Invariants of surfaces and properties of Hirzebruch surfaces

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- Let C be a smooth projective curve and E a vector bundle over C of rank 2. Then $S = \mathbb{P}_C(E)$, $p : S \to C$ is a geometrically ruled surface over C.

- p^*E contains a line bundle N as a sub-bundle: above a point $s \in S$ corresponding to a line $D \subset E_{p(s)}$, we have $N_s = D$.

- The tautological bundle $\mathcal{O}_S(1)$ is defined by the exact sequence:

$$0 \longrightarrow N \longrightarrow p^*E \xrightarrow{u} \mathcal{O}_S(1) \longrightarrow 0$$

Sections of C

- Assume that T is a variety and $f : T \to C$ a morphism.
- There is a one-to-one correspondence between the following two sets:

$$\begin{array}{c} \{g : T \to S | pg = f\} \\ \updownarrow \text{1:1} \\ \{L \in \mathrm{Pic}(T) \quad \text{and} \quad \text{surjective} \quad v : f^* E \to L\} \end{array}$$

  where the map from upstairs to downstairs is given by taking $L = g^* \mathcal{O}_S(1)$ and $v = g^* u$, while the inverse is by $g : t \mapsto [\ker(v_t)]$.
- If we have shown this, by taking $T = C$ and $f = \mathrm{id}$, we see that giving a section $s : C \to S$ is equivalent to giving a rank 1 quotient of E.

- Given a C-morphism $g : T \to S$, let $L = g^*\mathcal{O}_S(1)$ and $v = g^*u$.
- For the exact sequence

$$0 \longrightarrow N \longrightarrow p^*E \xrightarrow{u} \mathcal{O}_S(1) \longrightarrow 0$$

  taking fibers at $g(t)$, we see that $\ker(v_t) = N_{g(t)}$.
- Via the map from downstairs to upstairs, t is assigned to

  $[N_{g(t)}] = g(t)$ by the definition of N.

- Conversely, given a line bundle L and a surjective morphism $v : f^*E \to L$, we can immediately see that $g(t)$ is corresponding to a subspace of $E_{f(t)}$, so it is on the fiber of $f(t)$ along p, i.e. $f = pg$.
- Furthermore, L_t is the cokernel of the embedding $N_{g(t)} \subset E_{f(t)}$. So $L = g^*\mathcal{O}_S(1)$.

Remark

WLOG, let $s \in \mathbb{P}(E)$ corresponds to $[1:0]$ of $\mathbb{P}(E_{p(s)})$. Then $\mathcal{O}(1)_s$ is generated by $(0,1)$, or in other words, it is parametrized by $\frac{x_1}{x_0}$, whose vanishing degree is 1. This means that the restriction of $\mathcal{O}_S(1)$ at each fiber is equal to $\mathcal{O}_{\mathbb{P}^1}(1)$.

Proposition(III.18)

Let $S = \mathbb{P}_C(E)$ be a geometrically ruled surface over C, $p : S \to C$ the structure map. Write h for the class of $\mathcal{O}_S(1)$ in $\mathrm{Pic}(S)$(or in $H^2(S, \mathbb{Z})$). Then:

(1)$\mathrm{Pic}(S) = p^*\mathrm{Pic}(C) \oplus \mathbb{Z}$;
(2)$H^2(S, \mathbb{Z}) = \mathbb{Z}h \oplus \mathbb{Z}f$, where f is the class of a fiber;
(3)$h^2 = \deg(E)$, where $\deg(E) = \deg(\det(E))$;
(4)$[K] = -2h + (\deg(E) + 2g(C) - 2)f$ in $H^2(S, \mathbb{Z})$.

- We show that $\mathrm{Pic}(S) = p^*\mathrm{Pic}(C) \oplus \mathbb{Z}$.
- Let F be a fiber of p. Since $\mathcal{O}_S(1)|_F \cong \mathcal{O}_{\mathbb{P}^1}(1)$, $h.F = 1$
- Let $D \in \mathrm{Pic}(S)$ such that $D.F = m$ and $D' := D - mh$. Then $D'.F = 0$.
- For any $x \in C$, we have $p_*\mathcal{O}(D')_x \otimes k(x) \cong H^0(p^{-1}(x), \mathcal{O}(D')_{p^{-1}(x)})$ by Grauert's Theorem. The latter group is isomorphic to $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$. So $p_*\mathcal{O}(D')$ is a line bundle.
- $\mathcal{O}(D') = p^*p_*\mathcal{O}(D')$
- We have a decomposition $D = (D - mh) + mh$ with $D - mh \in p^*\mathrm{Pic}(C)$, which is clearly unique. We proved (1).

