$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ non-negative integers

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ integers

The symbols $\sum_{n \geqslant 0}$, $\sum_{n=0}^{+\infty}$, $\sum_{n \in \mathbb{N}}$ are equivalent.

The symbols $\sum_{n=-\infty}^{+\infty}$, $\sum_{n \in \mathbb{Z}}$ are equivalent.

A POLYNOMIAL is $\sum_{n=0}^{d} a_n z^n$ where $a_0, \dots, a_d \in \mathbb{C}$.

A POWER SERIES is a sort of an infinite polynomial:

$$\sum_{n\geq 0} a_n z^n = \sum_{n=0}^{+\infty} a_n z^n = \sum_{n\in\mathbb{N}} a_n z^n$$

where $(a_n)_{n\geq 0} = (a_n)_{n\in\mathbb{N}}$ is a non-negatively indexed sequence of complex numbers.

A polynomial can be evaluated at any $z\in\mathbb{C}$.
The set of $z\in\mathbb{C}$ s.t. $\sum_{n=0}^{+\infty} a_n z^n$ converges is called the REGION OF CONVERGENCE of the power series.

REGION OF CONVERGENCE OF A POWER SERIES

indexed by non-negative integers

Theorem (Cauchy-Hadamard) Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers.

Let $R = \dfrac{1}{\limsup\limits_{n\to+\infty} \sqrt[n]{|a_n|}} \in [0, +\infty]$

Let $\Gamma = \left\{ z \in \mathbb{C} \mid \sum\limits_{n=0}^{+\infty} a_n z^n \text{ converges} \right\}$. Then:

- if $R = 0$ then $\Gamma = \{0\}$
- if $0 < R < +\infty$ then $\{z \in \mathbb{C} \mid |z| < R\} \subseteq \Gamma \subseteq \{z \in \mathbb{C} \mid |z| \leq R\}$
- if $R = +\infty$ then $\Gamma = \mathbb{C}$

POWER SERIES AND HOLOMORPHIC FUNCTIONS ON AN OPEN BALL

<u>Thm</u> Fix $r \in (0, +\infty]$ and $p \in \mathbb{C}$. Let $B_r(p) = \{z \in \mathbb{C} \mid |z-p| < r\}$ be the open ball with centre p and radius r. Then there is a 1-1 correspondence (bijection):

$$\left\{ (a_n)_{n\geq 0} \;\middle|\; \frac{1}{\limsup\limits_{n\to+\infty} \sqrt[n]{|a_n|}} \geq r \right\} \overset{1:1}{\simeq} \left\{ \begin{array}{c} \text{holomorphic functions} \\ B_r(p) \longrightarrow \mathbb{C} \end{array} \right\}$$

$$(a_n)_{n\geq 0} \longmapsto \left[\begin{array}{l} B_r(p) \longrightarrow \mathbb{C} \\ z \longmapsto \sum_{n=0}^{+\infty} a_n (z-p)^n \end{array} \right]$$

$$\left(\frac{f^{(n)}(p)}{n!} \right)_{n\geq 0} \longleftarrow\!\!\!\shortmid \left[f : B_r(p) \longrightarrow \mathbb{C} \right]$$

Note: $\displaystyle \frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi} \int_0^{2\pi} f(p+se^{it})(se^{it})^{-n}\,dt$ where s is any real number s.t. $0 < s < r$

Properties of the correspondence between power series and holomorphic functions on an open ball

- **ADDITIVITY**: if $(a_n)_n$ corresponds to f and $(b_n)_n$ corresponds to g, then $(a_n+b_n)_n$ corresponds to $f+g$
- **MULTIPLICATION BY SCALAR**: if $(a_n)_n$ corresponds to f and $\lambda \in \mathbb{C}$, then $(\lambda a_n)_n$ corresponds to λf
- **CAUCHY/CONVOLUTION PRODUCT CORRESPONDS TO PRODUCT OF FUNCTION**

Let $(a_n)_{n\geq 0}$ (resp. $(b_n)_{n\geq 0}$) correspond to f (resp. g).
Define $c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$ for every $n \geq 0$. Then $(c_n)_{n\geq 0}$ corresponds to $f \cdot g$; indeed

$$f(z)\cdot g(z) = \Big(\sum_{k\geq 0} a_k (z-p)^k\Big)\Big(\sum_{m\geq 0} b_m (z-p)^m\Big) = \sum_{\substack{k\geq 0\\ m\geq 0}} a_k \cdot b_m (z-p)^{k+m}$$

$$= \sum_{n\geq 0}\Big(\sum_{\substack{k,m\geq 0\\ \text{s.t. } k+m=n}} a_k \cdot b_m\Big)(z-p)^n = \sum_{n\geq 0} c_n (z-p)^n.$$

