SOLUTIONS OF 1st CALL, 12/01/2026

Problem 1

$$(3-i)^{-1} = \frac{1}{3-i}\,\frac{3+i}{3+i} = \frac{3+i}{9+1} = \frac{3}{10} + i\,\frac{1}{10}$$

$$e^{2-i\frac{\pi}{2}} = e^2 e^{-i\frac{\pi}{2}} = e^2 \cdot (-i)$$

$$(3-i)^{-1} e^{2-i\frac{\pi}{2}} = \left(\frac{3}{10} + i\,\frac{1}{10}\right) e^2(-i) = \frac{-3e^2}{10}\, i + \frac{e^2}{10}$$

$$a = \frac{e^2}{10}\,, \quad b = \frac{-3e^2}{10}$$

Problem 2

$z^4 - 3i z^2 - 2 = 0$. Set $w = z^2$. Then

$$w^2 - 3iw - 2 = 0 \qquad \Delta = (-3i)^2 - 4\cdot(-2) = -9+8 = -1$$

$$w = \frac{3i \pm i}{2} = \begin{cases} \dfrac{4i}{2} = 2i = 2e^{i\frac{\pi}{2}} \\[2ex] \dfrac{2i}{2} = i = e^{i\frac{\pi}{2}} \end{cases}$$

$$z^2 = 2e^{i\frac{\pi}{2}}$$

$$z = \sqrt{2} \cdot e^{i\frac{\pi}{4}} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1+i$$

$$z = \sqrt{2} \cdot e^{i\left(\frac{\pi}{4}+\pi\right)} = -1-i$$

$$z^2 = e^{i\frac{\pi}{2}}$$

$$z = e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$z = e^{i\left(\frac{\pi}{4}+\pi\right)} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

Problem 3 (1) $\det A = 0$ because the 2nd column is zero

(2) $$P_A(t) = \det(A - tI) = \det\begin{pmatrix} -1-t & 0 & 0 \\ -81 & -t & 43 \\ 0 & 0 & -1-t \end{pmatrix} = (-t)\cdot\det\begin{pmatrix} -1-t & 0 \\ 0 & -1-t \end{pmatrix} =$$

Laplace wrt 2nd column

$$= (-t)(t+1)^2$$

(3) The eigenvalues of A are

0 with algebraic multiplicity 1

−1 2

(4) $$E_0 = \operatorname{Ker} A = \operatorname{Ker}\begin{pmatrix} -1 & 0 & 0 \\ -81 & 0 & 43 \\ 0 & 0 & -1 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$E_{-1} = \operatorname{Ker}(A + I) = \operatorname{Ker}\begin{pmatrix} 0 & 0 & 0 \\ -81 & 1 & 43 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -81 & 1 & 43 \end{pmatrix}$$

$$-81x + y + 43z = 0 \qquad y = 81x - 43z$$

A basis of E_{-1} is

$$\begin{pmatrix} 1 \\ 81 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -43 \\ 1 \end{pmatrix}$$

(5) $$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 81 & -43 \\ 0 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Problem 4

$$\begin{cases} a_0 = 5+i \\ a_1 = -5+2i \\ a_{n+2} = a_{n+1} + 2a_n \quad \forall n \geq 0 \end{cases}$$

(1) $t^2 = t+2$, $t^2 - t - 2 = 0$ The solutions are $2, -1$. Hence there exist $\gamma_1, \gamma_2 \in \mathbb{C}$ s.t.

$$a_n = \gamma_1 \cdot 2^n + \gamma_2 \cdot (-1)^n \quad \forall n \geq 0.$$

$$\begin{cases} 5+i = a_0 = \gamma_1 + \gamma_2 \\ -5+2i = a_1 = 2\gamma_1 - \gamma_2 \end{cases}$$

$3\gamma_1 = 5+i-5+2i = 3i \quad \gamma_1 = i$

$\gamma_2 = 5.$

$$\Rightarrow a_n = i \cdot 2^n + 5 \cdot (-1)^n \quad \forall n \geq 0$$

(2) $|a_n|$ goes as 2^n for $n \to +\infty$

$$\sqrt[n]{|a_n|} \longrightarrow 2 \quad \Rightarrow \quad R = \frac{1}{2}$$

(3) If $|z| < \frac{1}{2}$ then

$$\sum_{n\geq 0} a_n z^n = i \cdot \sum_{n\geq 0} (2z)^n + 5 \cdot \sum_{n\geq 0} (-z)^n = i \cdot \frac{1}{1-2z} + 5 \cdot \frac{1}{1+z} =$$

