Minimal Models of $C \times \mathbb{P}^1$ for a Smooth Irrational Curve C

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Plan

Plan for today:

1. Technical preliminaries
2. Define ruled and geometrically ruled surfaces
3. Prove the Noether-Enriques theorem
4. Classify Minimal Models for $C \times \mathbb{P}^1$ where C is an irrational smooth curve

Conventions

surface $:=$ smooth projective surface over $\mathbb{C}$

curve $:=$ projective curve over $\mathbb{C}$

point $:=$ closed point

k is an algebraically closed field

If S is a surface, C a curve an $p\colon S \to C$ a surjective morphism. We always consider fibres with multiplicites i.e. as pull-back of divisors (or scheme-theoretically).

Non-Abelian Čech Cohomology

X topological space,

G a sheaf of (not necessarily abelian) groups on X,

$\mathcal{U} = (U_i)_{i \in I}$ an open cover of X.

A Čech 1-cocycle of G on $\mathcal{U}$ is a tuple

$$(g_{ij})_{i,j \in I}, \text{ where } g_{ij} \in \Gamma(U_i \cap U_j, G)$$

that satisfies the cocycle condition

$$g_{kj} g_{ji} = g_{ki} \text{ for all } i, j, k \in I$$

wherever defined.

Non-Abelian Čech Cohomology

Two Čech 1-cocycles (g_{ij}) and (g'_{ij}) are called cohomologically equivalent if there is $(h_i)_{i \in I}$ where $h_i \in \Gamma(U_i, G)$ such that

$$h_i g_{ij} = g'_{ij} h_j$$

for all $i, j \in I$ wherever defined.

This is an equivalence relation,

$$\check{\mathrm{H}}^1(\mathcal{U}, G) := \{\text{Čech 1-cocycles for } G \text{ on } \mathcal{U}\}/\text{cohomological equivalence}$$

first Čech cohomology of G on $\mathcal{U}$.

It is a pointed set with distinguished element being the cohomology class of the Čech 1-cocycle given by $g_{ij} = 1$ for all i, j.

Non-Abelian Čech Cohomology

Another open cover $\mathcal{V} = (V_j)_{j \in J}$ is a refinement of $\mathcal{U}$ there is a map $\tau\colon J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for all $j \in J$.

We get a well-defined map (does not depend on the choice of τ)

$$\begin{aligned} \tau^*\colon \check{\mathrm{H}}^1(\mathcal{U}, G) &\to \check{\mathrm{H}}^1(\mathcal{V}, G) \\ (g_{ii'}) &\mapsto \left(g_{\tau(j)\tau(j')}\big|_{V_j \cap V_{j'}} \right) \end{aligned}$$

$\mathcal{U}$ and $\mathcal{V}$ are equivalent if both are refinements of each other

$$\mathcal{U} \text{ and } \mathcal{V} \text{ equivalent} \implies \tau^* \text{ is an isomorphism}$$

Non-Abelian Čech Cohomology

The first Čech cohomology of G on X is

$$\check{\mathrm{H}}^1(X, G) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^1(\mathcal{U}, G),$$

where the direct limit is taken over equivalence classes of covers.

If G is a sheaf of abelian groups on X, we have an isomorphism

$$\check{\mathrm{H}}^1(X, G) \cong \mathrm{H}^1(X, G).$$

Vector Bundles and Projective Bundles

Now X and F are varieties over k,

G an algebraic group over k with action on F which is regular and effective i.e,

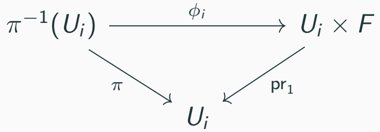
$$G \times F \to F \text{ is a morphism and if } gf = f \text{ for all } f \in F \text{ then } g = 1.$$

Denote by G_r the sheaf of groups on X given by

$$\Gamma(U, G_r) = \{\text{morphisms } U \to G\}.$$

An algebraic fibre bundle over X with structure group G and fibre F is a pair (W, π) where W is a variety and $\pi\colon W \to X$ is a morphism such that there exist

i. an open cover $\mathcal{U} = (U_i)_{i \in I}$ of X and morphisms $\phi_i\colon \pi^{-1}(U_i) \to U_i \times F$ such that the diagram



commutes,

ii. for each pair $(i, j) \in I \times I$ a section $g_{ij} \in \Gamma(U_i \cap U_j, G_r)$ such that for all $x \in U_i \cap U_j$ and all $f \in F$ we have

$$\phi_i \phi_j^{-1}(x, f) = (x, g_{ij}(x) f).$$

Vector Bundles and Projective Bundles

Notice that since we assumed the action to be effective, the sections g_{ij} satisfy the coycle conditions.

