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The hyperelliptic locus in the moduli stack of curves of given genus

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But what have I done with my life? thought Mrs. Ramsay, taking her place at the head of the table, and looking at all the plates making white circles on it. ‘William, sit by me,’ she said. ‘Lily,’ she said, wearily, ‘over there.’ They had that – Paul Rayley and Minta Doyle – she, only this – an infinitely long table and plates and knives. At the far end, was her husband, sitting down, all in a heap, frowning. What at? She did not know. She did not mind. She could not understand how she had ever felt any emotion or affection for him. She had a sense of being past everything, through everything, out of everything, as she helped the soup, as if there was an eddy – there – and one could be in it, or one could be out of it, and she was out of it. It’s all come to an end, she thought, while they came in one after another, Charles Tansley – ‘Sit there, please,’ she said – Augustus Carmichael – and sat down. And meanwhile she waited, passively, for some one to answer her, for something to happen. But this is not a thing, she thought, ladling out soup, that one says.

To the Lighthouse, Virginia Woolf

George Gray

I have studied many times
The marble which was chiseled for me –
A boat with a furled sail at rest in a harbor.
In truth it pictures not my destination
But my life.
For love was offered me and I shrank from its disillusionment;
Sorrow knocked at my door, but I was afraid;
Ambition called to me, but I dreaded the chances.
Yet all the while I hungered for meaning in my life.
And now I know that we must lift the sail
And catch the winds of destiny
Wherever they drive the boat.
To put meaning in one's life may end in madness,
But life without meaning is the torture
Of restlessness and vague desire –
It is a boat longing for the sea and yet afraid.

Spoon River Anthology, Edgar Lee Masters

Abstract

In this thesis we recall the construction of $\mathcal{M}_g$, the stack of smooth curves of genus g. Then we define the stack $\mathcal{H}_g$ of hyperelliptic curves of genus g as the fibred category whose objects are scheme morphisms $X \to P \to S$, where $X \to P$ is faithfully flat of degree 2, $X \to S$ is a smooth curve of genus g and $P \to S$ is a smooth curve of genus 0. We prove that $\mathcal{H}_g$ is a closed substack of $\mathcal{M}_g$ if $g \geq 2$. If H_g and M_g are the moduli spaces of the stacks $\mathcal{H}_g$ and $\mathcal{M}_g$, we prove that H_g is a closed subscheme of M_g, when $g \geq 2$ and the characteristic is zero.

Contents

Introduction

Moduli of curves

Moduli of curves have fascinated mathematicians since the nineteenth century, thanks to the work of Bernhard Riemann ([**Rie57**]) who in 1857 was the first to consider a space, M_g, whose points correspond to isomorphism classes of smooth complex curves of genus g. By viewing curves as branched covers of $\mathbb{P}^1$, Riemann correctly computed the number of *moduli*, i.e. he showed that $\dim M_g = 3g - 3$ for all $g \geq 2$. In the first half of the twentieth century the study of M_g was tackled by several mathematicians, such as Lars Ahlfors, Oswald Teichmüller and Lipman Bers. In what follows the letter g stands for an integer which is greater than 1.

The first rigorous construction of M_g was carried out by David Mumford in 1965. His method rests on geometric invariant theory ([**MFK94**]) and works in every characteristic. More precisely, let us consider the functor

$$F_{\mathcal{M}_g} \colon (\mathrm{Sch})^{\mathrm{op}} \to (\mathrm{Set})$$

defined as follows: for every scheme S, $F_{\mathcal{M}_g}(S)$ is the set of isomorphism classes of smooth curves of genus g over S, i.e. proper smooth morphisms $X \to S$ whose geometric fibres are connected curves of genus g (Definition 3.1). Since the class of smooth curves of genus g is stable under base change (Proposition 3.3), the functor $F_{\mathcal{M}_g}$ is well defined. Now one may ask whether the functor $F_{\mathcal{M}_g}$ is representable, i.e. there exists a scheme $\mathbb{M}$ such that $F_{\mathcal{M}_g} \simeq \mathrm{Hom}(-, \mathbb{M})$; in other words such an $\mathbb{M}$ has to satisfy the following property: there exists a smooth curve $\mathbb{X} \to \mathbb{M}$ of genus g (called the *universal curve*) such that every smooth curve $X \to S$ is the pull-back of $\mathbb{X} \to \mathbb{M}$ along a unique morphism $S \to \mathbb{M}$.

