Math Methods M, Call 6, 17th September 2026

Problem 1

$$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$(e+ei)^4 = e^4(1+i)^4 = e^4 \cdot \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^4 = e^4 \cdot 4 \cdot e^{i\pi}$$

$$= -4e^4$$

$$\frac{25}{6}\pi = 4\pi + \frac{\pi}{6} \qquad e^{4+\frac{25}{6}\pi i} = e^4 \cdot e^{\frac{\pi}{6}i}$$

$$\frac{(e+ei)^4}{e^{4+\frac{25}{6}\pi i}} = \frac{-4e^4}{e^4 e^{\frac{\pi}{6}i}} = -4e^{-\frac{\pi}{6}i} = -4 \cdot \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -2\sqrt{3} + 2i$$

$$\Rightarrow \quad a = -2\sqrt{3}$$
$$b = 2$$

Problem 2

$$(z+1)(z-2) + 4 = i(z-2)$$

$$z^2 - z - 2 + 4 = iz - 2i$$

$$z^2 - z + 2 - iz + 2i = 0$$

$$z^2 + (-1-i)z + 2 + 2i = 0 \qquad \Delta = (-1-i)^2 - 4(2+2i) =$$

$$= \cancel{1} - \cancel{1} + 2i - 8 - 8i = -8 - 6i = 1 - 6i - 9 = 1 - 2\cdot 3i + (3i)^2 =$$

$$= (1-3i)^2$$

$$z = \frac{1+i \pm (1-3i)}{2} = \begin{cases} \frac{2-2i}{2} = 1-i \\ \frac{4i}{2} = 2i \end{cases}$$

Problem 3

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -10 & 22 & 1 \end{pmatrix}$$

(a) $\det A = (-1)(-1)\,1 = 1$

(b) $p_A(t) = (-1-t)^2(1-t) = (t+1)^2(1-t)$

$= -(t^2-1)(t+1) = -(t^3+t^2-t-1)$

$= -t^3 - t^2 + t + 1$

(c) -1 with alg. mult 2

1 1

(d) $E_{-1} = \operatorname{Ker}(A+I) = \operatorname{Ker}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -10 & 22 & 2 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -5 & 11 & 1 \end{pmatrix} =$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, z = 5x - 11y \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -11 \end{pmatrix} \right)$

$E_1 = \operatorname{Ker}(A-I) = \operatorname{Ker}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -10 & 22 & 0 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(e) yes, $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -11 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Problem 4

$$\begin{cases} a_0 = 1 \\ a_1 = -1+2i \\ a_{n+2} = (1+i)a_{n+1} - ia_n \quad \forall n \geq 0 \end{cases}$$

(a) $\lambda^2 = (1+i)\lambda - i \qquad (\lambda-1)(\lambda-i) = 0$

$a_n = c_1 \cdot 1^n + c_2 \cdot i^n \quad \forall n \geq 0.$

$1 = a_0 = c_1 + c_2 \qquad c_2 - ic_2 = 1 - (-1+2i) = 2-2i = 2(1-i)$

$-1+2i = c_1 + c_2 i \qquad (1-i)c_2 = 2(1-i) \Rightarrow c_2 = 2.$

$c_1 = 1 - c_2 = -1$

$\Rightarrow a_n = -1 + 2i^n \quad \forall n \geq 0$

(b) $R = 1.$

(c) For $|z| < 1$, $\displaystyle\sum_{n=0}^{+\infty} a_n z^n = -\sum_{n=0}^{+\infty} z^n + 2 \cdot \sum_{n=0}^{+\infty} (iz)^n =$

$$= -\frac{1}{1-z} + 2\frac{1}{1-iz} = \frac{-1+iz+2-2z}{(1-z)(1-iz)}$$

$p(z) = 1 + (-2+i)z$

$q(z) = (1-z)(1-iz)$

(d) $|2i| > 1 = R \Rightarrow$ doesn't converge

(e) $\left|\frac{i}{2}\right| < 1 = R \Rightarrow$ does converge

$$p\left(\frac{i}{2}\right) = 1 + (-2+i)\cdot\frac{i}{2} = 1 - i - \frac{1}{2} = \frac{1}{2} - i = \frac{1-2i}{2}$$

$$q\left(\frac{i}{2}\right) = \left(1 - \frac{i}{2}\right)\left(1 + \frac{1}{2}\right) = \frac{2-i}{2}\cdot\frac{3}{2}$$

$$\sum_{n=0}^{+\infty} a_n \left(\frac{i}{2}\right)^n = \frac{p\left(\frac{i}{2}\right)}{q\left(\frac{i}{2}\right)} = \frac{\frac{1-2i}{2}}{\frac{2-i}{2}\cdot\frac{3}{2}} = \frac{2}{3}\,\frac{1-2i}{2-i}\,\frac{2+i}{2+i} = \frac{2}{3}\,\frac{2+i-4i+2}{5}$$

$$= \frac{2}{3}\,\frac{4-3i}{5}$$

$$\alpha = \frac{8}{15} \qquad \beta = -\frac{2}{5}$$

Problem 5

$$f(z) = -5z^4 + \frac{2}{z-\frac{1}{2}} + \frac{3}{3-z}$$

$A = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$

For $z \in A$ $\quad \left|\frac{1}{2z}\right| < 1$, $\left|\frac{z}{3}\right| < 1$

$$\frac{2}{z-\frac{1}{2}} = \frac{2}{z} \cdot \frac{1}{1-\frac{1}{2z}} = \frac{2}{z} \cdot \sum_{m \geq 0} \left(\frac{1}{2z}\right)^m = \sum_{m \geq 0} 2^{1-m} z^{-1-m} =$$

$$= \sum_{n \geq 1} 2^{2-n} z^{-n}$$

$$\frac{3}{3-z} = \frac{1}{1-\frac{z}{3}} = \sum_{m \geq 0} \left(\frac{z}{3}\right)^m = \sum_{n \leq 0} \left(\frac{z}{3}\right)^{-n} = \sum_{n \leq 0} 3^n z^{-n}$$

$$u_n = \begin{cases} 2^{2-n} & n \geq 1 \\ 3^n & n \leq 0 \text{ and } n \neq -4 \\ 3^{-4} - 5 & n = -4 \end{cases}$$