- For the claim $H^2(S,\mathbb{Z}) = \mathbb{Z}h \oplus \mathbb{Z}f$, recall that $H^2(S,\mathbb{Z})$ is a quotient of $\mathrm{Pic}(S)$ by taking the induced long exact sequence of cohomology for the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^\times \to 0.$$

- Two points on C have the same cohomology class in $H^2(C,\mathbb{Z}) \cong \mathbb{Z}$.

- Therefore $H^2(S,\mathbb{Z})$ is generated by f and h, which are linearly independent since $f^2 = 0$ and $f.h = 1$.

- To prove $h^2 = \deg(E)$, we use the following result: let E' be a vector bundle on a surface S with the exact sequence $0 \to L \to E' \to M \to 0$ with $L, M \in \mathrm{Pic}(S)$.
  Then

$$
\begin{aligned}
L.M &= (-L).(-M) \\
&= \chi(\mathcal{O}_S) - \chi(L) - \chi(M) + \chi(L \otimes M) \\
&= \chi(\mathcal{O}_S) - \chi(E') + \chi(\det E')
\end{aligned}
$$

- In particular, $L.M$ is independent of the choice of L and M. Denote it by $c_2(E')$.

- Whenever there is an exact sequence $0 \to L \to E \to M \to 0$, we have $c_2(p^*E) = p^*L.p^*M = 0$.

- Such an extension $0 \to L \to E \to M \to 0$ exists, because p admits a section. As a result, $c_2(p^*E) = 0$.

- Locally the morphism p looks like $\mathbb{P}^1_A \to \mathrm{Spec}(A) \subset C$, so there is a section $s : C \dashrightarrow S$. But every rational map starting from a curve is a morphism. So there is a section of p.

- On the other hand, the exact sequence $0 \to N \to p^*E \to \mathcal{O}_S(1) \to 0$ gives $h.N = c_2(p^*E) = 0$.

- $N = p^* \det E - h \Rightarrow h^2 = h.p^* \det E = \deg(E)$.

- To calculate $[K]$, write $[K] = ah + bf$ in $H^2(S, \mathbb{Z})$. Then $-2 = \deg(K_F) = K.f + f^2 = a \Rightarrow a = -2$.
- Denote the class of the image of s in $\mathrm{Pic}(S)$(or in $H^2(S, \mathbb{Z})$) by $[C]$.
- Since $[C].f = 1$, we write $[C] = h + rf$ in $H^2(S, \mathbb{Z})$. Then $2g(C) - 2 = ([K] + [C]).[C]$ gives $b = \deg(E) + 2g(C) - 2$.

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- Recall that every geometrically ruled surface over C is C-isomorphic to $\mathbb{P}_C(E)$ for some rank 2 vector bundle E.
- $\mathbb{P}_C(E)$ and $\mathbb{P}_C(E')$ are C-isomorphic if and only if there exists a line bundle L on C such that $E' \cong E \otimes L$.
- (Birkhoff-Grothendieck)Every vector bundle on $\mathbb{P}^1$ is decomposable i.e. in the form of $\mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_m)$.
- A Hirzebruch surface is a geometrically ruled surface over $\mathbb{P}^1$. It is isomorphic to $\mathbb{F}_n := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ for some $n \geq 0$.

Proposition IV.1

(1)$\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}h \oplus \mathbb{Z}f$ with $f^2 = 0$, $f.h = 1$ and $h^2 = n$.
(2)If $n > 0$, there is a unique irreducible curve B on $\mathbb{F}_n$ with negative self-intersection. If b is its class in $\mathrm{Pic}(\mathbb{F}_n)$, then $b = h - nf$, $b^2 = -n$.
(3)$\mathbb{F}_n$ and $\mathbb{F}_m$ are not isomorphic unless $m = n$. $\mathbb{F}_n$ is minimal except if $n = 1$. $\mathbb{F}_1$ is isomorphic to $\mathrm{Bl}_p\mathbb{P}^2$.