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{X} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathbb{M} \end{array}$$

Such an $\mathbb{M}$ is called a *fine moduli space* for smooth curves of genus g and is unique up to unique isomorphism, if it exists. Unfortunately the functor $F_{\mathcal{M}_g}$ is not representable, that is there does not exist a fine moduli space for smooth curves of genus g. The reason is that there are non-trivial families of smooth curves of genus g, all of whose fibres are isomorphic to each other, as explained in the example below.

Example 0.1. Let k be an algebraically closed field, let C be a smooth proper connected curve over k of genus g, and let σ be a non-trivial automorphism of C. Consider the nodal curve S over k made up of two integral curves S_1 and S_2 meeting at two distinct points p and q. Consider the trivial families $C \times S_1$ and $C \times S_2$ over S_1 and S_2 and glue them by the identity over p and by σ over q. Obtain a smooth curve $X \to S$ of genus g whose fibres are C, but X is not isomorphic to the trivial family $C \times S$.

Suppose absurdly that there exists a fine moduli space $\mathbb{M}$ for smooth curves of genus g. Let $f\colon S \to \mathbb{M}$ be the morphism corresponding to the family X and let $f_0\colon S \to \mathbb{M}$ the morphism corresponding to the trivial family $C \times S$. Since $X|_{S\setminus\{p\}}$ and $X|_{S\setminus\{q\}}$ are trivial, $f|_{S\setminus\{p\}} = f_0|_{S\setminus\{p\}}$ and $f|_{S\setminus\{q\}} = f_0|_{S\setminus\{q\}}$, then $f = f_0$, but this is absurd because $X \not\cong C \times S$.

Since the functor $F_{\mathcal{M}_g}$ is not representable, there are two possibilities to face the problem: considering a representable functor which is quite near to $F_{\mathcal{M}_g}$ or enlarging the category of schemes to a category of geometric objects that may represent $F_{\mathcal{M}_g}$.

The first solution deals with the possibility to find a scheme M with a natural transformation $\phi\colon F_{\mathcal{M}_g} \to \mathrm{Hom}(-, M)$ of contravariant functors from the category of schemes to the category of sets such that:

(1) for all algebraically closed fields Ω, the function
$$\phi(\operatorname{Spec}\Omega)\colon F_{\mathcal{M}_g}(\operatorname{Spec}\Omega) \to \mathrm{Hom}(\operatorname{Spec}\Omega, M)$$
is bijective;

(2) given any scheme N and any natural transformation $\psi\colon F_{\mathcal{M}_g} \to \mathrm{Hom}(-, N)$, there is a unique scheme morphism $\chi\colon M \to N$ such that $\psi = \mathrm{Hom}(-, \chi) \circ \phi$.

If such an M exists, it is unique up to unique homomorphism and is called the *coarse moduli space* of smooth curves of genus g. Using geometric invariant theory, David Mumford has proven that such an M exists ([**MFK94**, Theorem 5.11]) and is quasi-projective over any open subset of $\operatorname{Spec}\mathbb{Z}$ which is different from the whole $\operatorname{Spec}\mathbb{Z}$ ([**MFK94**, Corollary 7.14, p. 143]). The coarse moduli space of curves of genus g is denoted by M_g. Condition (1) says that the geometric points of M_g correspond bijectively to isomorphism classes of smooth curves of genus g; this is the modern formulation of Riemann's intuition about the moduli space of curves of genus g. Condition (2) says that the functor represented by M_g is the nearest to $F_{\mathcal{M}_g}$ among the representable functors.

The second solution consists of enlarging the category of schemes to the category of *stacks*. With stacks moduli problems become often tautologically representable because the stack is essentially the moduli problem itself. Algebraic stacks, which are morally "sheaves of categories" which satisfy some finiteness hypotheses and are near to be schemes with respect to the étale topology, were introduced by David Mumford in the seminal papers [**Mum65**] and [**DM69**] and then studied by Micheal Artin in [**Art74**]. The stack $\mathcal{M}_g$ of smooth curves of genus g is the category whose objects are families $X \to S$ of smooth curves of genus g and whose arrows are cartesian diagrams

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}$$

of smooth curves of genus g. If $\mathcal{M}_g$ is equipped with the functor $\mathcal{M}_g \to$ (Sch) that sends a smooth curve $(X \to S)$ into the scheme S, it is easily seen that $\mathcal{M}_g$ is a category fibred in groupoids over (Sch). Besides one may prove that étale descent data of $\mathcal{M}_g$ are effective and that the functor of isomorphisms of two smooth curves of genus $g \geq 2$ is finite and unramified (Proposition 3.33); in the language of stacks this means that $\mathcal{M}_g$ is a separated Deligne-Mumford stack (Theorem 3.34).

