LINEAR RECURRENCE SEQUENCES

Fix $r \in \mathbb{N}^+$, $b_1, \dots, b_r \in \mathbb{C}$, $a_0, a_1, \dots, a_{r-1} \in \mathbb{C}$.

Consider the sequence of complex numbers $(a_n)_{n \in \mathbb{N}}$ defined by

$$\begin{cases} a_0, a_1, \dots, a_{r-1} \\ a_{n+r} = b_1 a_{n+r-1} + b_2 a_{n+r-2} + \dots + b_{r-1} a_{n+1} + b_r a_n \quad \forall n \geq 0 \end{cases}$$

We want to study properties of this sequence.

Consider the matrix $A \in M_r(\mathbb{C})$ given by

$$A = \left(\begin{array}{cccc|c} b_1 & b_2 & \cdots & b_{r-1} & b_r \\ \hline 1 & & & & 0 \\ & 1 & & & 0 \\ & & & & \vdots \\ & & & 1 & 0 \end{array}\right)$$

and consider the vector

$$v_n = \begin{pmatrix} a_{n+r-1} \\ \vdots \\ a_{n+1} \\ a_n \end{pmatrix} \in \mathbb{C}^r . \qquad \text{In particular} \quad v_0 = \begin{pmatrix} a_{r-1} \\ \vdots \\ a_1 \\ a_0 \end{pmatrix}$$

Prop $\forall n \geq 0 \quad v_{n+1} = A v_n$

Pf

$$v_{n+1} = \begin{pmatrix} a_{n+r} \\ a_{n+r-1} \\ \vdots \\ a_{n+2} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \cdots & b_{r-1} & b_r \\ 1 & & & & 0 \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+r-1} \\ a_{n+r-2} \\ \vdots \\ a_{n+1} \\ a_n \end{pmatrix} = A v_n .$$

$\square$

<u>Prop</u> $\forall n \geq 0 \quad v_n = A^n v_0$

<u>Pf</u> $v_1 = Av_0$

$v_2 = Av_1 = A(Av_0) = A^2 v_0$

$v_3 = Av_2 = A(A^2 v_0) = A^3 v_0$

$v_4 = Av_3 = A(A^3 v_0) = A^4 v_0 \ \cdots \qquad \square$

<u>Prop</u> The characteristic polynomial of A is

$P_A(t) = \det(A - tI) = (-1)^r \cdot [t^r - (b_1 t^{r-1} + \dots + b_{r-1} t + b_r)]$

<u>Pf</u> Omitted. It follows from the properties of the companion matrix. $\square$

Algorithm to find a closed formula for a_n

Write the matrix A.

Try to diagonalise A: compute the eigenvalues and the eigenvectors of A. Try to find $M \in M_r(\mathbb{C})$ invertible such that $D = M^{-1}AM$ is diagonal.

Then $A = MDM^{-1}$, so $A^n = MD^nM^{-1}$ for all $n \geq 0$.

The powers D^n are easy to compute because D is diagonal.

Then

$$v_n = A^n v_0 = MD^nM^{-1}v_0$$

Read the last entry and get a_n in terms of n.

If A is not diagonalisable, then you need to find $M \in M_r(\mathbb{C})$ s.t. $M^{-1}AM$ is upper triangular, e.g. in Jordan form.

Prop In most cases we have:

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } A\}$$

Pf Assume that A has distinct eigenvalues $\lambda_1,\dots,\lambda_r$ such that $|\lambda_1|>|\lambda_2|\geq\dots\geq|\lambda_r|$.

Then A is diagonalisable: $\exists M\in M_r(\mathbb{C})$ invertible such that $M^{-1}AM = D = \mathrm{diag}(\lambda_1,\dots,\lambda_r)$. Then $A = MDM^{-1}$

$A^n = MD^nM^{-1}$ $\quad v_n = MD^nM^{-1}v_0$, $\quad D^n = \mathrm{diag}(\lambda_1^n,\dots,\lambda_r^n)$

so there exist $c_1,\dots,c_r\in\mathbb{C}$ s.t.

$a_n = c_1\lambda_1^n + \dots + c_r\lambda_r^n$ $\forall n\geq 0$. Then

$$\frac{a_{n+1}}{a_n} = \frac{c_1\lambda_1^{n+1}+\dots+c_r\lambda_r^{n+1}}{c_1\lambda_1^n+\dots+c_r\lambda_r^n} = \lambda_1\,\frac{c_1\lambda_1^n + c_2\frac{\lambda_2^{n+1}}{\lambda_1}+\dots+c_r\frac{\lambda_r^{n+1}}{\lambda_1}}{c_1\lambda_1^n + c_2\lambda_2^n+\dots+c_r\lambda_r^n} =$$

$$= \lambda_1 \cdot \frac{c_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{n+1} + \cdots + c_r\left(\frac{\lambda_r}{\lambda_1}\right)^{n+1}}{c_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{n} + \cdots + c_r\left(\frac{\lambda_r}{\lambda_1}\right)^{n}}$$

For $j = 2, \dots, r$, $\left|\frac{\lambda_j}{\lambda_1}\right| < 1 \implies \lim_{n\to+\infty} \left(\frac{\lambda_j}{\lambda_1}\right)^n = 0$

Now assume $c_1 \neq 0$, hence $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = |\lambda_1|$. $\square$

<u>Remark</u> The strategy of the proof can be used to compute $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|$ even when the assumptions are not satisfied.

What happens if A doesn't have r distinct eigenvalues?

Let $\lambda_1, \dots, \lambda_s$ be the distinct eigenvalues of A, i.e. the zeroes of $P_A(t) = (-1)^r \cdot [t^r - (b_1 t^{r-1} + \dots + b_{r-1} t + b_r)]$.

For each $i = 1, \dots, s$, let m_i be the multiplicity of λ_i as zero of P_A.

Then there exist polynomials $c_1(n), \dots, c_s(n)$ such that:

- $\deg c_i(n) < m_i \quad \forall i = 1, \dots, s$
- $\forall n \geq 0, \quad a_n = c_1(n) \cdot \lambda_1^n + \dots + c_s(n) \cdot \lambda_s^n$

Since the polynomial $c_i(n)$ has degree $< m_i$, we can write

$$c_i(n) = \gamma_{i,0} + \gamma_{i,1} n + \dots + \gamma_{i,m_i-1} n^{m_i - 1}$$

The $\gamma_{i,j}$ can be determined from knowing $a_0, \dots, a_{r-1}$.