Example

1. $F = \mathbb{A}^r$, $G = \mathrm{GL}(r, k) \rightsquigarrow$ rank r vector bundle over X.
2. $F = \mathbb{P}^r$, $G = \mathrm{PGL}(r+1, k)$, we speak of a $\mathbb{P}^r$-bundle over X.
3. If $\mathcal{E}$ is a rank $r+1$ vector bundle we can simply replace each fibre by $\mathbb{P}^r$ and replace the sections g_{ij} by their classes in $\mathrm{PGL}(n+1, k)$. This is now a $\mathbb{P}^r$-bundle.
   We denote this bundle by $\mathbb{P}_X(\mathcal{E})$. It is called the projectivization of $\mathcal{E}$.

Vector Bundles and Projective Bundles

Sending

$$[(\phi_i\phi_j^{-1}\big|_{U_i\cap U_j})_{i,j}] \mapsto [(g_{ij})_{ij}]$$

gives an isomorphism of isomorphism classes of fibre bundles with structure group G, fibre F over X that are trivialised by $\mathcal{U}$ and $\check{\mathrm{H}}^1(\mathcal{U}, G)$.

$\rightsquigarrow$ isomorphism of isomorphism classes of fibre bundles with structure group G, fibre F over X and $\check{\mathrm{H}}^1(X, G)$. The distinguished element on the left hand side is the trivial bundle $X \times F$.

This gives several important identifications for a variety X:

1. $\check{\mathrm{H}}^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$,
2. $\check{\mathrm{H}}^1(X, \mathrm{GL}(r, \mathcal{O}_X)) \cong$
   $\{$iso. classes of rank r vector bundles on $X\}$,
3. $\check{\mathrm{H}}^1(X, \mathrm{PGL}(r+1, \mathcal{O}_X)) \cong \{$iso. classes of $\mathbb{P}^r$-bundles on $X\}$.

GAGA Theorems

Let X be a smooth quasi-projective variety over $\mathbb{C}$. We can view X as a subset of some $\mathbb{C}P^n$ and endow the set X with the subspace topology,

$$\rightsquigarrow X^{\mathrm{an}} \text{ the analytification of } X.$$

It is a complex manifold in a natural way and if X is projective then X^{an} is compact.

The analytic topology is finer than the Zariski topology and every regular function is holomorphic, hence we have a morphism of ringed spaces $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$.

GAGA Theorems

A coherent sheaf on X^{an} is a coherent analytic sheaf. To any coherent sheaf $\mathcal{F}$ on X we can assign a coherent analytic sheaf by

$$\mathcal{F} \mapsto i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\text{an}}} = i^*\mathcal{F} =: \mathcal{F}^{\text{an}}$$

called the analytification of $\mathcal{F}$.

Notice that this maps locally free sheaves to locally free sheaves and $\mathcal{O}_X^{\text{an}} = \mathcal{O}_{X^{\text{an}}}$.

> It is a non-trivial theorem of Oka that $\mathcal{O}_{X^{\text{an}}}$ is indeed a coherent analytic sheaf.

GAGA Theorems

Theorem (Serre '56)
Let X be a smooth projective variety over $\mathbb{C}$, then

1. *For every coherent sheaf $\mathcal{F}$ on X and every $q \geq 0$, there is an isomorphism*
$$\mathrm{H}^q(X, \mathcal{F}) \to \mathrm{H}^q(X^{an}, \mathcal{F}^{an}).$$
2. *There is a bijection*
$$\mathsf{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathsf{Hom}_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an}, \mathcal{G}^{an}).$$
3. *For every coherent analytic sheaf $\mathcal{G}$ on X^{an} there is a coherent sheaf $\mathcal{F}$ on X unique up to unique isomorphism such that $\mathcal{F}^{an} = \mathcal{G}$. Furthermore, if $\mathcal{G}$ is locally free, so is $\mathcal{F}$.*

The Exponential Sequence for a Surface

In particular:

$$\mathrm{H}^q(X, \mathcal{O}_X) \cong \mathrm{H}^q(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \text{ for all } q \geq 0 \text{ and } \mathrm{Pic}(X) \cong \mathrm{Pic}(X^{\mathrm{an}})$$

Now let S be a smooth projective surface we have the exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_{S^{\mathrm{an}}} \longrightarrow \mathcal{O}_{S^{\mathrm{an}}} \xrightarrow{f \mapsto e^{2\pi i f}} \mathcal{O}^*_{S^{\mathrm{an}}} \longrightarrow 0$$

from which we obtain the exact sequence

$$\mathrm{H}^1(S^{\mathrm{an}}, \mathcal{O}_{S^{\mathrm{an}}}) \longrightarrow \mathrm{H}^1(S^{\mathrm{an}}, \mathcal{O}^*_{S^{\mathrm{an}}}) \longrightarrow \mathrm{H}^2(S^{\mathrm{an}}, \underline{\mathbb{Z}}_{S^{\mathrm{an}}}) \longrightarrow \mathrm{H}^2(S^{\mathrm{an}}, \mathcal{O}_{S^{\mathrm{an}}})$$

The Exponential Sequence for a Surface

We have an isomorphism $\mathrm{Pic}(S) \cong \mathrm{H}^1(S^{\mathrm{an}}, \mathcal{O}^*_{S^{\mathrm{an}}})$. Since S^{an} is locally contractible we have $\mathrm{H}^2(S^{\mathrm{an}}, \underline{\mathbb{Z}}_{S^{\mathrm{an}}}) \cong \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})$.