Nowadays, thanks to the work of Seán Keel and Shigefumi Mori ([**KM97**]), one may reconstruct the coarse moduli space M_g from the stack $\mathcal{M}_g$: there exists a scheme M_g with a morphism $\phi\colon \mathcal{M}_g \to M_g$ such that:

(1) for all algebraically closed fields Ω, the function

$$\phi(\operatorname{Spec}\Omega)\colon \mathcal{M}_g(\operatorname{Spec}\Omega)/\text{isomorphism} \to M_g(\operatorname{Spec}\Omega)$$

is bijective;

(2) given any scheme N and any morphism $\psi\colon \mathcal{M}_g \to N$, there is a unique scheme morphism $\chi\colon M_g \to N$ such that $\psi = \chi \circ \phi$;

(3) ϕ is proper and quasi-finite.

Then $\mathcal{M}_g$ and M_g have the same geometric points and the same connected components, but $\mathcal{M}_g$ is able to keep track of automorphisms of curves and "represents tautologically" the moduli problem of smooth curves of genus g.

Moduli of hyperelliptic curves

When we talk about hyperelliptic curves, the characteristic of the residue field of every point of all schemes is different from 2, that is we work over $\mathbb{Z}[1/2]$. A *hyperelliptic curve* over an algebraically closed field is a smooth connected curve which possesses a finite morphism of degree two onto the projective line $\mathbb{P}^1$. What is a sensible moduli problem about hyperelliptic curves of genus g?

A first attempt might be to consider the category fibred in groupoids $\mathcal{H}_g^*$ defined as follows: objects of $\mathcal{H}_g^*$ over a scheme S are families

$X \to S$ of smooth curves of genus g such that the geometric fibres are hyperelliptic. Clearly $\mathcal{H}_g^*$ is a subcategory of $\mathcal{M}_g$, but one may prove that $\mathcal{H}_g^*$ is not a stack because there is no link among the various double covers of the fibres, so $\mathcal{H}_g^*$ is very wild.

A second attempt is to consider the category fibred in groupoids $\mathcal{H}_g$ defined as follows: objects of $\mathcal{H}_g$ over a scheme S are pairs of scheme morphisms $X \to P \to S$ such that $X \to P$ is a double cover (i.e. finite finitely presented faithfully flat morphism of degree 2), $P \to S$ is a family of smooth curves of genus 0 and the composite $X \to S$ is a family of smooth curves of genus g. In other words, $\mathcal{H}_g$ comes from $\mathcal{H}_g^*$ by fixing the double cover for each fibre of the family of curves of genus g. This choice is undertaken by Knud Lønsted and Steven Kleiman in [**LK79**]. Alessandro Arsie and Angelo Vistoli have proven in [**AV04**] that $\mathcal{H}_g$ is an algebraic stack (Theorem 4.23) and have determined its Picard group.

One may consider the morphism $\mathcal{H}_g \to \mathcal{M}_g$ which maps $(X \to P \to S)$ into $(X \to S)$, i.e. it forgets the double cover. It is well known that this is a closed immersion, but no proof exists in literature. We shall prove this in Theorem 4.26. Our method lies on a useful characterization of closed immersions (Theorem 1.30): a scheme morphism is a closed immersion if and only if it is proper, unramified and injective on geometric points. The morphism $\mathcal{H}_g \to \mathcal{M}_g$ is proper because the valuative criterion is satisfied. It is unramified because it factors naturally as $\mathcal{H}_g \to \mathcal{I}_{\mathcal{M}_g} \to \mathcal{M}_g$, where $\mathcal{I}_{\mathcal{M}_g}$ is the inertia stack of $\mathcal{M}_g$, which is unramified over $\mathcal{M}_g$ because $\mathcal{M}_g$ is a Deligne-Mumford stack, and $\mathcal{H}_g \to \mathcal{I}_{\mathcal{M}_g}$ is fully faithful. The fact that the morphism $\mathcal{H}_g \to \mathcal{M}_g$ is injective on geometric points is equivalent to the uniqueness of the g_2^1 for a hyperelliptic curve of genus g over an algebraically closed field.

