Castelnuovo Theorem

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Goal: Introduce rational surfaces and a criteria for rational surfaces.

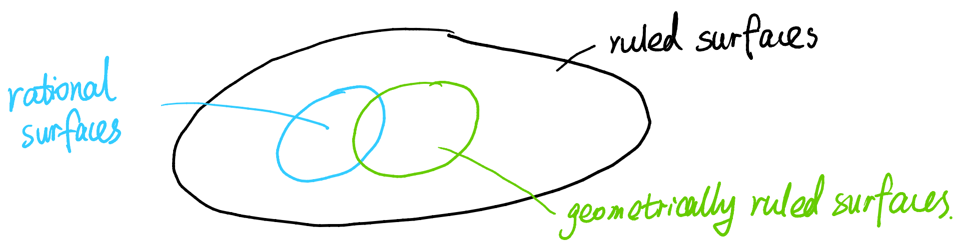
Def: Let V be a variety of dimension n.

- V is **rational** if there exists a birational map $\mathbb{P}^n_k \overset{\sim}{\dashrightarrow} V$.
  (i.e. $k(V) \cong k(x_1, \dots, x_n)$)
- V is **unirational** if there exists a dominant rational map $\mathbb{P}^n_k \dashrightarrow V$.
  (i.e. $k(V) \subseteq k(x_1, \dots, x_n)$)
- $n = 2 \rightsquigarrow$ rational surfaces and unirational surfaces.

eg: $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ rational surfaces; blow ups of them.

Remark: **rational** surfaces are **ruled** surfaces.

recall: A surface is **ruled** if it is birational equivalent to $C \times \mathbb{P}^1$ where C is a smooth curve.
If $C = \mathbb{P}^1$, then it is rational surface.



Castelnuovo Theorem: Let S be a smooth projective surface over $\mathbb{C}$.
Then S is **rational** iff $q = P_2 = 0$

Lüroth: Every unirational curve is rational.
(true over a field of any characteristic, not necessarily algebraically closed)

Lüroth (Corollary of Castelnuovo Theorem): Every unirational surface is rational
(true in char $= 0$, false in positive characteristic).

pf: Let S be a unirational surface, then we have $\mathbb{P}^2 \overset{\phi}{\dashrightarrow} S$ dominant rational map.

By the theorem of "elimination of indetermunancy", there exists a surface R and a morphism $\eta: R \to \mathbb{P}^2$ which is a composite of a finite number of blow-ups, and a morphism $f: R \to S$ s.t the diagram commutes:

$$\begin{tikzcd} & R \arrow[dl, "\eta"'] \arrow[dr, "f"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi"'] & & S \end{tikzcd}$$

> Since R is birational to $\mathbb{P}^2$ it is rational. and $f = \phi \circ \eta$ is surjective on R, $q = P_2 = 0$.

Suppose

$q = h^1(S, \mathcal{O}_S) = h^0(S, \Omega_S) > 0$

$P_2 = h^0(S, K^{\otimes 2}) > 0$.

pullback of nonzero sections of Ω_S and $K^{\otimes 2}$ are nonzero sections of the corresponding bundles on R

$\rightsquigarrow$ on R, $q > 0$, $P_2 > 0$. $\lightning$.

So $q = P_2 = 0$ on S $\rightsquigarrow$ S is rational by Castelnuovo.

To prove Castelnuovo Theorem, we need two lemmas:

Lemma 1. Let S be a minimal surface. For all $a > 0$, there exists an effective divisor D on S s.t. $K.D \leq -a$ and $|K+D| = \phi$.

Lemma 2. Let S be a minimal surface with $q = P_2 = 0$. Then there exists a smooth rational curve C on S s.t. $C^2 \geq 0$.

