SOLUTIONS OF 2nd CALL, 28/01/2026

Problem 1

$$\frac{8}{3}\pi = 2\pi + \frac{2}{3}\pi$$

$$e^{\frac{8}{3}\pi i} = \cos\left(\frac{8}{3}\pi\right) + i\sin\left(\frac{8}{3}\pi\right) = \cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$e^{1+\frac{8\pi}{3}i} = e \cdot e^{\frac{8\pi}{3}i} = e\cdot\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{e}{2}(-1+\sqrt{3}i)$$

$$\frac{1}{i+\sqrt{3}} = \frac{1}{i+\sqrt{3}}\,\frac{-i+\sqrt{3}}{-i+\sqrt{3}} = \frac{-i+\sqrt{3}}{1+3} = \frac{-i+\sqrt{3}}{4}$$

$$\frac{e^{1+\frac{2\pi}{3}i}}{i+\sqrt{3}} = \frac{e}{2}(-1+\sqrt{3}i)\cdot\frac{-i+\sqrt{3}}{4} = \frac{e}{8}\left[i - \cancel{\sqrt{3}} + \cancel{\sqrt{3}} + 3i\right]$$

$$= \frac{e}{8}\cdot 4i = \frac{ei}{2} \implies a = 0,\ b = \frac{e}{2}$$

Problem 2 $z^6 - 7iz^3 + 8 = 0$. Set $w = z^3$. Then

$0 = w^2 - 7iw + 8 = (w-8i)(w+i) \Longrightarrow w = 8i$ or $w = -i$

i.e. $z^3 = 8i$ or $z^3 = -i$

$$z^3 = 8i = 2^3 \cdot e^{i\frac{\pi}{2}} \Longrightarrow z = 2e^{i\left(\frac{\pi}{6} + k\frac{2\pi}{3}\right)} \text{ for } k = 0,1,2$$

$z = 2e^{i\frac{\pi}{6}} = \sqrt{3} + i$

$z = 2e^{i\frac{5\pi}{6}} = -\sqrt{3} + i$

$z = 2e^{i\frac{3}{2}\pi} = -2i$

$$z^3 = -i = e^{i\frac{3\pi}{2}} \Longrightarrow z = e^{i\left(\frac{\pi}{2} + k\frac{2\pi}{3}\right)} \text{ for } k = 0,1,2$$

$z = e^{i\frac{\pi}{2}} = i$

$z = e^{i\frac{7}{6}\pi} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$z = e^{i\frac{11}{6}\pi} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

Problem 3

$$A = \begin{pmatrix} -8 & 13 & 20 \\ 0 & -1 & 0 \\ -4 & 7 & 10 \end{pmatrix}$$

a) Use Laplace w.r.t. 2nd row $\det A = -\det\begin{pmatrix} -8 & 20 \\ -4 & 10 \end{pmatrix} =$

$= -(-80+80) = 0$

b) $P_A(t) = \det(A - tI) = \det\begin{pmatrix} -8-t & 13 & 20 \\ 0 & -1-t & 0 \\ -4 & 7 & 10-t \end{pmatrix} =$

$= (-1-t)\cdot\det\begin{pmatrix} -8-t & 20 \\ -4 & 10-t \end{pmatrix} = -(t+1)\left[(t+8)(t-10)+80\right]$

$= -(t+1)(t^2-2t-\cancel{80}+\cancel{80}) = -(t+1)\,t\,(t-2)$

c) The eigenvalues are $2, 0, -1$

d) $E_2 = \operatorname{Ker}(A-2I) = \operatorname{Ker}\begin{pmatrix} -10 & 13 & 20 \\ 0 & -3 & 0 \\ -4 & 7 & 8 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix} =$