Thanks to Keel-Mori theorem (Theorem 2.28), the stack $\mathcal{H}_g$ admits a coarse moduli space H_g. Historically the moduli space H_g of hyperelliptic curves of genus g had been studied before that the stack $\mathcal{H}_g$ was introduced. For example see [**Igu60**], [**Løn76**] or [**LL78**]. We prove that, in characteristic zero, the moduli space H_g is a closed subscheme of M_g (Proposition 4.28).

Outline of the thesis

In Chapter 1 we study covers, which are finite flat finitely presented scheme morphisms of constant degree, we study the connection with locally free sheaves of algebras and we classify double covers. In Section 1.4 we study radicial and unramified morphisms and we give a useful characterization of closed immersions (Theorem 1.30) which will be used to prove that the stack of hyperelliptic curves is a closed substack of the stack of curves (Theorem 4.26).

In Chapter 2 we review the most important facts of the theory of algebraic spaces and algebraic stacks, following [**LMB00**]. In Section

2.5 we state the Keel-Mori theorem about the coarse moduli space of a separated Deligne-Mumford stack (Theorem 2.28), we determine the coarse moduli space of a quotient stack (Proposition 2.34), and we prove that a closed immersion of separated Deligne-Mumford stacks induces, in characteristic zero, a closed immersion on moduli spaces (Proposition 2.35). This last result will be useful to prove that, in characteristic zero, the coarse moduli space of hyperelliptic curves of genus g is a closed subscheme of the coarse moduli space of smooth curves of genus g (Proposition 4.28).

In Chapter 3 we recall the properties of families of smooth curves of fixed genus and their quotients under the action of a finite group (Theorem 3.21), we introduce $\mathcal{M}_g$ the stack of smooth curves of genus g and we prove that it is a separated Deligne-Mumford stack if $g \geq 2$ (Theorem 3.34). The proof exposed by us avoids to appeal to the theory of minimal surfaces, but uses elementary arguments about blowing-ups of surfaces over discrete valuation rings.

In Chapter 4 we present the classical facts about hyperelliptic curves over an algebraically closed field and we study families of hyperelliptic curves. Then we define the stack $\mathcal{H}_g$ of hyperelliptic curves of genus g and we prove that it is a closed substack of the stack $\mathcal{M}_g$ (Theorem 4.26). Finally we see that this induces a closed immersion on moduli spaces in characteristic zero (Proposition 4.28).

In Appendix A we recall the rudiments of descent theory: we define Grothendieck topologies and sheaves on a site, we introduce categories fibred in groupoids, and we define stacks on a site, which are morally "sheaves of categories".

In Appendix B we summarize some facts about base change theory of a flat coherent sheaf with respect to a proper morphism and we recall the basics of Grothendieck duality theory.

Conventions and terminology

We assume the standard terminology of commutative algebra and of algebraic geometry, like in *Éléments de géométrie algébrique* [**EGA**], with the customary exception of calling a "scheme" what is called there a "préschéma". By a ring we always understand a commutative ring with unit.

An *algebraic curve*, or simply a *curve*, over the field k is a scheme of finite type over k whose irreducible components are of dimension 1.

CHAPTER 1

Some remarks on morphisms of schemes

In this chapter we study covers, which are finite flat finitely presented scheme morphisms of constant degree, we study the connection with locally free sheaves of algebras and we classify double covers. In Section 1.4 we study radicial and unramified morphisms and we give a useful characterization of closed immersions (Theorem 1.30) which will be used to prove that the stack of hyperelliptic curves is a closed substack of the stack of curves (Theorem 4.26).

1.1. Covers

If $f\colon X \to Y$ is a finite morphism of schemes, then $f_*\mathcal{O}_X$ is a quasi-coherent finitely generated $\mathcal{O}_Y$-module, hence for every point $y \in Y$ the stalk $(f_*\mathcal{O}_X)_y$ is a finite $\mathcal{O}_{Y,y}$-module. Therefore we may make the following definition.

