Today: every surface is projective, smooth / $\mathbb{C}$.

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among smooth projective curves:

- birational map = isomorphism
- rational map = morphism

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among surfaces:

- proper birational morphism $S \to S'$ = finite composition of blowups $S \to S_1 \to \dots \to S_n = S'$

- birational map $S \dashrightarrow S'$ = finite composition of blowups and blowdowns

$$\begin{tikzcd} & S'' \arrow[dl, "\text{proper birational}"'] \arrow[dr, "\text{proper birational}"] & \\ S \arrow[rr, dashed] & & S' \end{tikzcd}$$

- birational map $\Leftarrow$ isomorphism
  birational map $\not\Rightarrow$ isomorphism

Def A surface S is MINIMAL if
$\forall S'$ surface, $S \xrightarrow{f} S'$ proper birational $\Rightarrow$ f isomorphism

Castelnuovo thm:
S is minimal $\iff$ $\nexists\, C$ (-1)-curve in S
$(C \simeq \mathbb{P}^1,\ C^2 = -1)$

Theorem (Existence of minimal models)
S surface $\Rightarrow$ $\exists\, S_{min}$ s.t. · S_{min} minimal surface
· $S \to S_{min}$ proper birational

Idea of the proof If S doesn't contain a (-1)-curve, then S is minimal, $S_{min} = S$. So we are done.
If S contains a (-1)-curve, let's contract it: $S \longrightarrow S'$
rank Num S > rank Num S'. $\square$

S surface: $q(S) = h^1(\mathcal{O}_S) = h^0(\Omega^1_S)$ irregularity

$p_g(S) = h^2(\mathcal{O}_S) = h^0(\omega_S)$ geometric genus

$P_m(S) = h^0(\omega_S^{\otimes m})$ $\forall m \geq 1$ plurigenera. $P_1 = p_g$.

Prop S, S' minimal, $\exists m \geq 1 : P_m(S) \neq 0$ $(\kappa(S) \neq -\infty)$

$\Rightarrow$ Every rational map $S \dashrightarrow S'$ is an isomorphism.

Cor (Uniqueness of minimal models)

S surface s.t. $\exists m \geq 1 : P_m(S) \neq 0$. $(\kappa(S) \neq -\infty)$

$\Rightarrow \exists!$ S_{min} minimal surface s.t. $S \to S_{min}$

unique

proper birational

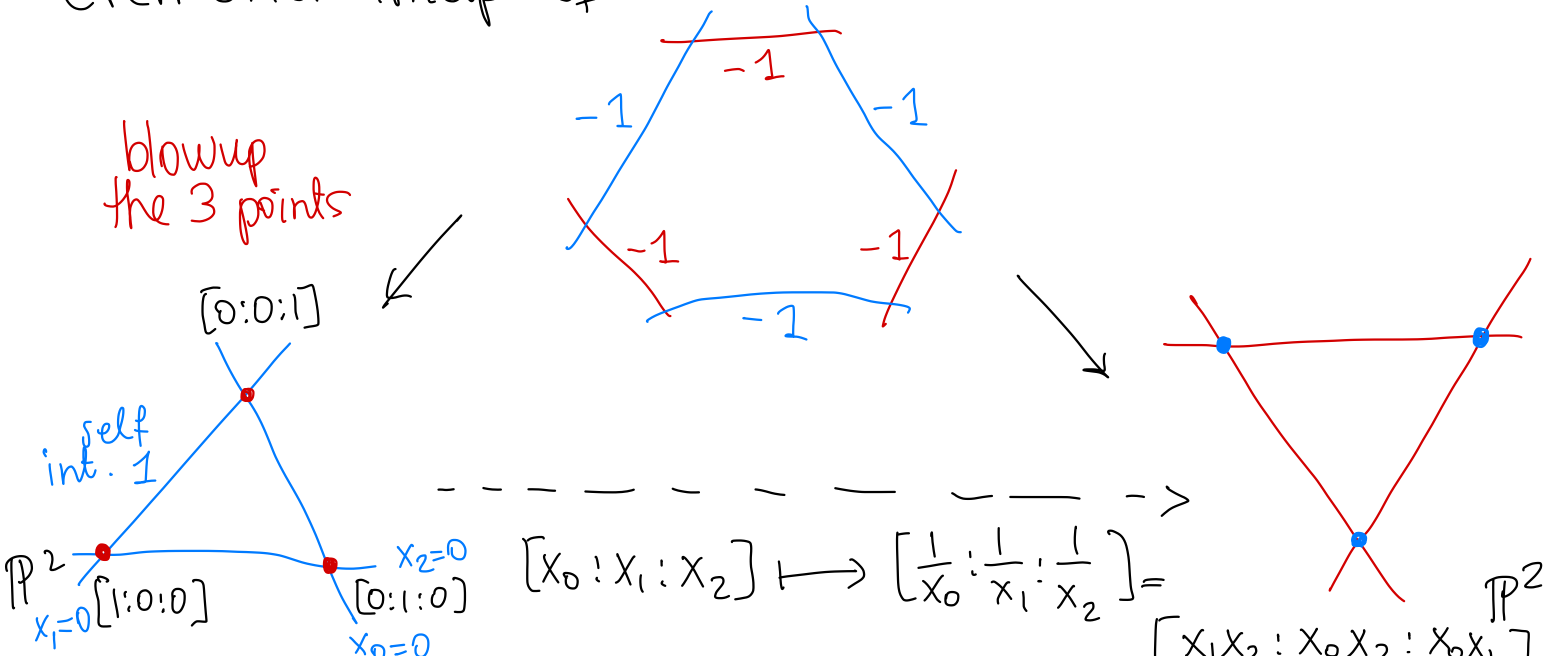
Is the assumption on P_m necessary?