III. Lemma 1 $\overset{\text{II}}{\Rightarrow}$ Lemma 2 $\overset{\text{I}}{\Rightarrow}$ Castelnuovo

I. Consider exact sequence: $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to N_{C/S} \to 0$

then we have long sequence: $0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(C)) \to H^0(C, \mathcal{O}_C(C)) \to \underbrace{H^1(S, \mathcal{O}_S)}_{}^{=0} \cdots$

- $q = \dim H^1(S, \mathcal{O}_S) = 0$
- $\deg \mathcal{O}_C(C) = C^2 \geq 0$

positive degree bundles on projective variety have non-zero sections. $\rightsquigarrow H^0(C, \mathcal{O}_C(C)) \neq \phi \rightsquigarrow H^0(S, \mathcal{O}_S(C)) \neq \phi$

$$\rightsquigarrow \quad h^0(S, \mathcal{O}_S(C)) = h^0(S, \mathcal{O}_S) + h^0(C, \mathcal{O}_C(C))$$

- $H^0(S, \mathcal{O}_S) \simeq K \rightsquigarrow h^0(S, \mathcal{O}_S) = 1.$
- Riemann-Roch for $\mathcal{O}_C(C)$ on C:

$$h^0(C, \mathcal{O}_C(C)) - h^1(C, \mathcal{O}_C(C)) = \deg(\mathcal{O}_C(C)) + 1 - g(C).$$

$$= 1 \quad + C^2 - g(C) + 1 + h^1(C, \mathcal{O}_C(C))$$

$$= 2 + C^2 - g(C) + h^1(C, \mathcal{O}_C(C)).$$

$C \sim \mathbb{P}^1$, $\mathcal{O}_C(C)$ has positive degree.

$\rightsquigarrow$ $g(C) = 0$, $h^1(C, \mathcal{O}_C(C)) = 0$.

$$\geq 2$$

$\rightsquigarrow \dim H^0(S, \mathcal{O}_S(C)) \geq 2$

then we take two sections $s_0, s_1 \in H^0(S, \mathcal{O}_S(C))$

$$\begin{array}{ccc} S & \overset{\phi}{\dashrightarrow} & \mathbb{P}^1 \\ x & \longmapsto & [s_0(x) : s_1(x)] \end{array}$$

defined on $S \setminus V(s_0, s_1)$.

$$\rightsquigarrow \quad \begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\eta"', "\text{blow-ups}"] & \mathbb{P}^1 \\ S \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

$\phi^{-1}([0,1]) = V(s_0)$

$f^{-1}([0,1]) \cong V(s_0) \simeq \mathbb{P}^1.$

So f is a morphism with one fibre iso- to $\mathbb{P}^1$.

By Noether – Enriques theorem, R is rational $\rightsquigarrow S$ rational.

Recall: Let S be a surface, p a morphism from S to a smooth curve C. Suppose there exists $x \in C$ such that p is smooth over x and the fibre $p^{-1}(x)$ is isomorphic to $\mathbb{P}^1$. Then S is ruled.

II: **Lemma 1** Let S be a minimal surface
For all $a>0$, there exists an effective divisor D on S
s.t. $K.D \leq -a$ and $|K+D| = \phi$.

$\Downarrow$

Lemma 2 Let S be a minimal surface with $q = P_2 = 0$.
Then there exists a smooth rational curve C on S s.t. $C^2 \geq 0$.

Pf: Assume Lemma 1,
then some component C of D satisfies $C.K<0$, $|C+K| = \phi$.

Apply Riemann-Roch to $K+C$

$$\underbrace{h^0(K+C)}_{=0 \text{ since } |C+K|=\phi} - h^1(K+C) + h^0(-C) = \tfrac{1}{2}(K+C).(K+C-K) + 1 + p_a^X$$

$$= \tfrac{1}{2}(K+C).(C) + \underbrace{h^0(\mathcal{O}_X)}_{=1} - \underbrace{h^1(\mathcal{O}_X)}_{q=0} + h^2(\mathcal{O}_X)$$

$$\rightsquigarrow \quad 0 = h^0(K+C) \geq 1 + \tfrac{1}{2}(C^2 + C.K) = g(C)$$

$$\rightsquigarrow \quad g(C) = 0 \quad \rightsquigarrow C \simeq \mathbb{P}^1, \text{ rational curve.}$$

