APPELLO 5, 14^{th} July 2026

PROBLEM 1 $\frac{\pi}{6}\cdot 37=\left(\frac{36}{6}+\frac{1}{6}\right)\pi=6\pi+\frac{\pi}{6}\Rightarrow\left(e^{i\frac{\pi}{6}}\right)^{37}=e^{i\frac{37}{6}\pi}=e^{i\frac{\pi}{6}}$

$\frac{1}{2}+\frac{\sqrt{3}}{2}i=e^{i\frac{\pi}{3}}$, $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{-2}=\left(e^{i\frac{\pi}{3}}\right)^{-2}=e^{i\frac{-2\pi}{3}}$

$a+bi=\left(e^{i\frac{\pi}{6}}\right)^{37}\cdot\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{-2}=e^{i\frac{\pi}{6}}\cdot e^{i\left(-\frac{2}{3}\right)\pi}=e^{i\pi\cdot\left(\frac{1}{6}-\frac{2}{3}\right)}=e^{i\pi\frac{1-4}{6}}=$

$=e^{i\pi\frac{-3}{6}}=e^{-i\frac{\pi}{2}}=-i\implies a=0,\ b=-1.$

PROBLEM 2 $z(z-2)+z+1=(i-1)z-1$

$z^2-z+1=(i-1)z-1$

$z^2-iz+2=0$

$$z=\frac{i\pm\sqrt{i^2-8}}{2}=\frac{i\pm\sqrt{-9}}{2}=\frac{i\pm 3i}{2}=\begin{cases}\frac{i+3i}{2}=2i\\ \frac{i-3i}{2}=-i\end{cases}$$

PROBLEM 3

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

(a) $\det A = \det \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} = 0$

1st row is zero

replace 1st row with 1st row + 3rd row

Laplace expansion wrt 2nd row

(b) $P_A(t) = \det \begin{pmatrix} -1-t & 0 & 1 \\ 0 & -2-t & 0 \\ 1 & 0 & -1-t \end{pmatrix} = (-2-t) \cdot \det \begin{pmatrix} -1-t & 1 \\ 1 & -1-t \end{pmatrix} =$

$= -(t+2) \cdot [(t+1)^2 - 1] = -(t+2)(t^2+2t) = -t(t+2)^2$

(c) Eigenvalues: 0 with alg. mult. 1

-2 2

(d) • Eigenvalue $\lambda = 0$:

$$E_0 = \operatorname{Ker} A = \operatorname{Ker}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \overset{\text{swap 1st and 3rd}}{=} \operatorname{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \overset{\text{replace 3rd by 3rd+1st}}{=} \operatorname{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{\text{3rd useless}}{=}$$

$$= \operatorname{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \end{pmatrix} \underset{\substack{\text{multiply} \\ \text{2nd by } -\frac{1}{2}}}{=} \operatorname{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a basis of E_0

• Eigenvalue $\lambda = -2$: $E_{-2} = \operatorname{Ker}(A + 2I) = \operatorname{Ker}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \operatorname{Ker}(1\ 0\ 1) =$

· 2nd useless

· 3rd useless because equal to 1st

$$= \operatorname{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right)$$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is a basis of E_{-2}

(e) A is diagonalisable because

eigen value	alg. mult	geom. mult.
0	1	1
-2	2	2

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad D = M^{-1}AM = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

PROBLEM 4 $\begin{cases} a_0 = 5 \\ a_1 = 2+3i \\ a_{n+2} = (1+i)a_{n+1} - ia_n \quad \forall n \geq 0 \end{cases}$

(a) The characteristic equation is $\lambda^2 = (1+i)\lambda - i \Rightarrow \lambda^2 - (1+i)\lambda + i = 0$ $(\lambda-1)(\lambda-i) = 0$ whose solutions are $\lambda = 1$ and $\lambda = i$. Hence there exist $c_1, c_2 \in \mathbb{C}$ s.t. $\forall n \geq 0$, $a_n = c_1 \cdot 1^n + c_2 \cdot i^n = c_1 + c_2 i^n$.