Definition 1.1. The *degree* of a finite scheme morphism $f\colon X \to Y$ at a point $y \in Y$ is the cardinality of every minimal basis of $(f_*\mathcal{O}_X)_y$ as a $\mathcal{O}_{Y,y}$-module, i.e.:

$$\deg_y f = \dim_{k(y)} (f_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} k(y).$$

With the notation of the previous definition, we have also

$$\deg_y f = \dim_{k(y)} \mathcal{O}_{X_y}(X_y),$$

because one can easily check that $(f_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} k(y)$ and $\mathcal{O}_{X_y}(X_y)$ are isomorphic as $k(y)$-algebras.

Proposition 1.2. *Let $f\colon X \to Y$ be a finite morphism, let $g\colon Y' \to Y$ be a morphism, let $f'\colon X \times_Y Y' \to Y'$ be the morphism induced by f by base change. For every point $y' \in Y'$, $\deg_{y'} f' = \deg_{g(y')} f$.*

Proof. We can suppose that Y and Y' are affine, hence X and $X' = X \times_Y Y'$ are affine. We have a finite ring homomorphism $A \to B$, an arbitrary ring homomorphism $A \to A'$, and a prime ideal $\mathfrak{p}' \in \operatorname{Spec} A'$. Set $\mathfrak{p} = \mathfrak{p}' \cap A$. The dimension over $k(\mathfrak{p}')$ of

$$(B \otimes_A A') \otimes_{A'} k(\mathfrak{p}') \simeq B \otimes_A k(\mathfrak{p}') \simeq (B \otimes_A k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} k(\mathfrak{p}')$$

is equal to the dimension of $B \otimes_A k(\mathfrak{p})$ over $k(\mathfrak{p})$. □

Definition 1.3. Let $d \geq 1$ be a natural number. A *cover of degree* d is a finite flat finitely presented morphism of schemes $f\colon X \to Y$ such that $\deg_y f = d$ for all $y \in Y$.

PROPOSITION 1.4. *For a morphism of schemes $f\colon X \to Y$, the following conditions are equivalent.*

(1) f is a cover of degree d.
(2) f is finite of finite presentation and the stalk $(f_*\mathcal{O}_X)_y$ is a free $\mathcal{O}_{Y,y}$-module of rank d for all $y \in Y$.
(3) f is affine and $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module of rank d.

If, in addition, Y is locally noetherian and reduced, then the previous conditions are also equivalent with the following one:

(4) f is finite and $\deg_y f = d$ for all $y \in Y$.

PROOF. All conditions are local on the image and in each case f is affine, so we can suppose that X and Y are affine: $A = \mathcal{O}_Y(Y)$, $B = \mathcal{O}_X(X)$ and f is associated to the ring homomorphism $A \to B$.

(1) $\Leftrightarrow$ (2): in this case B is a finite A-module and a finitely presented A-algebra. From [**EGA**, Proposition IV.1.4.7], B is a finitely presented A-module, hence B is flat if and only if it is projective ([**Mat89**, Corollary p.53]).

(3) $\Rightarrow$ (2): we can suppose that B is a free A-module of rank d. Therefore B is a finitely presented A-module, hence is a finitely presented A-algebra ([**EGA**, Proposition IV.1.4.7]) and the localizations of B at primes of A are free modules.

(2) $\Rightarrow$ (3): fix a prime $\mathfrak{p}$ of A and let $x_1, \dots, x_d \in B$ be elements which map to a basis of $B_\mathfrak{p}$ over $A_\mathfrak{p}$. Let $\phi\colon A^d \to B$ be the homomorphism defined by $x_1, \dots, x_d$. Since B is finitely presented A-module, $\ker\phi$ and $\operatorname{coker}\phi$ are finite A-modules; from $(\ker\phi)_\mathfrak{p} = 0$ and $(\operatorname{coker}\phi)_\mathfrak{p} = 0$, there exists $s \in A \setminus \mathfrak{p}$ such that $\phi_s\colon A_s^d \to B_s$ is an isomorphism. This proves that the sheaf $f_*\mathcal{O}_X|_{Y_s}$ is free of rank d.

(1) $\Rightarrow$ (4): obvious.

(4) $\Rightarrow$ (3): apply [**Har77**, Exercise II.5.8] to the coherent sheaf $f_*\mathcal{O}_X$. □

From the proposition above it follows that isomorphisms are precisely covers of degree 1.

