SOLUTIONS OF 3rd CALL 12/02/2026

Problem 1 $\frac{23\pi}{4} = \left(5 + \frac{3}{4}\right)\pi = 4\pi + \frac{7}{4}\pi = 2 \cdot 2\pi + \frac{7}{4}\pi$

$$e^{\frac{23\pi}{4}i} = e^{\frac{7}{4}\pi i} = \cos\left(\frac{7}{4}\pi\right) + i\sin\left(\frac{7}{4}\pi\right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$\frac{(e-ei)^2}{e^{2-\frac{23}{4}\pi i}} = \frac{\cancel{e^2}(1-i)^2}{\cancel{e^2}\cdot e^{-\frac{23}{4}\pi i}} = (1-i)^2 \cdot e^{\frac{23}{4}\pi i} = (1+i^2-2i)\cdot\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= -2i \cdot \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\sqrt{2}\,i\,(1-i) = -\sqrt{2}\,(i+1)$$

$$= -\sqrt{2} - \sqrt{2}i$$

$$\Longrightarrow \quad a = b = -\sqrt{2}$$

Problem 2

$$z^2 + 16z^{-2} = -4 \qquad z^4 + 16 = -4z^2 \qquad z^4 + 4z^2 + 16 = 0$$

$$z^4 + 4z^2 + 4 + 12 = 0 \qquad (z^2+2)^2 = -12$$

$$z^2 + 2 = \pm 2\sqrt{3}\, i \qquad z^2 = -2 \pm 2\sqrt{3}\, i$$

- $z^2 = -2 + 2\sqrt{3}\, i = 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 4e^{\frac{2}{3}\pi i}$

$$\begin{cases} z = 2e^{\frac{1}{3}\pi i} = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 1 + \sqrt{3}\, i \\ z = 2e^{\left(\frac{1}{3}\pi + \pi\right) i} = -1 - \sqrt{3}\, i \end{cases}$$

- $z^2 = -2 - 2\sqrt{3}\, i = 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 4e^{\frac{4}{3}\pi i}$

$$\begin{cases} z = 2e^{\frac{2}{3}\pi i} = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = -1 + \sqrt{3}\, i \\ z = 2e^{\left(\frac{2}{3}\pi + \pi\right) i} = 1 - \sqrt{3}\, i \end{cases}$$

Problem 3

$$A = \begin{pmatrix} 0 & -4 & 0 \\ -2 & -1 & 1 \\ -10 & -17 & 5 \end{pmatrix}$$

Laplace expansion w.r.t. 1st row

(a) $\det A = -(-4)\cdot \det\begin{pmatrix} -2 & 1 \\ -10 & 5 \end{pmatrix} = 4\cdot\left[(-2)\cdot 5 - (-10)\right] = 4\cdot 0 = 0$

(b) $P_A(t) = \det(A - tI) = \det\begin{pmatrix} -t & -4 & 0 \\ -2 & -1-t & 1 \\ -10 & -17 & 5-t \end{pmatrix} =$ Laplace expansion w.r.t. 3rd column

$$= -\det\begin{pmatrix} -t & -4 \\ -10 & -17 \end{pmatrix} + (5-t)\cdot\det\begin{pmatrix} -t & -4 \\ -2 & -1-t \end{pmatrix}$$

$$= -(17t - 40) + (5-t)\left[t(t+1) - 8\right]$$

$$= -17t + 40 + (5-t)(t^2 + t - 8)$$

$$= -17t + \cancel{40} + 5t^2 + 5t - \cancel{40} - t^3 - t^2 + 8t$$

$$= -t^3 + 4t^2 - 4t = -t(t^2 - 4t + 4) = -t(t-2)^2$$

(c) The eigenvalues of A are

0 with algebraic multiplicity 1

2 2

(d) $E_0 = \operatorname{Ker} A$

swap 1st and 2nd

multiply: 1st by -1, 2nd by $-1/4$, 3rd by -1

replace 3rd with 3rd $- 5 \cdot$ 1st

multiply 3rd by $1/12$

$$\begin{pmatrix} 0 & -4 & 0 \\ -2 & -1 & 1 \\ -10 & -17 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & -1 & 1 \\ 0 & -4 & 0 \\ -10 & -17 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 10 & 17 & -5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 12 & 0 \end{pmatrix} \rightsquigarrow$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

replace 3rd with 3rd $-$ 2nd

replace 1st with 1st $-$ 2nd

$$\begin{cases} 2x - z = 0 \\ y = 0 \end{cases}$$

$$\begin{cases} x = \frac{1}{2} z \\ y = 0 \end{cases}$$

$$E_0 = \operatorname{Span} \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} = \operatorname{Span} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$E_2 = \operatorname{Ker}(A-2I) = \operatorname{Ker}\begin{pmatrix} -2 & -4 & 0 \\ -2 & -3 & 1 \\ -10 & -17 & 3 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -2 & -4 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix} =$$

$$= \operatorname{Ker}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(e) The geometric multiplicity of the eigenvalue 2 is $\dim E_2 = 1$ and hence is strictly less than its algebraic multiplicity, which is 1.
Therefore A is not diagonalisable.