Birational maps between rational surfaces are often non-isomorphisms:

- $\mathbb{P}^2$, $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_n$ for $n \geq 2$ are rational and minimal

(Recall that $\mathbb{F}_1$ contains a (-1)-curve)

- Cremona map of $\mathbb{P}^2$

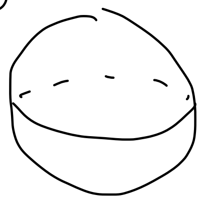


$$[x_0 : x_1 : x_2] \longmapsto \left[\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right] = [x_1x_2 : x_0x_2 : x_0x_1]$$

X smooth projective curve over $\mathbb{C}$.

GENUS of X: $g = h^1(\mathcal{O}_X) = h^0(\omega_X) = \frac{1}{2}\dim_{\mathbb{Q}} H^1(X, \mathbb{Q})$

$g=0$



Riemann sphere : $\mathbb{P}^1$

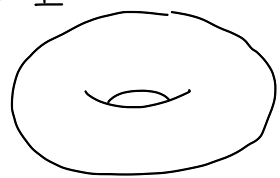
positive curvature

unique complex str.

$\deg \omega_X < 0$

$\omega_X^\vee$ ample

$g=1$



1-dim complex tori

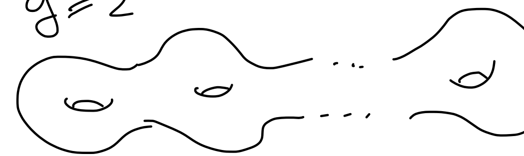
flat curvature
zero

reasonably few complex structure

$\deg \omega_X = 0$

$\omega_X \simeq \mathcal{O}_X$

$g \geq 2$



hyperbolic

negative curvature

many complex structures

$\deg \omega_X > 0$

ω_X ample

$\deg \omega_X = 2g-2$

X projective variety.

Plurigenera: $P_m(X) = h^0(\omega_X^{\otimes m})$ for $m \geq 1$.

The KODAIRA DIMENSION of X measures how $P_m(X)$ grows for $m \to +\infty$:

$$\kappa(X) := \begin{cases} -\infty & \text{if } \forall m \geq 1,\ P_m(X) = 0 \\ \min\left\{ k \in \mathbb{Z} \;\middle|\; \left(\dfrac{P_m(X)}{m^k}\right)_{k \geq 1} \text{ is bounded} \right\} & \text{othw.} \end{cases}$$

$$\kappa(X) \in \{-\infty, 0, 1, \dots, \dim X\}$$

<u>Def</u> X is OF GENERAL TYPE if $\kappa(X) = \dim X$

C smooth projective curve of genus g.

- $g=0$: $C \simeq \mathbb{P}^1$, $\omega_C = \mathcal{O}_{\mathbb{P}^1}(-2)$

  $P_m(C) = h^0(\omega_C^{\otimes m}) = h^0(\mathcal{O}_{\mathbb{P}^1}(-2m)) = 0 \quad \forall m \geq 1$

  $\kappa(C) = -\infty$

- $g=1$: C elliptic curve, $\omega_C \simeq \mathcal{O}_C$

  $P_m(C) = h^0(\omega_C^{\otimes m}) = h^0(\mathcal{O}_C) = 1 \quad \forall m \geq 1$

  $\kappa(C) = 0$.

- $g \geq 2$: $\deg \omega_C = 2g-2 > 0$

  $h^1(\omega_C^{\otimes m}) \underset{\substack{\text{serre} \\ \text{duality}}}{=} h^0(\omega_C^{\otimes(-m)} \otimes \omega_C) = h^0(\omega_C^{\otimes(1-m)}) = 0$ if $m \geq 2$

  $\deg L < 0 \Rightarrow h^0(L) = 0$

for $m \geq 2$ $\quad h^0(\omega_C^{\otimes m}) = \chi(\omega_C^{\otimes m}) \overset{R.R.}{=} \deg \omega_C^{\otimes m} + 1 - g = (2m-1)(g-1)$

$\kappa(C) = 1$.