$= \operatorname{Span}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, or more slowly

$$\begin{cases} -10x + 13y + 20z = 0 \\ -3y = 0 \\ -4x + 7y + 8z = 0 \end{cases} \qquad \begin{cases} y = 0 \\ -10x + 20z = 0 \\ -4x + 8z = 0 \end{cases} \qquad \begin{cases} y = 0 \\ x - 2z = 0 \end{cases}$$

$$\begin{cases} y = 0 \\ x = 2z \end{cases} \quad \text{so} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ is a multiple of } \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$E_0 = \operatorname{Ker} A = \operatorname{Ker} \begin{pmatrix} -8 & 13 & 20 \\ 0 & -1 & 0 \\ -4 & 7 & 10 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} -4 & 7 & 10 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} -4 & 7 & 10 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \operatorname{Ker} \begin{pmatrix} -4 & 0 & 10 \\ 0 & 1 & 0 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} 2 & 0 & -5 \\ 0 & 1 & 0 \end{pmatrix} = \operatorname{Span} \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$$

$$E_{-1} = \operatorname{Ker}(A + I) = \operatorname{Ker} \begin{pmatrix} -7 & 13 & 20 \\ 0 & 0 & 0 \\ -4 & 7 & 11 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} -7 & 13 & 20 \\ -4 & 7 & 11 \end{pmatrix} =$$

$$= \operatorname{Ker} \begin{pmatrix} 4 & -7 & -11 \\ 7 & -13 & -20 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} 1 & -\frac{7}{4} & -\frac{11}{4} \\ 7 & -13 & -20 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} 1 & -\frac{7}{4} & -\frac{11}{4} \\ 0 & -\frac{3}{4} & -\frac{3}{4} \end{pmatrix} =$$

$$= \mathrm{Ker}\begin{pmatrix} 1 & -\frac{7}{4} & -\frac{11}{4} \\ 0 & 1 & 1 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

e) A is diagonalisable.

$$M = \begin{pmatrix} 2 & 5 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Problem 4

$$\begin{cases} a_0 = 4 \\ a_1 = 12 \\ a_{n+2} = 6a_{n+1} - 25a_n & \text{for } n \geq 0 \end{cases}$$

a) The characteristic equation is $\lambda^2 = 6\lambda - 25$, $\lambda^2 - 6\lambda + 25 = 0$

$(\lambda - 3)^2 + 16 = 0 \qquad \lambda - 3 = \pm 4i \qquad \lambda = 3 \pm 4i$

hence $a_n = \gamma_1 (3+4i)^n + \gamma_2 (3-4i)^n \quad \forall n \geq 0$ for some $\gamma_1, \gamma_2 \in \mathbb{C}$.

$$\begin{cases} 4 = a_0 = \gamma_1 + \gamma_2 \\ 12 = a_1 = \gamma_1 (3+4i) + \gamma_2 (3-4i) \end{cases}$$

$$\begin{cases} \gamma_2 = 4 - \gamma_1 \\ 12 = \gamma_1 (3+4i) + (4-\gamma_1)(3-4i) \\ \quad = \gamma_1 \cdot 8i + 12 - 16i \end{cases}$$

$$\begin{cases} \gamma_2 = 4 - \gamma_1 \\ 8i\,\gamma_1 - 16i = 0 \end{cases} \implies \gamma_1 = \gamma_2 = 2$$

$$\implies a_n = 2(3+4i)^n + 2(3-4i)^n \quad \forall n \geq 0.$$

b) $|3+4i| = |3-4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

$|a_n|$ goes as 5^n as $n \to +\infty$

$\Rightarrow \lim_{n\to+\infty} \sqrt[n]{|a_n|} = 5 \Rightarrow$ the radius of convergence of

