SOLUTIONS TO MOCK EXAM 1

Problem 1

$$(2+3i)^{-1} = \frac{1}{2+3i} = \frac{1}{2+3i}\,\frac{2-3i}{2-3i} = \frac{2-3i}{4+9} = \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i$$

$$a = \frac{2}{13}, \quad b = -\frac{3}{13}$$

Problem 2

Look for $z \in \mathbb{C}$ s.t. $z^3 = 27i$.

$27 = 3^3$, $i = e^{i\frac{\pi}{2}}$

$z = re^{i\theta}$ $\quad z^3 = r^3 e^{3i\theta}$ $\quad$ so want $\quad r^3 e^{3i\theta} = 3^3 e^{i\frac{\pi}{2}}$

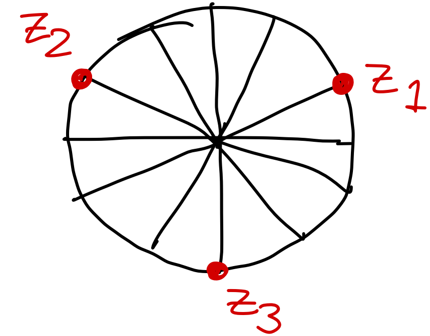
$r > 0$

$$\Longrightarrow \quad r = 3$$

$$3\theta = \frac{\pi}{2} + 2\pi k \qquad k \in \mathbb{Z}$$

$$\Longrightarrow \quad \theta = \frac{\pi}{6} + \frac{2\pi}{3}k \qquad k \in \mathbb{Z}$$

$$z_1 = 3e^{i\frac{\pi}{6}} = 3\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$



$$z_2 = 3e^{i\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)} = 3e^{i\frac{5}{6}\pi} = 3\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

$$z_3 = 3e^{i\left(\frac{\pi}{6} + \frac{4}{3}\pi\right)} = 3e^{i\frac{3}{2}\pi} = -3i$$

z_1, z_2, z_3 are all $z \in \mathbb{C}$ s.t. $z^3 = 27i$

Problem 3

$$A = \begin{pmatrix} 2 & 0 & 10 \\ 0 & 2 & -20 \\ 0 & 0 & -3 \end{pmatrix}$$

1) $\det A = 2 \cdot 2 \cdot (-3) = -12$

$\uparrow$ because A is upper triangular

2) The characteristic polynomial of A is

$$P_A(t) = \det(A - tI) = \det \begin{pmatrix} 2-t & 0 & 10 \\ 0 & 2-t & -20 \\ 0 & 0 & -3-t \end{pmatrix} = (2-t)(2-t)(-3-t) =$$

$$= -(t-2)^2(t+3)$$

3) 2 and -3 are the eigenvalues

4) $E_2 = \operatorname{Ker}(A - 2I) = \operatorname{Ker} \begin{pmatrix} 0 & 0 & 10 \\ 0 & 0 & -20 \\ 0 & 0 & -5 \end{pmatrix} = \operatorname{Ker}(0 \ 0 \ 1) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$

$$E_{-3} = \mathrm{Ker}(A+3I) = \mathrm{Ker}\begin{pmatrix} 5 & 0 & 10 \\ 0 & 5 & -20 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

5) A is diagonalizable because

$\dim E_2 = 2 =$ multiplicity of 2 as root of P_A

$\dim E_{-3} = 1 =$ . . . . -3 . . . . .

$$M = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad M^{-1}AM = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Problem 4

$$\begin{cases} a_0 = 2 \\ a_1 = 4 \\ a_{n+2} = 4a_{n+1} - 5a_n & \forall n \geq 0 \end{cases}$$

1) The characteristic polynomial is $t^2 - (4t - 5) = t^2 - 4t + 5$.

Its roots are $\dfrac{4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = 2 \pm \sqrt{4-5} = 2 \pm i$.

Both have multiplicity 1.

So there exist $\gamma_1, \gamma_2 \in \mathbb{C}$ s.t.

$\forall n \geq 0 \quad a_n = \gamma_1 \cdot (2+i)^n + \gamma_2 \cdot (2-i)^n$

Now find γ_1, γ_2 by considering a_0, a_1:

$$\begin{cases} 2 = a_0 = \gamma_1 + \gamma_2 \\ 4 = a_1 = \gamma_1(2+i) + \gamma_2(2-i) \end{cases} \Rightarrow \gamma_1 = \gamma_2 = 1$$

$\Rightarrow \forall n \geq 0 \quad a_n = (2+i)^n + (2-i)^n$.