The exponential sequence gives the first Chern class map

$$c_1 \colon \mathrm{Pic}(S) \to \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z}),\ \mathcal{O}_S(D) \mapsto c_1(\mathcal{O}_S(D))$$

which preserves the intersection product

$$D.D' = c_1(\mathcal{O}_S(D)) \cup c_1(\mathcal{O}_S(D'))$$

(because the intersection product is Poincaré dual to the cup product).

Rational Varieties

A variety X over k is called rational if X is birational to $\mathbb{P}^n_k$ for some n.

This is equivalent to saying that $K(X) \cong k(x_1, \ldots, x_n)$.

An example of a rational surface is $\mathbb{P}^1 \times \mathbb{P}^1$. It is birational to $\mathbb{P}^2$.

An example of a variety that is not rational is the curve given by $V(Y^2Z - X^3 - XZ^2)$, it has genus one while $\mathbb{P}^1$ has genus zero.

Two Useful Facts

1. Let $f\colon S \to C$ be a surjective morphism from a surface to a smooth curve. Then for any fibre F of f we have $F^2 = 0$:
Let $F = f^{-1}(x)$, then $N_{F/S} = f^* N_{x/C} = f^* \mathcal{O}_x = \mathcal{O}_F$ since any line bundle over a point is trivial. Thus $F^2 = \deg_F(N_{F/S}) = \deg_F \mathcal{O}_F = 0$.

2. Let S be a surface, D an effective divisor on S and C an irreducible curve on S with $C^2 \geq 0$. Then $D.C \geq 0$:
We write $D = D' + nC$ where D' does not contain C and $n \geq 0$ because D is effective. Then $D.C = D'.C + nC^2 \geq 0$.

Ruled and Geometrically Ruled Surfaces

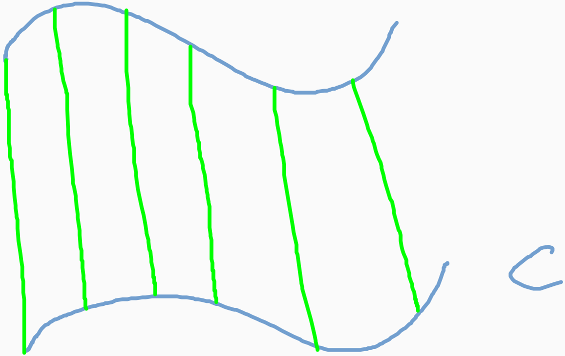
Definition
A surface S is called ruled if there is a smooth curve C and a birational map $S \dashrightarrow C \times \mathbb{P}^1$.

Definition
Let C be a smooth curve. A geometrically ruled surface over C is a pair (S, p) consisting of a surface S and a smooth morphism $p\colon S \to C$ whose fibres are isomorphic to $\mathbb{P}^1$.

In particular the smooth morphism p is surjective. Most of the time we suppress the morphism p and say: "Let S be a geometrically ruled surface over C".

Ruled and Geometrically Ruled Surfaces

An example of a geometrically ruled surface



Ruled and Geometrically Ruled Surfaces

Warning: Some authors (e.g. Harthorne) use the word ruled for what we call geometrically ruled and use birationally ruled for what we call ruled.

Ruled and Geometrically Ruled Surfaces

Let C be a smooth curve.

Example

1. The surface $C \times \mathbb{P}^1$ is a ruled surface and a geometrically ruled over C.
2. Let $\mathcal{E}$ be a rank 2 vector bundle over C, then the associated projective bundle $\mathbb{P}_C(\mathcal{E})$ is a ruled surface (since it is locally trivial) and via the bundle projection $\pi\colon \mathbb{P}_C(\mathcal{E}) \to C$ it becomes a geometrically ruled surface over C.
3. Every rational surface is ruled.

Not Every Ruled Surface is Geometrically Ruled

The surface $\mathbb{P}^2$ is not a geometrically ruled surface over any curve C:

non-constant

Suppose $p\colon \mathbb{P}^2 \to C$ is a smooth morphism to a smooth projective curve C. Let F be a fibre of p, then $F^2 = 0$.