- Let s denote the section $\mathbb{P}^1 \to \mathbb{F}_n$ corresponding to $\mathcal{O}_{\mathbb{P}^1}$ as a quotient of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$. Let b denote the class of $s(C)$ in $\mathrm{Pic}(\mathbb{F}_n)$. We show that $s(C)$ is the unique irreducible curve with negative self-intersection and calculate b.

- $b = h + rf$ since $b.f = 1$.

- Recall that for the commutative diagram

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[dr, "f"'] & S \arrow[d, "p"] \\ & C \end{tikzcd}$$

  we have $L = g^*\mathcal{O}_S(1)$ and the quotient $g^*u : f^*E \to L$. Replacing T by C, f by id and g by s, we find that $s^*\mathcal{O}_{\mathbb{F}_n}(1) = \mathcal{O}_{\mathbb{P}^1}$

- $h.b = \deg h|_b = \deg_C s^*h = 0$. So $r = -n$ and $b^2 = (h - nf)^2 = -n$.

- We now prove the uniqueness.
- Let C' be an irreducible curve with negative self-intersection on $\mathbb{F}_n$ other than $s(C)$. Write $[C'] = \alpha h + \beta f$ in $\mathrm{Pic}(\mathbb{F}_n)$.
- Since $f^2 = 0$ and the intersection number of two different irreducible curves are non-negative, we have $[C'].f \geq 0$ and $[C'].b \geq 0$.
- $[C'].f = (\alpha h + \beta f).f = \alpha \geq 0$ and
  $[C'].b = (\alpha h + \beta f).(h - nf) = \alpha h^2 + (\beta - n\alpha)h.f = \beta \geq 0$.
- $[C']^2 = (\alpha h + \beta f)^2 \geq 0$.

- We are going to see that n is uniquely determined by $\mathbb{F}_n$ and to show that $\mathbb{F}_n$ is minimal except if $n = 1$.

- When $n = 0$, the total space over $U_0 := \{x_0 \neq 0\}$ is $\mathbb{P}^1_{U_0} = U_0 \times \mathbb{P}^1$, while that over $U_1 := \{x_1 \neq 0\}$ is $U_1 \times \mathbb{P}^1$. The total space over $U_0 \cap U_1 = \mathrm{Spec} k[x, \frac{1}{x}]$ is $(U_0 \cap U_1) \times \mathbb{P}^1$. So they glue to $\mathbb{P}^1 \times \mathbb{P}^1$, on which the self-intersections of all irreducible curves are non-negative.
- n is uniquely determined by $\mathbb{F}_n$, because it is determined by the self-intersection of the unique irreducible curve or it is 0.

- $\mathbb{F}_n$ is minimal for $n \neq 1$ since the self-intersection of the exceptional divisor is -1.

- We are now to see $\mathbb{F}_1$ is $\mathrm{Bl}_p\mathbb{P}^2$.
- The blow-up of $\mathbb{P}^2$ at $[1:0:0]$ is defined to be $M := \{([x_0 : y_0 : z_0], [y_1 : z_1]) \in \mathbb{P}^2 \times \mathbb{P}^1 | y_0 z_1 = z_0 y_1\}$.
- Consider the map $M \to \mathbb{P}^1$ given by the projection onto the second component.
- The fiber at $[y_1 : z_1]$ is given by the linear equation $z_1 y = y_1 z$, so it is isomorphic to $\mathbb{P}^1$.
- M is a geometrically ruled surface over $\mathbb{P}^1$, not minimal. So the only possibility is that it is isomorphic to $\mathbb{F}_1$.

