MATH METHODS M, APPELLO 4, 11/06/2026

Problem 1

$$w = 3 + \frac{35}{4}\pi i = 3 + \frac{3\pi}{4} i + 8\pi i$$

$$(e^{w})^{-1} = e^{-w} = e^{-(3 + \frac{3}{4}\pi i + 8\pi i)} = e^{-3}\, e^{-\frac{3}{4}\pi i}\, e^{-8\pi i}$$

$$= e^{-3} e^{\frac{5}{4}\pi i} = e^{-3} \cdot \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)$$

$$(e + ei)^3 = e^3 (1+i)^3 = e^3 (1^3 + 3i + 3i^2 + i^3)$$

$$= e^3 (1 + 3i - 3 - i) = e^3 (-2 + 2i) = 2e^3 (-1+i)$$

$$(e+ei)^3 \cdot (e^{w})^{-1} = 2\cancel{e^3}(-1+i)\, \cancel{e^{-3}}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right) =$$

$$= (-1+i) \cdot \sqrt{2}\,(-1-i) = \sqrt{2}\,(1 - i^2) = 2\sqrt{2}$$

$$\Rightarrow \quad a = 2\sqrt{2}, \quad b = 0$$

Problem 2

$$\frac{iz}{z^2 - 3z + 5} = \frac{z}{2z - 1}$$

$$iz(2z-1) = z(z^2 - 3z + 5)$$

$$z \cdot \left(z^2 + (-3 - 2i)z + 5 + i\right) = 0$$

$$z = 0 \quad \text{or} \quad z^2 + (-3-2i)z + 5 + i = 0$$

$$\Delta = (-3-2i)^2 - 4(5+i)$$

$$= 9 - 4 + 12i - 20 - 4i$$

$$= -15 + 8i = 1 - 16 + 8i = 1^2 + (4i)^2 + 2 \cdot 4i =$$

$$= (1 + 4i)^2$$

$$\Rightarrow z = \frac{3 + 2i \pm (1 + 4i)}{2} = \begin{cases} \frac{4+6i}{2} = 2 + 3i \\ \frac{2-2i}{2} = 1 - i \end{cases}$$

Solutions: $0,\ 2+3i,\ 1-i$

Problem 3 $A = \begin{pmatrix} -2 & -2 & -25 \\ 0 & 0 & -10 \\ 0 & 0 & 5 \end{pmatrix}$

(a) $\det A = (-2)\cdot 0 \cdot 5 = 0$ because A is upper triangular

(b) $P_A(t) = (-2-t)(-t)(5-t)$

(c) $-2, 0, 5$ with alg mult. 1

(d) $E_{-2} = \operatorname{Ker}(A+2I) = \operatorname{Ker}\begin{pmatrix} 0 & -2 & -25 \\ 0 & 2 & -10 \\ 0 & 0 & 7 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$E_0 = \operatorname{Ker} A = \operatorname{Ker}\begin{pmatrix} -2 & -2 & -25 \\ 0 & 0 & -10 \\ 0 & 0 & 5 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -2 & -2 & -25 \\ 0 & 0 & 1 \end{pmatrix} =$

$= \operatorname{Ker}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$E_5 = \operatorname{Ker}(A-5I) = \operatorname{Ker}\begin{pmatrix} -7 & -2 & -25 \\ 0 & -5 & -10 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -7 & -2 & -25 \\ 0 & -5 & -10 \end{pmatrix} =$

$= \operatorname{Ker}\begin{pmatrix} -7 & 0 & -21 \\ 0 & 1 & 2 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

(e) A is diagonalisable

$M = \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Problem 4

$$\begin{cases} a_0 = 4 \\ a_1 = 4+2i \\ a_{n+2} = 2a_{n+1} - 2a_n \quad \forall n \geq 0 \end{cases}$$

(a) The characteristic equation is

$$\lambda^2 = 2\lambda - 2 \qquad \lambda^2 - 2\lambda + 2 = 0 \qquad (\lambda-1)^2 + 1 = 0$$

$$\Rightarrow \lambda - 1 = \pm i \Rightarrow \lambda = 1 \pm i$$

look for solution

$$a_n = \gamma_1 (1+i)^n + \gamma_2 (1-i)^n \quad \text{with } \gamma_1, \gamma_2 \in \mathbb{C}$$

$$\begin{cases} 4 = a_0 = \gamma_1 + \gamma_2 \\ 4+2i = a_1 = \gamma_1(1+i) + \gamma_2(1-i) \end{cases} \Rightarrow \begin{matrix} \gamma_1 = 3 \\ \gamma_2 = 1 \end{matrix}$$

$$\Rightarrow a_n = 3(1+i)^n + (1-i)^n \quad \forall n \geq 0.$$

(b) $|1+i| = |1-i| = \sqrt{2}$

$|a_n|$ goes to ∞ as $(\sqrt{2})^n$

By Cauchy-Hadamard the radius of convergence of $\sum_{n\geq 0} a_n z^n$ is $\dfrac{1}{\limsup_{n\to+\infty} \sqrt[n]{|a_n|}} = \dfrac{1}{\sqrt{2}}$