5) $|2+i| = |2-i| = \sqrt{5}$

For $n \to +\infty$ $|a_n|$ goes to $+\infty$ as $(\sqrt{5})^n$; hence $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \sqrt{5}$

Hence $R = \dfrac{1}{\sqrt{5}}$

6) For $z \in \mathbb{C}$ s.t. $|z| < \dfrac{1}{\sqrt{5}}$, we have $|(2+i)z| < 1$ and $|(2-i)z| < 1$

hence

$$\sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} ((2+i)z)^n + \sum_{n\geq 0} ((2-i)z)^n = \frac{1}{1-(2+i)z} + \frac{1}{1-(2-i)z}$$

$$= \frac{1-(2-i)z + 1-(2+i)z}{(1-(2+i)z)(1-(2-i)z)} = \frac{2-4z}{1-(2+i)z-(2-i)z+5z^2} = \frac{2-4z}{1-4z+5z^2}$$

hence $p(z) = 2-4z$

$q(z) = 1-4z+5z^2$

Problem 5 $A = \{ z \in \mathbb{C} \mid 0 < |z| < 2 \}$, $f : A \longrightarrow \mathbb{C}$

$$f(z) = \frac{1+3z}{z(z^2+4)}$$

Find $a, b, c \in \mathbb{C}$ s.t. $f(z) = \frac{a}{z} + \frac{b}{z-2i} + \frac{c}{z+2i} =$

$$= \frac{a}{z} + \frac{bz + 2ib + cz - 2ic}{z^2+4} = \frac{a(z^2+4) + (b+c)z^2 + 2i(b-c)z}{z(z^2+4)} =$$

$$= \frac{(a+b+c)z^2 + 2i(b-c)z + 4a}{z(z^2+4)}$$

$$\begin{cases} a+b+c = 0 \\ 2i(b-c) = 3 \\ 4a = 1 \end{cases} \qquad \begin{cases} a = 1/4 \\ b+c = -1/4 \\ b-c = \frac{3}{2i} = -\frac{3}{2}i \end{cases}$$

$$2b = -\frac{1}{4} - \frac{3}{2}i$$

$$b = -\frac{1}{8} - \frac{3}{4}i$$

$$c = -\frac{1}{8} + \frac{3}{4}i$$

If $0<|z|<2$ then

$$\frac{1}{z-2i}=\frac{1}{2i}\frac{1}{\frac{z}{2i}-1}=-\frac{1}{2i}\frac{1}{1-\frac{z}{2i}}=\frac{i}{2}\frac{1}{1+\frac{iz}{2}}=\frac{i}{2}\cdot\frac{1}{1-\left(-\frac{iz}{2}\right)}=$$

$$=\frac{i}{2}\cdot\sum_{m\geq 0}\left(-\frac{iz}{2}\right)^m=\sum_{m\geq 0}\frac{i}{2}\cdot\frac{(-i)^m}{2^m}z^m$$

$$\frac{1}{z+2i}=\frac{1}{2i}\frac{1}{\frac{z}{2i}+1}=-\frac{i}{2}\frac{1}{1-\frac{iz}{2}}=-\frac{i}{2}\sum_{m\geq 0}\left(\frac{iz}{2}\right)^m=\sum_{m\geq 0}\frac{-i\cdot i^m}{2^{m+1}}z^m$$

$$f(z)=az^{-1}+\sum_{m\geq 0}\left[b\frac{i}{2}\frac{(-i)^m}{2^m}+c\frac{(-i)\cdot i^m}{2^{m+1}}\right]z^m$$

$$=\sum_{n\in\mathbb{Z}}u_n z^{-n}$$

If $n \geqslant 2$ then $u_n = 0$.

If $n = 1$ then $u_1 = a = \frac{1}{4}$

If $n \leqslant 0$ then

$$u_n = \frac{1}{2^{1-n}} \cdot \left[b \cdot i \cdot (-i)^{-n} + c \cdot (-i) \cdot i^{-n} \right]$$

In particular:

- if $n = -2k$, with $k \geqslant 0$, then

$$u_{-2k} = \frac{1}{2^{1+2k}} \left[b\, i\, (-i)^{2k} + c\, (-i)\, i^{2k} \right]$$

$$= \frac{1}{2^{2k+1}} \left[b\, i\, (-1)^k + c\, (-i)\, (-1)^k \right]$$

$$= \frac{(-1)^k}{2^{2k+1}}(b-c)i = \frac{(-1)^k}{2^{2k+1}}\left(-\frac{3}{2}i\right)i = \frac{3\cdot(-1)^k}{2^{2k+2}}$$

- if $n=-2k-1$, with $k\geq 0$, then

$$u_{-2k-1} = \frac{1}{2^{2k+2}}\left[b\cdot i\cdot(-i)^{2k+1} + c\cdot(-i)\, i^{2k+1}\right]$$

$$= \frac{1}{2^{2k+2}}\left[b\cdot i\cdot(-1)\cdot(-1)^k\cdot i + c\cdot(-i)\cdot(-1)^k\cdot i\right]$$

$$= \frac{(-1)^k}{2^{2k+2}}\left[b+c\right] = \frac{(-1)^k}{2^{2k+2}}\left(-\frac{1}{4}\right) = \frac{(-1)^{k+1}}{2^{2k+4}}$$

Hence:

$$u_n = \begin{cases} 0 & \text{if } n \geq 2 \\ \frac{1}{4} & \text{if } n = 1 \\ \frac{3 \cdot (-1)^k}{2^{2k+2}} & \text{if } n = -2k,\ k \geq 0 \\ \frac{(-1)^{k+1}}{2^{2k+4}} & \text{if } n = -2k-1,\ k \geq 0 \end{cases}$$