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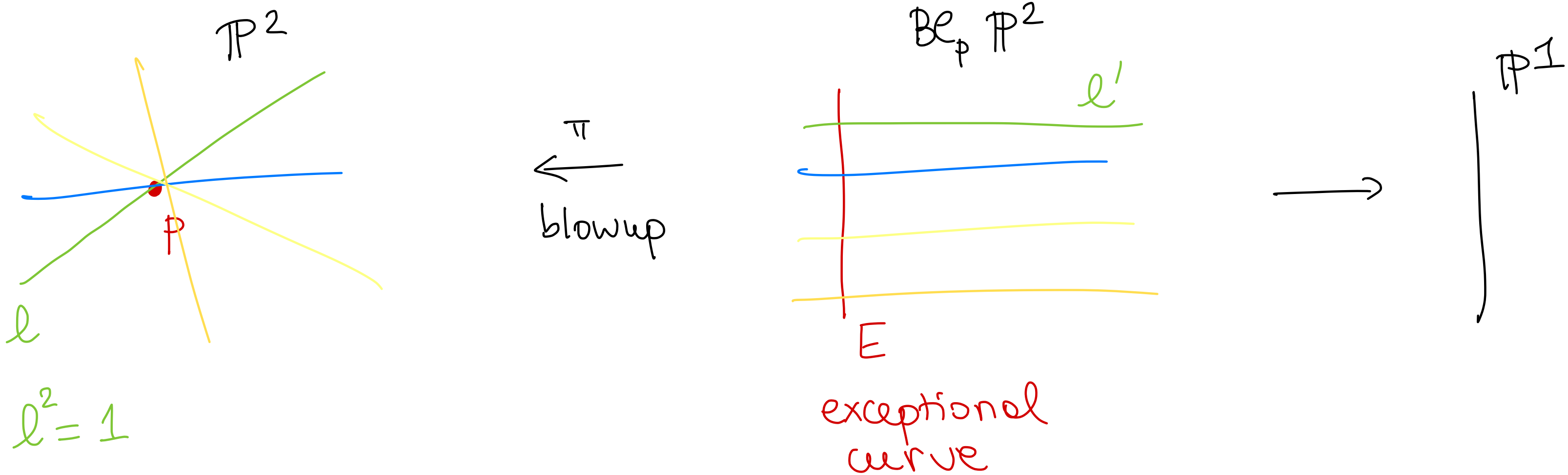
- Let X be a complex manifold of dimension n. Let $z_1, z_2, \cdots, z_n$ be the local coordinates.
- There is a decomposition $\Omega_{X,\mathbb{C}} := \Omega_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$, where $\Omega_X^{1,0}$ is generated by $dz_1, dz_2, \cdots, dz_n$ over $\mathcal{C}_X^\infty$ while $\Omega_X^{0,1}$ is generated by $d\bar{z}_1, d\bar{z}_2, \cdots, d\bar{z}_n$.
- This induces the decomposition of complex k-forms into forms of type (p, q) for $p + q = k$: $\bigwedge^k \Omega_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega_X^{p,q}$, where $\Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}$.



$\ell^2 = 1$

$(\ell')^2 = 0$

$$\pi : Bl_p\,\mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

$$\mathrm{Pic}(Bl_p\,\mathbb{P}^2) \simeq \mathrm{Pic}(\mathbb{P}^2) \oplus \mathbb{Z}\cdot E$$

$$\pi^*(\ell) = \ell' + E$$

$$\underset{\overset{\shortparallel}{1}}{\ell^2} = (\pi^*(\ell))^2 = (\ell' + E)^2 = (\ell')^2 + 2\underbrace{\ell'\cdot E}_{=1} + \underbrace{E^2}_{=-1}$$

$$\Rightarrow \quad (\ell')^2 = 0.$$

Hodge decomposition of Kähler manifolds

- A complex manifold is called Kähler manifold if the imaginary part of its Hermitian structure is a closed form.
- $\mathbb{P}^n$ is a Kähler manifold.
- A projective complex manifold is a Kähler manifold as a submanifold of $\mathbb{P}^n$.
- $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$, where $H^{p,q}(X) = H^q(X, \Omega_X^p)$.
- $H^{p,q} = \overline{H^{q,p}}$. If we write $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X)$, then $h^{p,q} = h^{q,p}$.

Hodge diamond

- Let X be a complex n-dimensional manifold.
- $\Omega_X^p \cong \mathcal{H}om(\Omega_X^{n-p}, \omega_X) \cong (\Omega_X^{n-p})^\vee \otimes \omega_X$. Hence $(\Omega_X^p)^\vee \cong \Omega_X^{n-p} \otimes \omega_X^\vee$.
- By Serre's duality, $H^q(X, \Omega_X^p) = H^{n-q}(X, (\Omega_X^p)^\vee \otimes \omega_X)^* = H^{n-q}(X, \Omega_X^{n-p})^*$.
- $H^{p,q}$ is dual to $H^{n-p,n-q}$
- (Poincaré duality)$H^{p+q}(X, \mathbb{C})$ is dual to $H^{2n-p-q}(X, \mathbb{C})$ by Hodge decomposition.