Since F is a non-zero effective divisor on $\mathbb{P}^2$ we have $F \sim dH$ for some $d > 0$ and some line H in $\mathbb{P}^2$. We find $F^2 = (dH)^2 = d^2 > 0$ which is a contradiction.

Noether-Enriques Theorem

Are there examples of surfaces that are geometrically ruled over some curve but are not ruled?

The Noether-Enriques Theorem

Theorem (Noether-Enriques)
Let S be a surface and $p\colon S \to C$ a morphism to a smooth curve C. Suppose that there exists $x \in C$ such that p is smooth over x and $p^{-1}(x) \cong \mathbb{P}^1$. Then there is an open neighbourhood U of x and an isomorphism $\phi\colon p^{-1}(U) \to U \times \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad\phi\quad} & U \times \mathbb{P}^1 \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

commutes. In particular S is ruled.

The Noether-Enriques Theorem

From the Noether-Enriques Theorem we immediately see that any geometrically ruled surface is ruled, we can choose an point of C and apply the theorem.

Denote the fibre of the point from the theorem by $F = p^{-1}(x)$. We will proceed in three steps:

1. Prove that $\mathrm{H}^2(S, \mathcal{O}_S) = 0$.
2. Use the previous fact to construct a divisor H of S with $H.F = 1$.
3. Use this H to construct the isomorphism.

Proof of the Noether-Enriques Theorem: Step 1

Proof.

Since F is a fibre we have $F^2=0$. The genus formula reads

$$0 = g(F) = 1 + \tfrac{1}{2}(F^2 + F.K) = 1 + \tfrac{1}{2} F.K$$

$$\implies F.K = -2$$

Suppose now that $H^2(S, \mathcal{O}_S) \neq 0$, by Serre duality $H^0(S, \omega_S)$ so there is $D \geq 0$ with $D \sim K$ (because $|K| \neq \emptyset$).

Then $F.D = F.K = -2$ but also $F.D \geq 0$ because D is effective, F is irreducible with $F^2 \geq 0$, a contradiction, so

$$\implies H^2(S, \mathcal{O}_S) = 0.$$

Proof of the Noether-Enriques Theorem: Step 2

Recall the homomorphism $c_1\colon \operatorname{Pic}(S) \cong \operatorname{Pic}(S^{an}) \to \mathrm{H}^2(S^{an}, \mathbb{Z})$ satisfies

$$D \cdot D' = c_1(\mathcal{O}_S(D)) \cup c_1(\mathcal{O}_S(D'))$$

Serre's GAGA Theorem 1 and step 1, give

$$0 = H^2(S, \mathcal{O}_S) \cong H^2(S^{an}, \mathcal{O}_{S^{an}})$$

$$\implies c_1 \text{ is surjective.}$$

Let $f = c_1(\mathcal{O}_S(F))$ it suffices to show that there is

$$h \in H^2(S^{an}, \mathbb{Z}) \text{ with } h \cup f = 1$$

(because we can choose any preimage to obtain the desired H).

Proof of the Noether-Enriques Theorem: Step 2

The cup product is bilinear

$$\implies I = \{a \cup f \mid a \in H^2(S^{an}, \mathbb{Z})\}$$

is an ideal in $\mathbb{Z}$. So there is a $d \in \mathbb{Z}$ with $I = d\mathbb{Z}$.

We want to show that $d = \pm 1$

Consider the $\mathbb{Z}$-linear map

$$\mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z}) \to \mathbb{Z}, \ a \mapsto \frac{1}{d}(a \cup f)$$

By Poincaré duality

$$\mathrm{Hom}_{\mathbb{Z}}(H^2(S^{an}, \mathbb{Z}), \mathbb{Z}) \cong H_2(S^{an}, \mathbb{Z})/\text{torsion} \cong H^2(S^{an}, \mathbb{Z})/\text{torsion}.$$

Proof of the Noether-Enriques Theorem: Step 2

$\implies$ there is $f' \in \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})$ with

$$a \cup f' = \frac{1}{2}(a \cup f) \quad \text{for all } a \in \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})$$

for all $a \in \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})$ and thus

$$2f' = f \quad \in \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})/\text{torsion}$$

since the cup product is a perfect pairing on $\mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})/\text{torsion}$.

