proj. surface over $k = \bar{k}$. $C \subset X$ irred. curve.
We say that C is CONTRACTIBLE if $\exists X \xrightarrow{\pi} X_0$ proper birational s.t. $\pi(C) = p$ point, $X \setminus C \xrightarrow{\pi|} X \setminus \{p\}$ isomorphism

$C \simeq \mathbb{P}^1$, $C^2 = -1$

C is a (-1)-curve

$\xRightarrow{\text{Castelnuovo}}$

$\Longleftarrow$ every proper birational morphism is a blow up

C contractible with X_0 smooth, projective

Exercise V.5.7 in Hartshorne

$C \simeq \mathbb{P}^1$, $C^2 < 0$ $\Longrightarrow$ C contractible with X_0 projective, not necessarily smooth $\Longrightarrow$ $C^2 < 0$

adapt the first part of the proof of Castelnuovo's theorem

$\Downarrow$ [Grauert, Artin]

C contractible with X_0 possibly singular, possibly non-algebraic