Hodge diamond

- $h^{p,q} = h^{q,p}$
- $h^{p,q} = h^{n-p,n-q}$
- We have a Hodge diamond(take $n = 4$ as an example):

$$
\begin{array}{ccccccccc}
 & & & & h^{4,4} & & & & \\
 & & & h^{3,4} & & h^{4,3} & & & \\
 & & h^{2,4} & & h^{3,3} & & h^{4,2} & & \\
 & h^{1,4} & & h^{2,3} & & h^{3,2} & & h^{4,1} & \\
h^{0,4} & & h^{1,3} & & h^{2,2} & & h^{3,1} & & h^{4,0} \\
 & h^{0,3} & & h^{1,2} & & h^{2,1} & & h^{3,0} & \\
 & & h^{0,2} & & h^{1,1} & & h^{2,0} & & \\
 & & & h^{0,1} & & h^{1,0} & & & \\
 & & & & h^{0,0} & & & &
\end{array}
$$

- Several numerical invariants on a complex projective surface S:

$$q(S) = h^1(S, \mathcal{O}_S) = h^0(S, \Omega_S^1)$$

$$p_g(S) = h^2(S, \mathcal{O}_S) = h^0(S, K)$$

$$P_n(S) = h^0(S, nK), n \geq 1.$$

  P_n are called plurigenera of S, $p_g = P_1$ is the geometric genus and q is the irregularity of S. We have $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$.

- We also consider topological invariants:

$$b_i(S) = \dim_{\mathbb{C}} H^i(S, \mathbb{C}), \chi_{top}(S) = \sum_i (-1)^i b_i(S).$$

- We have $b_0 = b_4 = 1$ and $b_1 = b_3$ by Poincaré duality, so that $\chi_{top}(S) = 2 - 2b_1 + b_2$.

- We now show the relation $q(S) = \frac{1}{2} b_1(S)$.
- In fact, we have $H^1(S, \mathbb{C}) = H^{0,1}(S) \oplus H^{1,0}(S)$. Therefore $b_1 = h^{0,1} + h^{1,0}$.
- On the other hand, $h^{0,1} = h^{1,0}$, we know that $q = h^{0,1} = \frac{1}{2} b_1$.

The Hodge diamond of a complex projective surface is in the form:

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

Propositon(III.20)

The integers q, p_g and P_n are birational invariants.

- Let $\phi : S' \dashrightarrow S$ be a birational map, corresponding to a morphism $f : S' \setminus F \to S$ with F finite.
- Let $\omega \in H^0(S, \Omega^1_S)$ be a 1-form. Then $f^*\omega$ defines a rational form on S', with poles lying in F.
- But the poles form a divisor, so $f^*\omega \in H^0(S, \Omega^1_S)$.
- We have a morphism $\phi^* : H^0(S, \Omega^1_S) \to H^0(S', \Omega^1_{S'})$.
- Since ϕ is birational, there is clearly an inverse of ϕ^*.
- The birational invariance of p_g and P_n is deduced similarly.

Theorem(Noether's formula)

$\chi(\mathcal{O}_S) = \frac{1}{12}(\chi_{top}(S) + K_S^2)$

Proposition

Let S be a ruled surface over C. Then

$$q(S) = g(C), p_g(S) = 0, P_n(S) = 0, \forall n \geq 2.$$

If S is geometrically ruled, then

$$K_S^2 = 8(1 - g(C)), b_2(S) = 2$$

.

- First we calculate q, p_g and P_n. Since we are calculating the birational invariants, we may assume that $S = C \times \mathbb{P}^1$.

- $H^0(S, \Omega^1_S) = H^0(C \times \mathbb{P}^1, \Omega^1_{C\times\mathbb{P}^1}) = H^0(C \times \mathbb{P}^1, p_1^*\omega_C \oplus p_2^*\omega_{\mathbb{P}^1}) = H^0(C, \omega_C) \oplus H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1})$

- $q(S) = g(C) + g(\mathbb{P}^1) = g(C)$.

