# SEMINAR ON ALGEBRAIC SURFACES

Freie Universität Berlin, Institut für Mathematik
Summer semester 2021

Andrea Petracci
andrea [dot] petracci [at] fu-berlin [dot] de
`https://userpage.fu-berlin.de/petracci/2021ForschungsmodulAlgebra/`

Updated on 8th June 2021

## Summary

**12th April.** (*Andrea Petracci*) Relation between quasi-projective varieties and schemes of finite type [Har77, Proposition II.2.6]. Definition of complex manifold [Huy05, §2.1]. Definition of tangent space and of smoothness of an algebraic variety [Har77, I.5]. The implicit function theorem implies that a smooth algebraic variety over $\mathbb{C}$, equipped with the analytic topology, is a complex manifold [Sha94b, VIII.3].

Definition of presheaf and of sheaf [Har77, II.1], [Voi02, 4.1]. Some basic examples: constant sheaf, continuous/smooth/holomorphic/regular functions. Homomorphism of sheaves, kernel and image. Short exact sequence of sheaves. The exponential sequence.

Sheaf cohomology (there are 3 approaches: derived functors [Har77, III.1-2], Čech cohomology [Har77, III.4], Godement resolution [God58])). Properties of sheaf cohomology: $\mathrm{H}^0$ and long exact sequence. If $X$ is a simplicial complex and $G$ is an abelian group, then the sheaf cohomology group $\mathrm{H}^i(X, \underline{G}_X)$ is isomorphic to the singular cohomology group $\mathrm{H}^i(X, G)$ [Voi02, Theorem 4.47]. If $X$ is a smooth manifold and $F$ is a sheaf of $C^\infty_X$-modules, then $F$ has no higher cohomology [Voi02, Proposition 4.36]. If $F_\bullet$ is an acyclic resolution of $G$, then $\mathrm{H}^i(X, G)$ is the $i$th cohomology group of $\Gamma(X, F_\bullet)$ [Voi02, Proposition 4.32].

The long exact sequence associated to the exponential sequence: interpretation of the connection homomorphism $\mathrm{H}^0(X, \mathcal{O}_X^*) \to \mathrm{H}^1(X, \mathbb{Z})$ in terms of the integration of the logarithmic derivative; the connection homomorphism $\mathrm{H}^1(X, \mathcal{O}_X^*) \to \mathrm{H}^2(X, \mathbb{Z})$ is called first Chern class. If $X$ is a smooth manifold, then the de Rham complex is an acyclic resolution of the constant sheaf $\underline{\mathbb{R}}_X$; de Rham theorem [Voi02, Corollary 4.37]. If $X$ is a complex manifold, then the Dolbeaut complex is an acyclic resolution of the sheaves of holomorphic forms [Voi02, Corollary 4.38].

**19th April.** (*Andrea Petracci*) (Quasi-)coherent sheaves [Har77, II.5] [Sha94b, VI.3]. Serre's theorem: a scheme is affine if and only if quasi-coherent sheaves do not have higher cohomology [Har77, Theorem III.3.7]. The sheaf cohomology groups of coherent sheaves on projective schemes over a field are finite dimensional vector spaces [Har77, Theorem III.5.2(a)]. Definition of Euler characteristic [Har77, Exercise III.5.1].

Definition of locally free sheaf and of invertible sheaf [Har77, II.5.1]. Cohomology of $\mathcal{O}(d)$ on $\mathbb{P}^n$ [Har77, Theorem III.5.1]. The short exact sequence given by a closed subscheme [Har77, Proposition II.5.9] (example of affine schemes, example of a hypersurface in $\mathbb{P}^n$).

Hom-sheaves and tensor product of quasi-coherent sheaves [Har77, p.109]. Picard group of a scheme [Har77, Proposition II.6.12].

Definition of discrete valuation ring [AM69, §9]. Prime divisor, discrete valuation associated to a prime divisor, Weil divisors, principal divisors, linear equivalence of divisors, divisor class group [Har77, p.129-131] [Sha94a, III.1.1]. Some examples on $\mathbb{P}^2$.

**26th April.** (*Andrea Petracci*) Effective divisors. The sheaf associated to a divisor on a smooth variety. Isomorphism between divisor class group and Picard group of a smooth variety [Har77, Corollary II.6.16]. Picard group of $\mathbb{P}^n$ [Har77, Proposition II.6.4] [Har77, Corollary II.6.16].

If $D$ is an effective divisor on $X$, then on the support of $D$ there is a scheme structure and $\mathcal{O}_X(-D)$ is the ideal sheaf of $D$ in $X$ [Har77, Proposition II.6.18].

The degree homomorphism from the Picard group of a curve to $\mathbb{Z}$ [Har77, Corollary II.6.10] [Sha94a, III.2.1].

Correspondence between effective divisors $D$ such that $\mathcal{O}_X(D) \simeq L$ and non-zero global sections of $L$ [Har77, Proposition II.7.7] [Sha94a, III.1.5]. Linear systems [Har77, p.157]. The morphism induced by a linear system.