Let $k = c_1(\omega_S)$ then

$$f \cup k = F.K = -2.$$

Let C' be an irreducible curve on S, write $c_1(\mathcal{O}_S(C')) = c'$, then $c' \cup c' + c' \cup k$ is even since

Proof of the Noether-Enriques Theorem: Step 2

$c' \cup c' + c' \cup k = C'^2 + C' \cdot K = 2(p_a(C') - 1)$

If $D = \sum_i n_i C_i'$ is any divisor on S and $c_i' = c_1(\mathcal{O}_S(C_i'))$. We compute

$$\left(\sum_i n_i c_i'\right) \cup \left(\sum_i n_i c_i'\right) + \left(\sum_i n_i c_i'\right) \cup k$$
$$= \sum_{i \neq j} n_i n_j (c_i' \cup c_j') + \sum_i n_i^2 (c_i' \cup c_i') + \sum_i n_i (c_i' \cup k)$$
$$\equiv \sum_i n_i (c_i' \cup c_i' + c_i' \cup k) \equiv 0 \bmod 2.$$

$\Rightarrow$ c' image of any divisor, the same holds

Proof of the Noether-Enriques Theorem: Step 2

We have $f \cup f = F^2 = 0$ and $f \cup k = F.K = -2$

$$\Longrightarrow f' \cup f' = 0 \text{ and } f' \cup k = \frac{1}{\alpha} f \cup k = -\frac{2}{\alpha}$$

$$\Longrightarrow f' \cup f' + f' \cup k = -\frac{2}{d}.$$

But $f' \cup f' + f' \cup k$ has to be an even number, so .

$$\Rightarrow \alpha = \pm 1.$$

Proof of the Noether-Enriques Theorem: Step 3

Let H be a divisor of S with $H.F = 1$. Since

$$\deg_F \mathcal{O}_S(H+rF)|_F = (H+rF).F = 1$$

for all $r \in \mathbb{Z}$ we have an exact sequence

$$0 \longrightarrow \mathcal{O}_S(H+(r-1)F) \longrightarrow \mathcal{O}_S(H+rF) \longrightarrow \mathcal{O}_F(1) \longrightarrow 0$$

which induces an exact sequence of $\mathbb{C}$-vector spaces

$$\mathrm{H}^0(S,\mathcal{O}_S(H+rF)) \xrightarrow{a_r} \mathrm{H}^0(F,\mathcal{O}_F(1)) \xrightarrow{c_r}$$

$$\mathrm{H}^1(S,\mathcal{O}_S(H+(r-1)F)) \xrightarrow{b_r} \mathrm{H}^1(S,\mathcal{O}_S(H+rF)) \longrightarrow 0$$

Proof of the Noether-Enriques Theorem: Step 3

$\mathcal{O}_S(H + rF)$ is coherent for every $r \in \mathbb{Z}$, so

$$h_r = \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S(H + rF)) < \infty.$$

b_r is surjective for every $r \in \mathbb{Z}$ we obtain a decreasing sequence

$$\cdots \geq h_{r-1} \geq h_r \geq \cdots$$

which must stabilise.

$\implies$ there is an $r \in \mathbb{Z}$ such that b_r is an isomorphism.

By exactness we have

$$\operatorname{im} a_r = \ker c_r \text{ and } \operatorname{im} c_r = \ker b_r = 0$$

$\implies \operatorname{im} a_r = \mathrm{H}^0(F, \mathcal{O}_F(1))$.

Proof of the Noether-Enriques Theorem: Step 3

Choose $s_0, s_1 \in H^0(S, \mathcal{O}_S(H + rF))$ with

$a_1(s_0) = x_0$ and $a_1(s_1) = x_1$

let $V = \text{span}(s_0, s_1)$, $\mathcal{V} = \mathbb{P}(V)$ and let P the corresponding linear system.

$$\implies (\text{Bs}\,\mathcal{V}) \cap F = \emptyset$$

Let Γ be an irreducible component in $\text{Bs}\,\mathcal{V}$, then

$$\Gamma \cdot F = 0$$

Claim: Γ is contained in a fibre.

Suppose Γ is not contained in a fibre then we can find a fibre F' s.t. Γ and F' have no common irreducible components

$$\Rightarrow F' \cdot \Gamma \geq 1$$

Contradicting $F \cdot \Gamma = 0$.

Proof of the Noether-Enriques Theorem: Step 3

Proof of the Noether-Enriques Theorem: Step 3

Suppose $\Gamma_1, \ldots, \Gamma_k$ are the irreducible fixed components of P and Γ_i is contained in $F_i = p^{-1}(x_i)$ distinct from F.

Every element of P is of the form

$$n_1\Gamma_1 + \cdots + n_k\Gamma_k + D' \text{ with } n_i \in \mathbb{Z}_{>0}$$

where D' moves in a 1-dimensional linear system without fixed ~~points~~ components

base locus of the mobile part = fin. many points

contained in fibres $F_{k+1} = p^{-1}(x_{k+1}), \ldots, F_l = p^{-1}(x_l)$ distinct from F.