$$\begin{cases} 5 = a_0 = c_1 + c_2 \\ 2+3i = a_1 = c_1 + c_2 i \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = 3 \end{cases} \Rightarrow \forall n \geq 0,\ a_n = 2 + 3 \cdot i^n$$

(b) $\limsup_{n \to +\infty} \sqrt[n]{|a_n|} = 1 \Rightarrow R = 1.$

(c) For $|z| < 1$ we have $\sum_{n=0}^{+\infty} a_n z^n = 2 \sum_{n=0}^{+\infty} z^n + 3 \cdot \sum_{n=0}^{+\infty} (iz)^n =$

$$= 2 \cdot \frac{1}{1-z} + 3 \cdot \frac{1}{1-iz} = \frac{2(1-iz) + 3(1-z)}{(1-z)(1-iz)} = \frac{5 + (-2i-3)z}{(1-z)(1-iz)}$$

$p(z) = 5 - (2i+3)z$

$q(z) = (1-z)(1-iz) = 1 - (1+i)z + iz^2$

(d) $\frac{1}{2} < R = 1 \Rightarrow \sum_{n=0}^{+\infty} a_n \left(\frac{1}{2}\right)^n$ exists

$$p\left(\tfrac{1}{2}\right) = 5 - (2i+3)\frac{1}{2} = 5 - i - \frac{3}{2} = \frac{7}{2} - i = \frac{7-2i}{2}$$

$$q\left(\tfrac{1}{2}\right) = 1 - (1+i)\frac{1}{2} + i\,\frac{1}{4} = 1 - \frac{1}{2} + i\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{4}i = \frac{2-i}{4}$$

$$\sum_{n=0}^{+\infty} a_n \left(\frac{1}{2}\right)^n = \frac{p\left(\frac{1}{2}\right)}{q\left(\frac{1}{2}\right)} = \frac{\frac{7-2i}{2}}{\frac{2-i}{4}} = \frac{-2i}{\cancel{2}} \cdot \frac{\cancel{4}^{\,2}}{2-i} \cdot \frac{2+i}{2+i} =$$

$$= \frac{2(7-2i)(2+i)}{5} = \frac{2}{5}(14 + 7i - 4i + 2) = \frac{2}{5}(16 + 3i) = \frac{32}{5} + \frac{6}{5}i$$

(e) $\frac{3}{2} > R = 1 \Rightarrow \sum_{n=0}^{+\infty} a_n \left(\frac{3}{2}\right)^n$ doesn't exist.

PROBLEM 5 $f(z) = \frac{3}{4-z} + 7z^{-2} + 8z^{10}$, $A = \{z \in \mathbb{C} \mid \frac{1}{2} < |z| < 2\}$

For $z \in A$, $\left|\frac{z}{4}\right| < 1$ hence $\frac{3}{4-z} = \frac{3}{4} \frac{1}{1-\frac{z}{4}} = \frac{3}{4} \sum_{m \geq 0} \left(\frac{z}{4}\right)^m =$

$= \sum_{m \geq 0} 3 \cdot 4^{-m-1} z^m \Rightarrow$ the Z-transform of $\frac{3}{4-z}$ on A is

$$\begin{cases} 3 \cdot 4^{n-1} & \text{if } n \leq 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Hence the Z-transform of f on A is

$$u_n = \begin{cases} 3 \cdot 4^{n-1} & \text{if } n \leq 0 \text{ and } n \neq -10 \text{, i.e. } n \leq -11 \text{ or } -9 \leq n \leq 0 \\ 3 \cdot 4^{-10-1} + 8 & \text{if } n = -10 \\ 0 & \text{if } n > 0 \text{ and } n \neq 2 \text{, i.e. } n = 1 \text{ or } n \geq 3 \\ 7 & \text{if } n = 2 \end{cases}$$