Definition of base point free, very ample [Har77, Definition p.120] and ample [Har77, Theorem II.7.6]. Veronese embedding and Segre embedding.

Relation between locally free sheaves and vector bundles [Har77, Exercise II.5.18]. Tangent bundle, cotangent bundle and canonical bundle on a smooth variety [Har77, Chapter II.8] [Sha94b, III.5].

Normal bundle of a smooth divisor in a smooth variety [Har77, Definition p.182]. Conormal sequence [Har77, Theorem II.8.17]. The canonical divisor of a smooth divisor: adjuntion formula [Har77, Proposition II.8.20].

Serre duality [Har77, Chapter III.7].

**3rd May.** (*Kristoffer Rank Rasmussen*) Intersection product on the Picard group of a surface [Har77, Theorem V.1.1].
Formula for the intersection product of two curves having no common irreducilble component [Har77, Proposition V.1.4].
Self-intersection of a curve [Har77, Example V.1.4.1].
The intersection product on $\mathbb{P}^2$ [Har77, Example V.1.4.2] and on $\mathbb{P}^1 \times \mathbb{P}^1$ [Har77, Example 1.4.3].
(If time permits, also [Har77, Exercise V.1.1]; the solution is the proof of [Bea96, Theorem I.4])

**10th May.** (*Sofía Garzón*) Canonical divisor of $\mathbb{P}^n$.

Self-intersection of the canonical divisor of a surface [Har77, Example V.1.4.4]: example of $\mathbb{P}^2$, of $\mathbb{P}^1 \times \mathbb{P}^1$.

Riemann–Roch for curves [Har77, Theorem IV.1.3]. Degree of the canonical divisor on a curve.

Adjunction formula for the genus of a smooth curve on a surface [Har77, Proposition V.1.5]. Genus of the smooth plane curves. Genus of curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Cotangent bundle and canonical bundle of a product. Self-intersection of the canonical divisor on the product of two curves [Har77, Exercise V.1.5].

Self-intersection of the canonical divisor of hypersurfaces in $\mathbb{P}^3$ and of the intersection of two quadrics in $\mathbb{P}^4$.

Riemann–Roch for surfaces [Har77, Theorem V.1.6] (or [Bea96, Theorem I.12]).

**17th May.** (*Sagi Rotfogel*) [Har77, Lemma V.1.7], [Har77, Corollary V.1.8].
Definition of numerical equivalence.
Hodge index theorem [Har77, Theorem V.1.9].
Definition of $\mathrm{Num}(X)$ [Har77, Remark 1.9.1].
The Néron–Severi group and topological interpretation of the intersection product [Bea96, I.10]. (see also [Har77, Exercise V.1.7])

**31st May.** (*Andrea Petracci*) Blowup of a surface at a point [Har77, Proposition V.3.1]. Algebraic definition and analytic definiton [Bea96, II.1].
Picard group and intersection pairing of a blown up surface [Har77, Proposition V.3.2].
Canonical divisor of a blown up surface [Har77, Proposition V.3.3], [Bea96, II.3].
Pullback of a curve along a blow up [Har77, Proposition V.3.6] [Bea96, II.2]

**7th June.** (*Emilio Montes de Oca*) Rational and birational maps between surfaces [Bea96, II.4-16]. Definition of minimal surface

## Plan

(F) **14th June.** (*Matija Blagojevic*) Castelnuovo's contractibility criterion [Bea96, II.17].

(G) **21st June.** (*Paul Brommer-Wierig*) Definition of rational variety. Definition of ruled surfaces and of geometrically ruled surfaces. Noether–Enriques theorem. Every geometrically ruled surface is a $\mathbb{P}^1$-bundle. Minimal models of irrational ruled surfaces. [Bea96, III.1-10]

(H) **28th June.** (*Li Li*) Invariants of surfaces: Hodge diamond, $q$, $p_g$, $\chi$, $\chi_{\mathrm{top}}$ and their relations [Bea96, end of III] (you can also consult some book on complex manifolds for the definition of the Hodge numbers). Invariants of ruled surfaces. Invariants of hypersurfaces of $\mathbb{P}^3$.
Definition of Hirzebruch surfaces and their properties [Bea96, IV.1]

(I) **5th July.** (*Yumeng Li*) Lüroth's problem and Castelnuovo's theorem [Bea96, V.1-9].

## References

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Bea96] A. Beauville, *Complex algebraic surfaces*, Second, London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid.

[God58] R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualit'es Sci. Ind. No. 1252. Publ. Math. Univ. Strasbourg. No. 13, Hermann, Paris, 1958.

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Huy05] D. Huybrechts, *Complex geometry*, Universitext, Springer-Verlag, Berlin, 2005. An introduction.

[Sha94a] I. R. Shafarevich, *Basic algebraic geometry. 1*, Second, Springer-Verlag, Berlin, 1994. Varieties in Projective Space, Translated from the 1988 Russian edition by Miles Reid.

[Sha94b] ———, *Basic algebraic geometry. 2*, Second, Springer-Verlag, Berlin, 1994. Schemes and complex manifolds, Translated from the 1988 Russian edition by Miles Reid.

[Voi02] C. Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.