Proof of the Noether-Enriques Theorem: Step 3

Let $x_{l+1}, \ldots, x_m$ be the points of X with reducible or non-reduced fibre and set

$$\mathcal{U} = C \setminus \{x_1, \ldots, x_m\}$$

don't want this

Proof of the Noether-Enriques Theorem: Step 3

P' = the restriction of P to $p^{-1}(U)$,

$\implies P'$ is base point free

Let $D \in V$ then

$D \geq 0, \; D \sim H + rF$

so

$D \cdot F = (H + rF) \cdot F = 1$

$\implies$ D intersects every fibre.

If F' is a fibre such that F' and D have no common irreducible component

$\implies$ F' and D meet transversally in one point.

Proof of the Noether-Enriques Theorem: Step 3

Claim: D is the union of a section and fibres.

We have

$$D = m_1 C_1 + \dots + m_s C_s + E$$

where $C_1, \dots, C_s$ are the irreducible components contained in a fibre, E has no common irred. component with any fibre.

$$\Rightarrow 1 = D.F = m_1 C_1.F + \dots + m_s C_s.F + E.F$$
$$= E.F$$

Proof of the Noether-Enriques Theorem: Step 3

⇒ E intersects every fiber transversally in one point and. E is irreducible

This cannot happen

We get the desired section by

$x \longmapsto$ point in $p^{-1}(x) \cap E$.

Proof of the Noether-Enriques Theorem: Step 3

Claim: Every element in P' is a section of p.

We have $P' \cong \mathbb{P}^1$ so $P' = \{C_t \mid t \in \mathbb{P}^1\}$

and if $C_t, C_{t'} \in P'$ with $C_t \neq C_{t'}$

generate P'.

Suppose C_t contains a fibre then

$C_{t'} \cdot F = 1 \Rightarrow C_{t'} \cap F \neq \emptyset \Rightarrow C_t \cap C_{t'} \neq \emptyset$

$\Rightarrow$ gives a base locus of P' ↯ to bpf.

Proof of the Noether-Enriques Theorem: Step 3

Let $g : p^{-1}(\mathcal{U}) \longrightarrow \mathbb{P}^1$, $y \longmapsto t$

where C_t is the unique element in P' passing through y

$\Longrightarrow g^{-1}(t) = C_t$

Let $h : p^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times \mathbb{P}^1$, $y \longmapsto (p(y), g(y))$,

then for $(x,t) \in \mathcal{U} \times \mathbb{P}^1$,

$h^{-1}(x,t) = p^{-1}(x) \cap g^{-1}(t) = p^{-1}(x) \cap C_t$

is one point hence h is an isomorphism.

Every Geometrically Ruled Surface is a $\mathbb{P}^1$-bundle

Proposition
Let C be a smooth curve and S a geometrically ruled surface over C. Then there is a rank 2 vector bundle $\mathcal{E}$ over C and an isomorphism $\phi\colon S \to \mathbb{P}_C(\mathcal{E})$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \phi\ } & \mathbb{P}_C(\mathcal{E}) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & C & \end{array}$$

commutes. Furthermore two $\mathbb{P}^1$-bundles $\mathbb{P}_C(\mathcal{E})$ and $\mathbb{P}_C(\mathcal{E}')$ are isomorphic (as bundles over C) if and only if there is a line bundle $\mathcal{L}$ on C with $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.

Every Geometrically Ruled Surface is a $\mathbb{P}^1$-bundle

Proof.

By the Noether-Enriques Theorem there is an open cover $\{U_i\}_{i \in I}$ of C and isomorphism $\phi_i \colon p^{-1}(U_i) \to U_i \times \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times \mathbb{P} \\ & \searrow \quad \swarrow & \\ & U_i & \end{array}$$

commutes, so S is a $\mathbb{P}^1$-bundle over C.

We have to show that every $\mathbb{P}^1$-bundle over C is the projectivization of a rank two vector bundle on C.

Recall the identifications of $\check{\mathrm{H}}^1(C, \mathrm{GL}(2, \mathcal{O}_C))$ and $\check{\mathrm{H}}^1(C, \mathrm{PGL}(2, \mathcal{O}_C))$ with the isomorphism classes of rank two vector bundles and $\mathbb{P}^1$-bundles on C respectively.

Every Geometrically Ruled Surface is a $\mathbb{P}^1$-bundle

From the exact sequence of sheaves of groups

$$1 \longrightarrow \mathcal{O}_C^* \longrightarrow \mathsf{GL}(2, \mathcal{O}_C) \longrightarrow \mathsf{PGL}(2, \mathcal{O}_C) \longrightarrow 1$$

we obtain an exact sequence of pointed sets

$$\check{\mathrm{H}}^1(C, \mathcal{O}_C^*) \longrightarrow \check{\mathrm{H}}^1(C, \mathsf{GL}(2, \mathcal{O}_C)) \longrightarrow \check{\mathrm{H}}^1(C, \mathsf{PGL}(2, \mathcal{O}_C)) \longrightarrow \check{\mathrm{H}}^2(C, \mathcal{O}_C^*)$$

By a theorem of Serre we have $\check{\mathrm{H}}^2(C, \mathcal{O}_C^*) = \mathrm{H}^2(C, \mathcal{O}_C^*) = 0$

$\implies$ every $\mathbb{P}^1$-bundle on C is the projectivization of a rank two vector bundle on C.