- Assume $p=0$. If $(a_n)_{n\geq 0}$ corresponds to f and $\gamma \in \mathbb{C}$, $|\gamma| \leq 1$, then $f(z) = \sum_{n\geq 0} a_n z^n$, hence
$$f(\gamma z) = \sum_{n\geq 0} a_n (\gamma z)^n = \sum_{n\geq 0} a_n \gamma^n z^n,$$ i.e.
$(\gamma^n a_n)_{n\geq 0}$ corresponds to the function $z \mapsto f(\gamma z)$

- The most important equality is $\sum_{n=0}^{+\infty} \alpha^n = \dfrac{1}{1-\alpha}$ if $\alpha \in \mathbb{C}$, $|\alpha| < 1$.
For example the constant sequence $(1)_{n\geq 0}$ corresponds to function
$$B_1(p) \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{1}{1-(z-p)}$$

- If $(a_n)_{n\geq 0}$ corresponds to f, then $(a_{n+1} \cdot (n+1))_{n\geq 0}$ corresponds to f'.

$$f(z) = \sum_{n\geq 0} a_n (z-p)^n, \quad f'(z) = \sum_{n\geq 1} a_n \cdot n\, (z-p)^{n-1} = \sum_{m\geq 0} a_{m+1} \cdot (m+1)(z-p)^m$$

A LAURENT SERIES is $\sum_{n=-\infty}^{+\infty} a_n z^n$ where $(a_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers indexed by all integers.

We split it into the two parts

- $\sum_{n\geq 0} a_n z^n$ converges if $|z| < \dfrac{1}{\limsup_{n\to+\infty} \sqrt[n]{|a_n|}}$

- $\sum_{n<0} a_n z^n = \sum_{k\geq 1} a_{-k} z^{-k} = \sum_{k\geq 1} a_{-k} (z^{-1})^k$ converges if

$$\frac{1}{|z|} = |z^{-1}| < \frac{1}{\limsup_{k\to+\infty} \sqrt[k]{|a_{-k}|}}, \text{ i.e.}$$

$$|z| > \limsup_{k\to+\infty} \sqrt[k]{|a_{-k}|} = \limsup_{n\to-\infty} \sqrt[-n]{|a_n|}$$

REGION OF CONVERGENCE OF A LAURENT SERIES

<u>Prop</u> Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers indexed by all integers. Let Γ be the region of convergence of the Laurent series $\sum_{n\in\mathbb{Z}} a_n z^n$.

Define:

$$R^+ := \frac{1}{\limsup_{n\to+\infty} \sqrt[n]{|a_n|}} \qquad R^- := \limsup_{n\to-\infty} \sqrt[-n]{|a_n|}$$

- If $R^- < R^+$ then $\{z\in\mathbb{C} \mid R^- < |z| < R^+\} \subseteq \Gamma \subseteq \{z\in\mathbb{C} \mid R^- \leq |z| \leq R^+\}$
- If $R^- = R^+$ then $\Gamma \subseteq \{z\in\mathbb{C} \mid |z| = R^+\}$
- If $R^- > R^+$ then $\Gamma = \emptyset$

LAURENT SERIES AND HOLOMORPHIC FUNCTIONS ON ANULI

Thm Fix $p \in \mathbb{C}$ and $0 \leq r^- < r^+ \leq +\infty$. Let $A = \{z \in \mathbb{C} \mid r^- < |z-p| < r^+\}$ be the anulus with centre p, inner radius r^-, inner radius r^+.
Then there is a 1-1 (bijective) correspondence:

$$\left\{ (a_n)_{n\in\mathbb{Z}} \;\middle|\; \begin{array}{l} \dfrac{1}{\limsup\limits_{n\to+\infty} \sqrt[n]{|a_n|}} \geq r^+ \\ \limsup\limits_{n\to-\infty} \sqrt[-n]{|a_n|} \leq r^- \end{array} \right\} \overset{1:1}{\simeq} \left\{ \begin{array}{c} \text{holomorphic functions} \\ A \to \mathbb{C} \end{array} \right\}$$

$$(a_n)_{n\in\mathbb{Z}} \longmapsto \left[\begin{array}{l} A \longrightarrow \mathbb{C} \\ z \longmapsto \sum\limits_{n\in\mathbb{Z}} a_n (z-p)^n \end{array} \right]$$

$$\left(\frac{1}{2\pi} \int_0^{2\pi} f(p+se^{it}) \cdot (se^{it})^{-n} \, dt \right)_{n\in\mathbb{Z}} \longleftarrow\!\!\!\shortmid \left[f : A \longrightarrow \mathbb{C} \right]$$

where $s \in \mathbb{R}$ is s.t.
$r^- < s < r^+$

Properties of the correspondence between Laurent series and holomorphic functions on anuli