- $\omega_S = \det \Omega^1_S = \det(p_1^*\omega_C \oplus p_2^*\omega_{\mathbb{P}^1}) \cong p_1^*\omega_C \otimes p_2^*\omega_{\mathbb{P}^1}$

- $H^0(S, \omega_S^{\otimes n}) \cong H^0(C \times \mathbb{P}^1, (p_1^*\omega_C)^{\otimes n} \otimes (p_2^*\omega_{\mathbb{P}^1})^{\otimes n}) \cong H^0(C \times \mathbb{P}^1, (p_1^*\omega_C^{\otimes n}) \otimes (p_2^*\omega_{\mathbb{P}^1}^{\otimes n}))$

- $P_n(S) = h^0(C, \omega_C^{\otimes n}) h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes n}) = 0$ for $n \geq 1$.

- Now assume that $S = \mathbb{P}_C(E)$ is geometrically ruled. We are going to show $K_S^2 = 8(1 - g(C))$ and $b_2(S) = 2$.
- Recall that $[K_S] = -2h + (\deg(E) + 2g(C) - 2)f$ in $H^2(S, \mathbb{Z})$ and that $K_S^2 = [K_S]^2$.
- $K_S^2 = 4h^2 - 4(\deg(E) + 2g(C) - 2)$

  .
- Recall that $h^2 = \deg(E)$. Therefore $K_S^2 = 8(1 - g(C))$.

- To calculate b_2, we substitute $\chi(\mathcal{O}_{\mathcal{S}})$ by $1 - q(S) + p_g(S)$ and $\chi_{top}(S)$ by $2 - 2b_1(S) + b_2(S)$ in the Noether's formula.

- $1 - q(S) + p_g(S) = \frac{1}{12}(2 - 2b_1(S) + b_2(S) + 8 - 8g(C))$ and recall that $q(S) = \frac{1}{2}b_1(S)$.

- $b_2(S) = 2$.

The Hodge diamond of a geometrically ruled surface is in the form:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & g(C) & & g(C) & \\
0 & & 2 & & 0 \\
 & g(C) & & g(C) & \\
 & & 1 & &
\end{array}
$$

Warning!

$h^{1,1}$ is not a birational invariant. For example a blow-up increases $h^{1,1}$ by 1.

Example

Calculate the invariants of a hypersurface in $\mathbb{P}^3$.

- Assume that $S \subset \mathbb{P}^3$ is a hypersurface of degree d. There is an exact sequence of sheaves over $\mathbb{P}^3$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0.$$

- Taking cohomology, we have the long exact sequence:

$$\cdots \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(S, \mathcal{O}_S) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-d)) \to \cdots$$

- $q(S) = h^1(S, \mathcal{O}_S) = 0$

- By adjunction formula, we have $K_S = (K_{\mathbb{P}^3} + S)|_S$.
- Therefore $\omega_S = \mathcal{O}_S(d-4)$.
- Twisting the short exact sequence defining $\mathcal{O}_S$ by $n(d-4)$, we get

$$0 \to \mathcal{O}_{\mathbb{P}^3}(nd-4n-d) \to \mathcal{O}_{\mathbb{P}^3}(nd-4n) \to \omega_S^{\otimes n} \to 0.$$

- Since $h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) = 0$ for all integers m, we have

$$\begin{aligned} P_n(S) &= h^0(\mathcal{O}_{\mathbb{P}^3}(nd-4n)) - h^0(\mathcal{O}_{\mathbb{P}^3}(nd-4n-d)) \\ &= \binom{nd-4n+3}{3} - \binom{nd-d-4n+3}{3}. \end{aligned}$$

- In particular, for $n=1$, we have $p_g = \binom{d-1}{3}$.

- We now calculate K_S^2 and $h^{1,1}$.
- To calculate K_S^2, we use the formula $L.M = \chi(\mathcal{O}_S) - \chi(L) - \chi(M) + \chi(L \otimes M)$.
- $K_S^2 = \chi(\mathcal{O}_S) - 2\chi(K_S) + \chi(2K_S)$.
- $\chi(\mathcal{O}_S) = 1 - q + p_g = 1 + \binom{d-1}{3}$.
- By Serre's duality, $\chi(K_S) = 1 + \binom{d-1}{3}$

- To calculate $\chi(2K_S)$, consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(d-8) \to \mathcal{O}_{\mathbb{P}^3}(2d-8) \to \mathcal{O}_S(2d-8) = \omega_S^{\otimes 2} \to 0.$$

- Since $h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) = 0$,

$$\begin{aligned} h^0(S, 2K_S) &= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2d-8)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-8)) \\ &= \binom{2d-5}{3} - \binom{d-5}{3}. \end{aligned}$$

- $h^1(S, 2K_S) = 0$ since $h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) = 0$ and $h^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) = 0$.