$\sum_{n\geq 0} a_n z^n$ is $R = \frac{1}{5}$

c) if $|z| < \frac{1}{5}$ then $|(3+4i)z| = |(3-4i)z| = 5|z| < 1$

$$\sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} \left[2\cdot(3+4i)^n z^n + 2(3-4i)^n z^n\right] =$$

$$= 2\cdot \sum_{n\geq 0} ((3+4i)z)^n + 2\sum_{n\geq 0}((3-4i)z)^n =$$

$$= 2\,\frac{1}{1-(3+4i)z} + 2\,\frac{1}{1-(3-4i)z} =$$

$$= 2\left[\frac{1}{(1-3z)-4iz} + \frac{1}{(1-3z)+4iz}\right]$$

$$= 2\ \frac{1-3z+\cancel{4iz}+1-3z-\cancel{4iz}}{(1-3z)^2+16z^2} = 2\ \frac{2-6z}{1+9z^2-6z+16z^2}$$

$$= \frac{4-12z}{1-6z+25z^2}$$

$$p(z) = 4-12z$$

$$q(z) = 1-6z+25z^2$$

d) $$\sum_{n\geq 0} a_n\left(\frac{i}{6}\right)^n = \frac{p\left(\frac{i}{6}\right)}{q\left(\frac{i}{6}\right)} = \frac{4-12\cdot\frac{i}{6}}{1-6\cdot\frac{i}{6}+25\left(\frac{i}{6}\right)^2} = \frac{4-2i}{1-i-\frac{25}{36}}$$

$$= \frac{4-2i}{\frac{11}{36}-i} = 36\ \frac{4-2i}{11-36i}\ \frac{11+36i}{11+36i} =$$

$$= 36\,\frac{44 + 144i - 22i + 72}{11^2 + 36^2}$$

$$= 36\,\frac{116 + 122i}{11^2 + 36^2}$$

$$\alpha = \frac{36 \cdot 116}{11^2 + 36^2} = \frac{4176}{1417}$$

$$\beta = \frac{36 \cdot 122}{11^2 + 36^2} = \frac{4392}{1417}$$

Problem 5

$$f(z) = 5z^7 + \frac{1}{4z^2+1} + \frac{2}{z-5}$$

$$A = \{ z \in \mathbb{C} \mid 1 < |z| < 2 \}$$

- The inverse Z-transform of $5z^7$ on A is $\begin{cases} 5 & \text{if } n = -7 \\ 0 & \text{if } n \neq -7 \end{cases}$

- For $z \in A$ $\quad |z| > 1 > \frac{1}{2} \quad |4z^2| > 1 \quad \left| \frac{1}{4z^2} \right| < 1$

$$\frac{1}{1+4z^2} = \frac{1}{4z^2} \frac{1}{1 - \left(-\frac{1}{4z^2}\right)} = \frac{1}{4z^2} \sum_{m \geq 0} \left(-\frac{1}{4z^2}\right)^m =$$

$$= \sum_{m \geq 0} (-1)^m \frac{1}{4^{m+1}} z^{-2(m+1)}$$

So the inverse Z-transform of $\frac{1}{1+4z^2}$ on A is

$$\begin{cases} (-1)^m \dfrac{1}{4^{m+1}} & \text{if } n = 2m+2 \text{ if } m \geq 0 \\ 0 & \text{if } (n \geq 1, n \text{ odd}) \text{ or } n \leq 0 \end{cases}$$

- For $z \in A$ $|z| < 2 < 5$ hence $\left|\frac{z}{5}\right| < 1$

$$\frac{2}{z-5} = -\frac{2}{5} \frac{1}{1-\frac{z}{5}} = -\frac{2}{5} \sum_{m \geq 0} \left(\frac{z}{5}\right)^m = \sum_{m \geq 0} \frac{-2}{5^{m+1}} z^m$$

so the inverse z-transform of $\frac{2}{z-5}$ on A is

$$\begin{cases} -\dfrac{2}{5^{m+1}} & \text{if } n = -m, m \geq 0 \\ 0 & \text{if } n > 0 \end{cases}$$

By summing these results we get that the inverse Z-transform of f on A is

$$\begin{cases} (-1)^m \dfrac{1}{4^{m+1}} & n>0,\ n \text{ even},\ n=2m+2,\ m\geq 0 \\ 0 & n>0,\ n \text{ odd} \\ -\dfrac{2}{5^{m+1}} & n\leq 0,\ n\neq -7,\ n=-m,\ m\neq 7 \\ -\dfrac{2}{5^8}+5 & n=-7 \end{cases}$$