$$= \frac{i(1+z)+5(1-2z)}{(1-2z)(1+z)} = \frac{i+iz+5-10z}{1+z-2z-2z^2} = \frac{(i+5)+(i-10)z}{1-z-2z^2}$$

$$p(z) = i+5+(i-10)z$$

$$q(z) = 1-z-2z^2$$

(4) $$\sum_{n\geq 0} a_n \left(\frac{i}{3}\right)^n = \frac{p\left(\frac{i}{3}\right)}{q\left(\frac{i}{3}\right)}$$

$$p\left(\frac{i}{3}\right) = i+5+(i-10)\frac{i}{3} = i+5-\frac{1}{3}-\frac{10}{3}i = \frac{14}{3}-\frac{7}{3}i$$

$$q\left(\frac{i}{3}\right) = 1-\frac{i}{3}-2\cdot\left(\frac{i}{3}\right)^2 = 1-\frac{i}{3}-2\frac{(-1)}{9} = \frac{11}{9}-\frac{i}{3}$$

$$q\left(\frac{i}{3}\right)^{-1} = \frac{\frac{11}{9}+\frac{i}{3}}{\left(\frac{11}{9}\right)^2+\left(\frac{1}{3}\right)^2} = \frac{\frac{11}{9}+\frac{i}{3}}{\frac{121+9}{81}} = \frac{81\cdot\left(\frac{11}{9}+\frac{i}{3}\right)}{130} = \frac{99+27i}{130}$$

$$\sum_{n\geq 0} a_n\left(\frac{i}{3}\right)^{-1} = p\left(\frac{i}{3}\right)\cdot q\left(\frac{i}{3}\right)^{-1} = \left(\frac{14}{3}-\frac{7}{3}i\right)\left(\frac{99}{130}+\frac{27}{130}i\right) =$$

$$= \frac{7}{\cancel{3}}\cdot\frac{\cancel{9}^{\,3}}{130}(2-i)(11+3i) = \frac{21}{130}(22+6i-11i+3)$$

$$= \frac{21}{130}(25-5i) = \frac{21}{26}(5-i) = \frac{105}{26}-\frac{21}{26}i$$

$$\alpha = \frac{105}{26}, \quad \beta = -\frac{21}{26}$$

Problem 5

$$z^2-8z+15=(z-3)(z-5)$$

$$\frac{75-23z}{z^2-8z+15}=\frac{a}{z-3}+\frac{b}{z-5}=\frac{az-5a+bz-3b}{(z-3)(z-5)}$$

$$\begin{cases} a+b=-23 \\ -5a-3b=75 \end{cases} \Rightarrow a=-3,\ b=-20$$

$$\frac{2}{z(z-1)}=\frac{c}{z}+\frac{d}{z-1}=\frac{cz-c+dz}{z(z-1)}$$

$$\begin{cases} c+d=0 \\ -c=2 \end{cases} \qquad \begin{matrix} c=-2 \\ d=2 \end{matrix}$$

$\Rightarrow$

$$f(z)=i-\frac{2}{z}+\frac{2}{z-1}-\frac{3}{z-3}-\frac{20}{z-5}$$

If $1<|z|<2$ then

$$\frac{1}{z-1} = \frac{1}{z} \frac{1}{1-\frac{1}{z}} = \frac{1}{z} \cdot \sum_{m\geq 0} \left(\frac{1}{z}\right)^m = \sum_{m\geq 1} \left(\frac{1}{z}\right)^m = \sum_{m\geq 1} z^{-m}$$

$$\frac{1}{z-3} = -\frac{1}{3} \frac{1}{1-\frac{z}{3}} = -\frac{1}{3} \sum_{m\geq 0} \left(\frac{z}{3}\right)^m = \sum_{m\geq 0} \left(-\frac{1}{3^{m+1}}\right) z^m$$

$$\frac{1}{z-5} = \sum_{m\geq 0} \left(-\frac{1}{5^{m+1}}\right) z^m$$

If $f(z) = \sum_{n\in\mathbb{Z}} u_n z^{-n}$ then

$$n \geqslant 2 \Rightarrow u_n = 2$$

$$n = 1 \Rightarrow u_{-1} = -2+2 = 0$$

$$n = 0 \Rightarrow u_0 = i - 3\cdot\left(-\frac{1}{3}\right) - 20\cdot\left(-\frac{1}{5}\right) = i+1+4 = i+5$$

$$n \leqslant -1 \Rightarrow u_n = -3\cdot\left(-\frac{1}{3^{-n+1}}\right) - 20\cdot\left(-\frac{1}{5^{-n+1}}\right) =$$

$$= 3^n + 4\cdot 5^n$$