Example $X \subset \mathbb{P}^n$ hypersurface of degree d. $n \geq 2$.

$\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$

adjunction $\Longrightarrow$ $\omega_X = \mathcal{O}_X(d-n-1)$

$\omega_X^{\otimes m} = \mathcal{O}_X(m(d-n-1))$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d+m(d-n-1)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(m(d-n-1)) \longrightarrow \omega_X^{\otimes m} \longrightarrow 0$$

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^n}(-d+m(d-n-1))) \to H^0(\mathcal{O}_{\mathbb{P}^n}(m(d-n-1))) \longrightarrow H^0(\omega_X^{\otimes m}) \to 0$$

- $1 \leq d \leq n \Longrightarrow \omega_X^\vee$ ample, $\kappa(X) = -\infty$

  X is Fano

- $d = n+1 \Longrightarrow \omega_X \simeq \mathcal{O}_X$, $\kappa(X) = 0$
- $d \geq n+2 \Longrightarrow \omega_X$ ample, $\kappa(X) = \dim X$

  X general type

- $\mathbb{P}^n$ $\quad \omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$

$$\omega_{\mathbb{P}^n}^{\otimes m} = \mathcal{O}_{\mathbb{P}^n}(-m(n+1)) \quad \text{negative}$$

$$h^0(\omega_{\mathbb{P}^n}^{\otimes m}) = 0 \quad \forall m \geq 1$$

$$\Rightarrow \kappa(\mathbb{P}^n) = -\infty.$$

X rational variety $\Longrightarrow \kappa(X) = -\infty.$

$$\not\Longleftarrow$$

related to
Lüroth problem.

Open question $X \subset \mathbb{P}^5$ cubic hypersurface defined by a general polynomial. Is X rational? $\quad (\kappa(X) = -\infty)$

<u>Theorem</u> (Enriques' classification) S minimal surface / $\mathbb{C}$

Then exactly one of the following holds:

- $S = \mathbb{P}^2$, $\kappa(S) = -\infty$
- S geometrically ruled, $\kappa = -\infty$, $S \longrightarrow C$ with fibres $\mathbb{P}^1$
- (K3 surface) $\kappa = 0$, $p_g = 1$, $q = 0$, $\omega_S \simeq \mathcal{O}_S$
- (Enriques) $\kappa = 0$, $p_g = 0$, $q = 0$, $\omega_S^{\otimes 2} \simeq \mathcal{O}_S$ (gives a torsion class in $\mathrm{Pic}(S)$)
- (abelian) $S = \mathbb{C}^2/\Lambda$, $\kappa = 0$, $p_g = 1$, $q = 2$, $\omega_S \simeq \mathcal{O}_S$
- (bielliptic) $\kappa = 0$, $p_g = 0$, $q = 1$
- (elliptic) $\kappa = 1$, $K_S^2 = 0$, $S \longrightarrow B$ elliptic fibration
- (general type) $\kappa = 2$: too many, still mysterious.

What happens in higher dimension?

In dim 3: $\exists$ examples $X \overset{\phi}{\dashrightarrow} X'$
birational maps

s.t. ϕ are <u>not</u> compositions of blowups
and blowdowns

small contraction.

$f: X \to X'$ birational proper which are iso in codim 2.

$\dim X = \dim X' = 3$

$Exc(f) = \{x \in X \mid f$ is not an iso at $x\}$ might be a curve.

Atiyah flop.

X proj. variety. $L \in \mathrm{Pic}(X)$.

L is called NEF (numerically effective).

if $\forall C$ smooth proj. curve, $\forall f : C \to X$

$\deg_C f^* L \geq 0$.

Prop S (sm. proj. surface) $\kappa(S) \neq -\infty$.

S is minimal $\iff$ K_S nef

Pf $\Leftarrow$) E (-1)-curve. $E^2 = -1$, $p_a(E) = 0$ $\xRightarrow{\text{adjunction}}$ $K_S \cdot E = -1$

$i : E \hookrightarrow S$ $\quad K_S \cdot E = \deg_E i^* \omega_S$ contradicts that K_S nef.

$\Rightarrow$) exercise. $\square$

<u>Def</u> (Mori) A projective variety X is a MINIMAL MODEL if K_X is nef.

<u>Theorem / Conjecture</u> (Mori's program / $\boxed{\text{MMP}}$ minimal model program)

X projective variety. Then:

- $\kappa(X) = -\infty \implies X$ is birational to X' and $X' \to T$ with fibres Fano varieties of dim > 0. T proj. variety, $0 \leq \dim T < \dim X$
- $\kappa(X) \geq 0 \implies X$ is birational to a minimal model

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- dim 3 : Mori
- arbitrary dim, $\kappa(X) = \dim X$: BCHM Birkar - Cascini - Hacon - McKernan