$C^2 \geq -1$ if $C^2 = -1$, C exceptional divisor
S minimal ↯

$$\rightsquigarrow \quad C^2 \geq 0$$

III. **Lemma 1** there exist an effective divisor D on S such that
$K.D<0$, $|K+D| = \phi$.

① $K^2=0$. let $D=-K$, Riemann-Roch:

$$h^0(-K) - h^1(-K) + \underbrace{l(2K)}_{P_2=0} = \tfrac{1}{2}(-K)(-2K) + 1 + p_a^S$$

$$= K^2 + 1 + \underbrace{h^0(\mathcal{O}_X)}_{=1} - \underbrace{h^1(\mathcal{O}_X)}_{q=0} + h^0(K) - 1.$$

$$\rightsquigarrow \quad h^0(-K) \geq 1 + K^2$$

So $|-K| \neq \phi$, then we can choose $0 \neq D \in |-K|$,
and we have $0 < D.H = -K.H \rightsquigarrow K.H < 0$

$\rightsquigarrow$ $(H+nK).H = a + nK.H$ for some $a>0$, then there exist $n \geq 0$ s.t.

$(H+nK).H \geq 0$, $(H+(n+1)K).H < 0$.

recall (lemma): Let S be a surface, D effective divisor on S.

C an irreducible curve on S w/ $C^2 \geq 0$.

then $D.C \geq 0$.

(If $D.C<0$, $\rightsquigarrow$ $|D| = \phi$.).

$\rightsquigarrow$ $|H+nK| \neq \phi$, $|H+(n+1)K| = \phi$.

let $D \in |H+nK|$. then $|K+D| = \phi$. $K.D = K.H < 0$.

2) $K^2 > 0$.

$h^0(-K) \geq 1+K^2 \geq 2$ $\quad |-K| \neq \phi$.

1. Suppose there exists a reducible divisor $D \in |-K|$.

$D = A+B$.

$D.K < 0$ $\rightsquigarrow$ $A.K < 0$ (or $B.K<0$).

$|K+A| = |-B| = \phi$.

2. Suppose $D \in |-K|$ irreducible

Let H be an effective divisor. since $|-K| \neq \phi$, $D.(-K) \geq 0$

and there exists $n>0$ such that $|H+nK| \neq 0$, $|H+(n+1)K| = \phi$.

$$D.(H+nK) = D.H + nD.K. = -HK - n\underbrace{K^2}_{K^2>0}.$$

$\exists n$ s.t. $D.(H+nK) \geq 0$

$D.(H+(n+1)K) < 0$.

$\rightsquigarrow$ $|H+nK| \neq \phi$

$|H+(n+1)K| = \phi$

Then we consider the two cases seperately in 2:

(i) $H+nK \not\equiv 0$ i.e. $H \not\sim nK$ for some $n \in \mathbb{Z}$

let $E \in |H+nK|$, $E = \sum n_i C_i$, $\rightsquigarrow$ $K.E = -D.E \leq 0$

$D \in |-K| \rightsquigarrow D^2 \geq 0$, D irreducible $\Big\} \Rightarrow D.E \geq 0$ (due to lemma)

$\rightsquigarrow K.C_i \leq 0$ for some i

Let $C = C_i$. then $|K+C| = \phi$, $g(C) = 0$, $C^2 = -2 - K.C$.

If $K.C \leq -2 \rightsquigarrow C^2 \geq 0$. ✓

If $K.C = -1$, $C^2 = -1$. exceptional curve ×.

If $K.C = 0$, $C^2 = -2$.

$h^0(2K+C) \leq h^0(K+C) = 0$

$\overset{\Downarrow}{(f)}$, $(f)+2K+C \geq 0$, $(f)+K+C \geq -K \geq 0$

Riemann-Roch for $D = -K-C$.

$h^0(-K-C) + h^0(2K+C) \geq 1 + \frac{1}{2}(-K-C)(-K-C-K) + \underbrace{\chi(\mathcal{O}_X)}_{1+0+h^0(K)} - 1$.

$\rightsquigarrow h^0(-K-C) \geq 1 + \frac{1}{2}\left[(K+C)^2 + K.(K+C)\right]$

$= 1 + \frac{1}{2}(\underset{-2}{C^2} + 3\underset{0}{K.C} + 2K^2) \geq K^2 \geq 1$

$C^2 = -2$, $K^2 \geq 1$, so $C \not\equiv -K$, then there exists a non-zero effective divisor A s.t. $A + C \in |-K|$.

$C \not\equiv -K \rightsquigarrow -K-C \not\equiv 0$,

$h^0(-K-C) \geq 1$, $\rightsquigarrow \underset{\text{eff. div.}}{\exists A \neq 0}$ s.t. $A \sim -K-C$, $A+C \sim (-K)$.