- We are left to calculate $h^2(S, 2K_S)$. It is equivalent to calculate $h^0(S, -K_S)$.
- Consider

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2d+4) \to \mathcal{O}_{\mathbb{P}^3}(-d+4) \to \mathcal{O}_S(-d+4) = \omega_S^\vee \to 0.$$

- Similar to the previous discussion,

$$\begin{aligned} h^0(S, -K_S) &= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-d+4)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2d+4)) \\ &= \binom{7-d}{3} - \binom{7-2d}{3}. \end{aligned}$$

- $\chi(2K_S) = \binom{2d-5}{3} - \binom{d-5}{3} + \binom{7-d}{3} - \binom{7-2d}{3}$

- We conclude that

$$
\begin{aligned}
K_S^2 &= \chi(\mathcal{O}_S) - 2\chi(K_S) + \chi(2K_S) \\
&= -1 - \binom{d-1}{3} + \binom{2d-5}{3} - \binom{d-5}{3} \\
&\quad + \binom{7-d}{3} - \binom{7-2d}{3}
\end{aligned}
$$

- We are left to calculate $h^{1,1}$.

- By Noether's formula,

$$
\begin{aligned}
\chi_{top}(S) &= 12\chi(\mathcal{O}_S) - K_S^2 \\
&= 13 + 13\binom{d-1}{3} - \binom{2d-5}{3} + \binom{d-5}{3} \\
&\quad - \binom{7-d}{3} + \binom{7-2d}{3}
\end{aligned}
$$

- On the other hand,

$$\begin{aligned}\chi_{top}(S) &= 2 - 2b_1 + b_2 \\ &= 2 - 4q + 2p_g + h^{1,1} \\ &= 2 + 2\binom{d-1}{3} + h^{1,1}\end{aligned}$$

- $h^{1,1} = 11 + 11\binom{d-1}{3} - \binom{2d-5}{3} + \binom{d-5}{3} - \binom{7-d}{3} + \binom{7-2d}{3}$

Conclusively, a hypersurface of degree d in $\mathbb{P}^3$ has the Hodge diamond:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
\binom{d-1}{3} & & 11 + 11\binom{d-1}{3} - \binom{2d-5}{3} + \binom{d-5}{3} - \binom{7-d}{3} + \binom{7-2d}{3} & & \binom{d-1}{3} \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

$X \subset \mathbb{P}^3$ smooth projective surface of degree d:

- $d=1$: $X = \mathbb{P}^2$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$K_X^2 = 9$

- $d=2$: $X = \mathbb{P}^1 \times \mathbb{P}^1$ quadric

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$K_X^2 = 8$

- $d=3$: $X \subset \mathbb{P}^3$ cubic surface

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$K_X^2 = 3$

- $d=4$: $X \subset \mathbb{P}^3$ quartic surface

K3 surface

$1 \leq \rho \leq 20$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$K_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4)$

$K_X = (K_{\mathbb{P}^3} + X)\big|_X = \mathcal{O}_X$

$K_X^2 = 0$

References

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Thank you for listening!

X smooth projective surface over $\mathbb{C}$

$$\mathrm{Div}\, X \twoheadrightarrow \underbrace{\frac{\mathrm{Div}\, X}{\text{lin. eq.}}}_{= \mathrm{Pic}\, X} \twoheadrightarrow \underbrace{\frac{\mathrm{Div}\, X}{\text{alg. eq.}}}_{= NS\, X} \xrightarrow[\text{has finite kernel}]{} \underbrace{\frac{\mathrm{Div}\, X}{\text{num. eq.}}}_{= N^1 X \simeq \mathbb{Z}^\rho \text{ for some } \rho}$$

$$H^2(X,\mathbb{Z}) \xrightarrow[\text{has finite kernel}]{} H^2(X,\mathbb{C})$$

$$N^1 X \xrightarrow{\sim} \left(\text{image of } H^2(X,\mathbb{Z})\right) \cap H^{1,1}(X)$$

Picard rank

$$\boxed{\rho \leq h^{1,1}(X)}$$