The first map is given by the tensor product action of $\mathsf{Pic}(C)$ on $\check{\mathrm{H}}^1(C, \mathsf{GL}(2, \mathcal{O}_C))$. By exactness we yield the second claim.

Minimal Models of $C \times \mathbb{P}^1$ for C Smooth Irrational

We want to classify the minimal models of surfaces birational to $C \times \mathbb{P}^1$ where C is a smooth irrational curve.

Lemma
Let C be a smooth curve and $p\colon S \to C$ a surjective morphism with connected fibres. Let $F = \sum_i n_i C_i$ be a reducible fibre, where the C_i's are distinct and irreducible. Then $C_i^2 < 0$.

Proof.
We have $n_i \geq 0$ for all i. Notice that

$$n_i C_i^2 = C_i.(F - \sum_{i \neq j} n_j C_j).$$

Since F is a fibre we have $C_i.F = 0$ and furthermore $C_i.C_j \geq 0$ for all $i \neq j$. Finally there must be at least one j with $C_i \cap C_j \neq \emptyset$ because F is connected.

Minimal Models of $C \times \mathbb{P}^1$ for C Smooth Irrational

Lemma
Let S be a minimal surface, C a smooth curve and $p\colon S \to C$ a surjective morphism with general fibre isomorphic to $\mathbb{P}^1$. Then S is geometrically ruled by p.

Proof.
Let F be a general fibre of p, then $F^2 = 0$

$$\implies F.K = -2 \text{ by the genus formula.}$$

Since all fibres are algebraically equivalent

$\implies$ Both equalities hold for any fibre

Minimal Models of $C \times \mathbb{P}^1$ for C Smooth Irrational

Case 1: F is an irreducible fibre.

Let F be irreducible and suppose $F = nF'$ with $n > 1$

$$\Rightarrow -2 = K.F = n(K.F')$$

$$\Rightarrow n = 2 \text{ and } K.F' = -1$$

Also

$$n^2 F'^2 = F^2 = 0 \Rightarrow F'^2 = 0$$

So $F'^2 + F'.K = -1$ but we know $F'^2 + F'.K$ has to be even thus $n = 1$.
The genus formula then gives $p_a(F) = 0$

$$\Rightarrow F \cong \mathbb{P}^1$$

Minimal Models of $C \times \mathbb{P}^1$ for C Smooth Irrational

Case 2: F is a reducible fibre.

Let $F = \sum_i n_i C_i$ be reducible. By the previous lemma $C_i^2 < 0$ for all i.
The genus formula gives

$$C_i . K > 2(p_a(C_i) - 1) \geq -2$$

thus

$$C_i . K \geq -1 \text{ and } = -1$$

$$\Longleftrightarrow p_a(C_i) = 0$$

$$\Longleftrightarrow C_i^2 = -1$$

But then C_i is a (-1)-curve on S, contradicting the minimality of S.

Minimal Models of $C \times \mathbb{P}^1$ for C Smooth Irrational

Therefore $C_i . K \geq 0$ and hence $F . K \geq 0$ which contradicts $F . K = -2$.

We have shown that there can't be any reducible fibres, so we are done.

Minimal Models of $C \times \mathbb{P}^1$ for C Smooth Irrational

Theorem
Let C be a smooth irrational curve. The minimal models of $C \times \mathbb{P}^1$ are exactly the geometrically ruled surfaces over C, i.e. the projective bundles $\mathbb{P}_C(\mathcal{E})$ for some rank two vector bundle $\mathcal{E}$ on C.

Proof.
Let $p \colon S \to C$ be a geometrically ruled surface.

Suppose S contains an exceptional curve E. Since $E^2 = -1$, the curve E cannot be a fibre of p.

Thus $p(E) = C$ which is impossible since C is irrational $\implies$ S is a minimal surface.

Minimal Models of $C \times \mathbb{P}^1$ for C Smooth Irrational

Now let S be a minimal surface, $\phi\colon S \dashrightarrow C \times \mathbb{P}^1$ a birational map and $q : C \times \mathbb{P}^1 \to C$ the projection onto the first factor.

By composing, we obtain a rational map $q \circ \phi\colon S \dashrightarrow C$.