$\rightsquigarrow |-K|$ contains reducible div. ↯.

(ii) $H = nK$, every effective divisor is a multiple of K.

$\rightsquigarrow \operatorname{Pic} S = \mathbb{Z}.[K]$.

$|-K| \neq \phi$

$$\underset{q=0}{\underbrace{H^1(S,\mathcal{O}_S)}} \to \operatorname{Pic}(S) \xrightarrow{\cong} H^2(S,\mathbb{Z}) \to \underset{\cong H^0(K)=0}{H^2(S,\mathcal{O}_S)}$$

$b_2 = \dim_{\mathbb{R}} H^2(S,\mathbb{R}) = 1$.

Intersection product: $H^2(S,\mathbb{Z}) \times \underbrace{H^2(S,\mathbb{Z})}_{\cong H_2(S,\mathbb{Z})} \longrightarrow H_0(X,\mathbb{Z}) \simeq \mathbb{Z}$

$\exists [X] \in H_4(S,\mathbb{Z})$, $(\alpha, \beta) \longmapsto (\alpha \cup \beta) \frown [X]$

$H^2(S,\mathbb{Z}) \simeq \mathbb{Z}$ have zero torsion $\rightsquigarrow$ the intersetion product is unimodular.

$([K], [K]) \longmapsto \pm 1$

$\rightsquigarrow K^2 = 1$.

Noether's formula:

$$\chi(\mathcal{O}_S) = \tfrac{1}{12}\left(K^2 + \chi_{top}(S)\right) = \tfrac{1}{12}(K^2 + 2 - 2b_1 + b_2)$$

$$\chi(\mathcal{O}_S) = \underbrace{h^0(\mathcal{O}_S)}_{=1} - \underbrace{h^1(\mathcal{O}_S)}_{=q=0} + \underbrace{h^2(\mathcal{O}_S)}_{=h^0(K)}$$

$h^0(K) = 0$ since $h^0(-K) \geq 2$

$\Longrightarrow b_1 = -4$ $\qquad b_1 = \dim_{\mathbb{R}}(H^1(S,\mathbb{R}))$ ↯.

③ $K^2 < 0$

- It suffices to find an effective divisor E s.t. $K.E < 0$, then there is a component C of E s.t. $K.C < 0$

By genus formular of $C \subseteq S$, we have

$g(C) = h^1(C, \mathcal{O}_C) = 1 + \frac{1}{2}(C^2 + C.K) \geq 0$

$\rightsquigarrow C^2 \geq -1$.

S minimal $\rightsquigarrow C^2 \geq 0$.

$(aC+nK).C$ will eventually become negative as n grows.
so $|aC+(1+n)K| = \phi$ as n grows sufficiently large.
while $|aC+nK| \neq \phi$.

$D \in |aC+nK|$, we have $K.D \leq -a$ and $|K+D| = \phi$.

- Let H be a hyperplane section of S

① $K.H < 0$. take $E = H$.

② $K.H = 0$. $|K+nH|$ is non-empty for n sufficiently large, we can take $E \in |K+nH|$

③ $K.H > 0$

$$r_0 = \frac{K.H}{(-K^2)} > 0$$

then $(H+r_0K)^2 = H^2 + \dfrac{(K.H)^2}{-K^2} > 0$

$$(H+r_0K).K = 0$$

So we can chose r rational number, $r > r_0$ and sufficiently close to r_0, we have

$$(H+rK)^2 > 0; \quad (H+rK).K < 0; \quad (H+rK).H > 0.$$

If $r = \frac{p}{q}$, $p, q > 0$. let $D_m = mq(H+rK)$.

then $D_m^2 > 0$, $D_m \cdot K < 0$

And $(K-D_m).H = (K-mq(H+rK)).H$ become negative for m large enough, so $h^0(K-D_m) \to 0$

And Riemann Roch on (D_m).

$$h^0(D_m) + h^0(K-D_m) \geq \frac{1}{2} D_m(D_m-K) + \chi(\mathcal{O}_X).$$

$\to \infty$ as $m \to \infty$.

So $h^0(D_m) \neq 0$ for large m.
take $E = |D_m|$.