Problem 4 $\begin{cases} a_0 = 5+i \\ a_1 = 8+i \\ a_{n+2} = 3a_{n+1} - 2a_n \quad \forall n \geq 0 \end{cases}$

(a) $\lambda^2 = 3\lambda - 2 \quad \lambda^2 - 3\lambda + 2 = 0 \quad (\lambda-2)(\lambda-1) = 0$

$\Rightarrow \exists \gamma_1, \gamma_2 \in \mathbb{C}$ s.t. $\forall n \geq 0 \quad a_n = \gamma_1 \cdot 1^n + \gamma_2 \cdot 2^n$

$\begin{cases} 5+i = a_0 = \gamma_1 + \gamma_2 \\ 8+i = a_1 = \gamma_1 + 2\gamma_2 \end{cases} \qquad \begin{array}{l} \gamma_2 = 8+i-(5+i) = 3 \\ \gamma_1 = 5+i-3 = 2+i \end{array}$

$\Rightarrow \quad a_n = 2+i+3 \cdot 2^n \quad \forall n \geq 0$

(b) $|a_n|$ goes as 2^n for $n \to +\infty$, hence $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 2$

$\Rightarrow$ the radius of convergence of the power series

$\sum_{n\geq 0} a_n z^n$ is $R = \frac{1}{2}$

(c) For $|z| < \frac{1}{2}$ we have $|2z| < 1$ and $|z| < 1$. Hence

$$\sum_{n=0}^{+\infty} a_n z^n = \sum_{n=0}^{+\infty} (2+i+3\cdot 2^n) z^n = (2+i)\cdot \sum_{n=0}^{+\infty} z^n + 3\cdot \sum_{n=0}^{+\infty} (2z)^n =$$

$$= (2+i)\frac{1}{1-z} + 3\cdot \frac{1}{1-2z} = \frac{(2+i)(1-2z)+3(1-z)}{(1-z)(1-2z)} =$$

$$= \frac{2-4z+i-2iz+3-3z}{1-2z-z+2z^2} = \frac{5+i+(-7-2i)z}{1-3z+2z^2}$$

$$p(z) = 5+i-(7+2i)z$$
$$q(z) = 1-3z+2z^2$$

(d) $\left|\frac{1+i}{3}\right| = \frac{\sqrt{2}}{3} < \frac{1}{2} = R$ hence the series $\sum_{n=0}^{+\infty} a_n \left(\frac{1+i}{3}\right)^n$ converges in $\mathbb{C}$.

$$p\left(\frac{1+i}{3}\right) = 5+i-(7+2i)\frac{1+i}{3} = 5+i-\frac{1}{3}\left[7+7i+2i-2\right]$$

$$= 5+i-\frac{1}{3}(5+9i) = \frac{10}{3}-2i$$

$$q\left(\frac{1+i}{3}\right) = 1-\cancel{3}\frac{1+i}{\cancel{3}}+2\left(\frac{1+i}{3}\right)^2 = -i+\frac{2}{9}(\cancel{1}-\cancel{1}+2i)$$

$$= -i+\frac{4}{9}i = -\frac{5}{9}i$$

$$\sum_{n=0}^{+\infty} a_n\left(\frac{1+i}{3}\right)^n = \frac{p\left(\frac{1+i}{3}\right)}{q\left(\frac{1+i}{3}\right)} = \frac{\frac{10}{3}-2i}{-\frac{5}{9}i} = \frac{9}{5}i\left(\frac{10}{3}-2i\right)$$

$$= 6i+\frac{18}{5} \qquad \alpha = \frac{18}{5}\,,\ \beta = 6$$

(e) $|i| = 1 > \frac{1}{2} = R \Rightarrow \sum_{n=0} a_n i^n$ doesn't converge

Problem 5 $f(z) = 8iz^6 + \frac{i}{z^2+9} + \frac{3}{3z-1}$

$A = \{z \in \mathbb{C} \mid |z| > 5\}$

The Z-transform of $8iz^6$ on A is $\begin{cases} 8i & \text{if } n = -6 \\ 0 & \text{if } n \neq -6 \end{cases}$

If $z \in A$ then $|z| > 5$ and hence $\left|\frac{9}{z^2}\right| < \frac{9}{25} < 1$ and

$\left|\frac{1}{3z}\right| < \frac{1}{15} < 1$ so

$$\frac{i}{z^2+9} = \frac{i}{z^2} \cdot \frac{1}{1-\left(-\frac{9}{z^2}\right)} = \frac{i}{z^2} \cdot \sum_{m \geq 0} \left(-\frac{9}{z^2}\right)^m = \sum_{m \geq 0} \frac{(-9)^m i}{(z^2)^{m+1}} =$$

$$= \sum_{m \geq 0} i(-9)^m z^{-2m-2}$$

The Z-transform of $\dfrac{i}{z^2+9}$ is $\begin{cases} i(-9)^{\frac{n}{2}-1} & n \geq 2,\ n \text{ even} \\ 0 & n \leq 1 \text{ or } n \text{ odd} \end{cases}$

$$\frac{3}{3z-1} = \frac{1}{z} \frac{1}{1-\frac{1}{3z}} = \frac{1}{z} \sum_{m \geq 0} \left(\frac{1}{3z}\right)^m = \sum_{m \geq 0} 3^{-m} z^{-m-1}$$

The Z-transform of $\dfrac{3}{3z-1}$ on A

is $\begin{cases} 3^{-n+1} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$

The Z-transform of f on A is

$$u_n = \begin{cases} 8i & \text{if } n = -6 \\ 0 & \text{if } n \leq -7 \text{ or } -5 \leq n \leq 0 \\ 3^{-n+1} & \text{if } n \geq 1 \text{ and } n \text{ odd} \\ 3^{-n+1} + i(-9)^{\frac{n}{2}-1} & \text{if } n \geq 2 \text{ and } n \text{ even} \end{cases}$$