By elimination of indeterminacy there is a surface S', a morphism $\eta\colon S' \to S$ obtained from a finite number of blow-ups $\varepsilon_1, \dots, \varepsilon_n$ and a morphism $f\colon S' \to C$ such that the diagram

$$\begin{array}{ccc} & S' & \\ {\scriptstyle \eta}\swarrow & & \searrow{\scriptstyle f} \\ S & \overset{q\circ\phi}{\dashrightarrow} & C \end{array}$$

commutes.

Minimal Models of $C \times \mathbb{P}^1$ for C Smooth Irrational

Suppose $n > 1$ is the minimal number of blow-ups needed for such a diagram to exist.

Let E be the exceptional curve of ε_n, it is impossible that $f(E) = C$, because C is irrational

$\implies$ E is mapped to a point, so $f = f' \circ \varepsilon_n$ which contradicts the minimality of n

$\implies$ $n = 1$ and $q \circ \phi$ is a morphism with general fibre isomorphic to $\mathbb{P}^1$.

Now apply the previous lemma to conclude.

Summary

To sum up today's results:

1. Every geometrically ruled surface is ruled.
2. Every geometrically ruled surface is isomorphic to the projectivization of a rank two vector bundle over a smooth curve.
3. Every geometrically ruled surface over an irrational curve is minimal.
4. If C is an irrational curve, the minimal models of $C \times \mathbb{P}^1$ are exactly the geometrically ruled surfaces over C.

Questions?

References

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Thm [Noether–Enriques] C sm. proj. curve over $\mathbb{C}$, S sm. proj. surface over $\mathbb{C}$, $p : S \to C$ s.t. $x \in C$ a fibre $p^{-1}(x) \simeq \mathbb{P}^1$

$\Rightarrow$ $\exists\, U \subseteq C$ open Zariski nbd of x s.t. $p^{-1}(U) \simeq U \times \mathbb{P}^1$

$$\begin{array}{ccc} p^{-1}(U) & \simeq & U \times \mathbb{P}^1 \\ & {}_{p}\searrow \quad \swarrow_{pr_1} & \\ & U & \end{array}$$

More algebraically:

k algebraically closed field, $S \xrightarrow{p} C$ morphism over k (S: sm. proj. surface; C: sm. proj. curve)

$\exists\, x \in C(k)$ s.t. $p^{-1}(x) \simeq \mathbb{P}^1_k$

then

$$\begin{array}{ccc} p^{-1}(\eta) & \simeq & \mathbb{P}^1_{\kappa(\eta)} \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} \kappa(\eta) & \end{array}$$

where η is the generic point of C

$\kappa(\eta) = K(C)$ "field of rational functions over C"

($\kappa(\eta)$: residue field of η)

Def K field. A BRAUER-SEVERI VARIETY over K is $X \longrightarrow \operatorname{Spec} K$ proper scheme of finite type such that $\exists L \supseteq K$ field extension and $X_L := X \otimes_K L \simeq \mathbb{P}^1_L$

↑ isom. of L-schemes

It is enough to take $L = \overline{K}$ algebraic closure, or L a finite Galois extension of K.

Examples: • $X = \mathbb{P}^1_K$

• $K = \mathbb{R}$, $X = \operatorname{Proj} \mathbb{R}[x_0, x_1, x_2]/(x_0^2 + x_1^2 + x_2^2)$

$X_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x_0, x_1, x_2]/(x_0^2 + x_1^2 + x_2^2) \simeq \mathbb{P}^1_{\mathbb{C}}$

but $X \not\simeq \mathbb{P}^1_{\mathbb{R}}$ $\qquad X(\mathbb{R}) = \emptyset$

Prop X BS-variety over K of dim 1.

1) X is a conic in $\mathbb{P}^2_K$: $\exists\, q \in K[x_0, x_1, x_2]_2$ s.t.

$$X = \mathrm{Proj}\ K[x_0, x_1, x_2]/(q)$$

2) $X \simeq \mathbb{P}^1_K \iff X(K) \neq \emptyset$.

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Pf 1) $\omega_X^\vee$ anticanonical line bundle.

$\deg \omega_X^\vee = 2 \Rightarrow \omega_X^\vee$ very ample

$h^0(\omega_X^\vee) = 3 \Rightarrow$ the map associated to $\omega_X^\vee$ is

$X \hookrightarrow \mathbb{P}^2_K$

2) $\Rightarrow$) obvious.

$\Leftarrow$) by 1 you know that $X \hookrightarrow \mathbb{P}^2_K$. $p \in X(K)$.

Projection from p: $\mathbb{P}^2_K \setminus \{p\} \longrightarrow \mathbb{P}^1_K$, restrict to X. $\square$

Theorem (Tsen) k algebraically closed field
C smooth proj. curve over k
$K = K(C)$ function field of C
$\Rightarrow$ every Brauer-Severi variety over K of dim 1 is isom. to $\mathbb{P}^1_K$.

- Brauer-Severy varieties
- central simple algebras
- Brauer group