- ADDITIVITY
- MULTIPLICATION BY SCALAR
- CAUCHY / CONVOLUTION PRODUCT CORRESPONDS TO PRODUCT OF FUNCTIONS

$(a_n)_{n\in\mathbb{Z}} \longleftrightarrow f$, $(b_n)_{n\in\mathbb{Z}} \longleftrightarrow g$

Define $c_n = \sum_{k\in\mathbb{Z}} a_k b_{n-k}$ for every $n\in\mathbb{Z}$. Then $(c_n)_{n\in\mathbb{Z}} \longleftrightarrow f\cdot g$

- If $p=0$, $\gamma\in\mathbb{C}\setminus\{0\}$, $(a_n)_{n\in\mathbb{Z}} \longleftrightarrow f$, then $(\gamma^n a_n)_{n\geq 0}$ corresponds to the function $z\mapsto f(\gamma z)$
- If $(a_n)_{n\in\mathbb{Z}} \longleftrightarrow f$ then $(a_{n+1}\cdot(n+1))_{n\in\mathbb{Z}} \longleftrightarrow f'$

SLOGAN: The Z-transform in signal processing is a slight modification of the bijective correspondence between sequences of complex numbers indexed by integers and holomorphic functions on anuli. The two modifications are the following:

- $p = 0$. the centre of the anulus is the origin
- z is swapped with z^{-1} and is swapped with $-n$.

<u>Def</u> If $(u_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers indexed by all integers, then its Z-transform is the Laurent series $\sum_{n\in\mathbb{Z}} u_n z^{-n}$

REGION OF CONVERGENCE OF THE Z-TRANSFORM

<u>Prop</u> Let $(u_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers.
Let Γ be the region of convergence of $\sum_{n\in\mathbb{Z}} u_n z^{-n}$.

Define
$$\rho_- := \limsup_{n\to+\infty} \sqrt[n]{|u_n|} \qquad \text{and} \qquad \rho_+ := \frac{1}{\limsup_{n\to-\infty} \sqrt[-n]{|u_n|}}$$

Then:

- If $\rho_- < \rho_+$ then $\{z\in\mathbb{C} \mid \rho_- < |z| < \rho_+\} \subseteq \Gamma \subseteq \{z\in\mathbb{C} \mid \rho_- \leq |z| \leq \rho_+\}$
- If $\rho_- = \rho_+$ then $\Gamma \subseteq \{z\in\mathbb{C} \mid |z| = \rho_- = \rho_+\}$
- If $\rho_- > \rho_+$ then $\Gamma = \emptyset$

Z - TRANSFORM

<u>Thm</u> Fix $0 \leqslant r^- < r^+ \leq +\infty$. Let $A = \{z \in \mathbb{C} \mid r^- < |z| < r^+\}$.
The following is a bijective correspondence:

$$\left\{ (u_n)_{n\in\mathbb{Z}} \;\middle|\; \begin{array}{l} \limsup\limits_{n\to+\infty} \sqrt[n]{|u_n|} \leqslant r^- \\[2ex] \dfrac{1}{\limsup\limits_{n\to-\infty} \sqrt[-n]{|u_n|}} \geqslant r^+ \end{array} \right\} \overset{1:1}{\simeq} \left\{ \begin{array}{c} \text{holomorphic functions} \\ A \longrightarrow \mathbb{C} \end{array} \right\}$$

$$(u_n)_{n\in\mathbb{Z}} \xmapsto{\text{Z- transform}} \left[\begin{array}{l} A \longrightarrow \mathbb{C} \\ z \longmapsto \sum\limits_{n\in\mathbb{Z}} u_n \cdot z^{-n} \end{array} \right]$$

$$\left(\frac{1}{2\pi} \int_0^{2\pi} f(se^{it}) \cdot (se^{it})^n \, dt \right)_{n\in\mathbb{Z}} \xleftarrow[\quad]{\text{inverse Z- transform}} [f : A \longrightarrow \mathbb{C}]$$

where $s \in \mathbb{R}$ is s.t.
$r^- < s < r^+$

PROPERTIES OF THE Z-TRANSFORM

- Additivity and multiplication by scalar
- Cauchy/convolution product corresponds to product of functions
- If f is the Z-transform of $(u_n)_{n\in\mathbb{Z}}$ and $\gamma \in \mathbb{C}\setminus\{0\}$, then the Z-transform of $(\gamma^n u_n)_{n\in\mathbb{Z}}$ is the function
  $z \longmapsto f(\gamma^{-1} z)$
- The Z-transform of the sequence $u_n = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$ is the function $\{z\in\mathbb{C} \mid |z|>1\} \longrightarrow \mathbb{C}$
  $$z \longmapsto \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$
- The Z-transform of the sequence $u_n = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n \leq 0 \end{cases}$ is the function $\{z\in\mathbb{C} \mid |z|<1\} \longrightarrow \mathbb{C}$
  $$z \longmapsto \frac{1}{1-z}$$