(c) $R = \dfrac{1}{\sqrt{2}}$. If $|z| < R$ then

$|(1+i)z| < 1$, $|(1-i)z| < 1$ and

$$\sum_{n\geq 0} a_n z^n = 3 \cdot \sum_{n\geq 0} ((1+i)z)^n + \sum_{n \geq 0} ((1-i)z)^n =$$

$$= 3 \frac{1}{1-(1+i)z} + \frac{1}{1-(1-i)z} =$$

$$= \frac{3(1-(1-i)z) + 1 - (1+i)z}{(1-(1+i)z)(1-(1-i)z)} =$$

$$= \frac{3 - 3(1-i)z + 1 - (1+i)z}{(1-z-iz)(1-z+iz)}$$

$$= \frac{4 + (-4+2i)z}{(1-z)^2 + z^2}$$

$p(z) = 4 + (-4+2i)z$

$q(z) = (1-z)^2 + z^2 = 1 - 2z + 2z^2$

(d) $|i| = 1 > \frac{1}{\sqrt{2}} = R \Rightarrow \sum_{n\geq 0} a_n i^n$ doesn't exist.

(e) $\left|\frac{i}{2}\right| = \frac{1}{2} < \frac{1}{\sqrt{2}} = R \Rightarrow \sum_{n\geq 0} a_n \left(\frac{i}{2}\right)^n$ exists

$$\sum_{n\geq 0} a_n \left(\frac{i}{2}\right)^n = \frac{p\left(\frac{i}{2}\right)}{q\left(\frac{i}{2}\right)}$$

$p\left(\frac{i}{2}\right) = 4 + (-4+2i)\frac{i}{2} = 4 + (-2+i)i = 4 - 2i - 1 = 3 - 2i$

$q\left(\frac{i}{2}\right) = 1 - 2\cdot\frac{i}{2} + 2\left(\frac{i}{2}\right)^2 = 1 - i - \frac{2}{4} = \frac{1}{2} - i$

$$\sum_{n\geq 0} a_n \left(\frac{i}{2}\right)^n = \frac{p\left(\frac{i}{2}\right)}{q\left(\frac{i}{2}\right)} = \frac{3-2i}{\frac{1}{2} - i} = \frac{3-2i}{1-2i}\cdot 2 = 2\,\frac{3-2i}{1-2i}\,\frac{1+2i}{1+2i} =$$

$$= 2\,\frac{3 + 6i - 2i + 4}{1+4} = \frac{2}{5}(7+4i) = \frac{14}{5} + \frac{8}{5}i$$

$\alpha = \frac{14}{5}$, $\beta = \frac{8}{5}$

Problem 5 If $z \in A$ then $|z| > \frac{1}{2}$ hence

$|z^{-1}| < 2$ $\quad 1 > \left|\frac{z^{-1}}{2}\right| > |7^{-1} z^{-1}|$

$$\frac{7i}{7z-1} = \frac{7i}{7z} \frac{1}{1-(7z)^{-1}} = \frac{7i}{7z} \cdot \sum_{m \geq 0} \left((7z)^{-1}\right)^m =$$

because $|(7z)^{-1}| < 1$

$$= \frac{\cancel{7}i}{\cancel{7}z} \sum_{m \geq 0} 7^{-m} z^{-m} = \sum_{m \geq 0} i \cdot 7^{-m} z^{-m-1} =$$

$$= \sum_{k \geq 1} i\, 7^{-k+1} z^{-k}$$

$k = m+1$
$m = k-1$

$$10 z^6 = 10 \cdot z^{-(-6)}$$

$$u_n = \begin{cases} 10 & \text{if } n = -6 \\ i\, 7^{-n+1} & \text{if } n \geq 1 \\ 0 & \text{if } n \leq -7 \text{ or } -5 \leq n \leq 0 \end{cases}$$