PROPOSITION 1.5. *The class of covers of degree d is stable under base change and is local on the codomain in the fpqc topology.*

PROOF. It follows easily from Proposition 1.2, Proposition 1.4 and [**Vis05**, Proposition 2.36]. □

1.2. Locally free sheaves of algebras

Covers of schemes are closely related to locally free sheaves of algebras as follows from the proposition below.

PROPOSITION 1.6. *If X is a scheme, then the following two categories are equivalent:*

(i) *the category* $\mathrm{Cov}_d(X)$ *whose objects are covers* $Y \to X$ *of degree* d *and whose arrows are isomorphisms over* X;
(ii) *the category* $\mathrm{Alg}_d(X)$ *whose objects are* $\mathcal{O}_X$*-algebras which are locally free of rank* d *and whose arrows are isomorphisms.*

PROOF. The functor $\mathrm{Cov}_d(X) \to \mathrm{Alg}_d(X)$ maps the cover $f\colon Y \to X$ into the $\mathcal{O}_X$-algebra $f_*\mathcal{O}_Y$. One of its quasi-inverses is the functor $\mathrm{Alg}_d(X) \to \mathrm{Cov}_d(X)$ that maps the $\mathcal{O}_X$-algebra $\mathcal{A}$ into the cover $\mathcal{S}pec_{\mathcal{O}_X}\mathcal{A} \to X$ defined in [**EGA**, II.1.3]. □

If A is a ring and M is a finite free A-module, one may define the *trace* of an A-linear endomorphism of M. If A is a ring and B is a finite free A-algebra, the multiplication by an element $b \in B$ is a A-linear endomorphism of B and one may consider its trace $\mathrm{tr}_{B/A}(b)$ over A; so we have an A-linear map $\mathrm{tr}_{B/A}\colon B \to A$. Note that $\mathrm{tr}_{B/A}(1) \in A$ is the rank of B over A.

Now we want to globalize this construction to locally free sheaves of algebras over a scheme.

PROPOSITION 1.7. *If* X *is a scheme and* $\mathcal{A}$ *is a sheaf of* $\mathcal{O}_X$*-algebras that is locally free of finite rank, then there exists a unique homomorphism of* $\mathcal{O}_X$*-modules* $\mathrm{tr}_{\mathcal{A}/X}\colon \mathcal{A} \to \mathcal{O}_X$ *such that, for every affine open subset* $U \subseteq X$ *over which* $\mathcal{A}$ *is free,* $(\mathrm{tr}_{\mathcal{A}/X})_U = \mathrm{tr}_{\mathcal{A}(U)/\mathcal{O}_X(U)}$.

PROOF. The set of affine open subsets of X over which $\mathcal{A}$ is free is a base of the topology of X, so it suffices to prove that the homomorphisms defined in the statement are compatible with restrictions. But this is easy because if $A \to A'$ is a ring homomorphism and B is a free A-algebra of rank d, then $B' = B \otimes_A A'$ is a free A'-algebra of rank d and $\mathrm{tr}_{B/A}(b) \otimes 1 = \mathrm{tr}_{B'/A'}(b \otimes 1)$ for all $b \in B$. □

DEFINITION 1.8. If X is a scheme and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras which is locally free of finite rank, the homomorphism of $\mathcal{O}_X$-modules $\mathrm{tr}_{\mathcal{A}/X}\colon \mathcal{A} \to \mathcal{O}_X$ defined in the previous proposition is called the *trace* of $\mathcal{A}$ over X.

PROPOSITION 1.9. *Let* X *be a scheme and let* $\mathcal{A}$ *be a locally free* $\mathcal{O}_X$*-algebra with rank* d.

(1) *The composite of the structure homomorphism* $\mathcal{O}_X \to \mathcal{A}$ *with the trace* $\mathrm{tr}_{\mathcal{A}/X}\colon \mathcal{A} \to \mathcal{O}_X$ *is the homomorphism* $\mathcal{O}_X \to \mathcal{O}_X$ *given by the multiplication by* d.
(2) *If* d *is invertible in* $\mathcal{O}_X(X)$*, then*

$$0 \longrightarrow \ker \mathrm{tr}_{\mathcal{A}/X} \longrightarrow \mathcal{A} \xrightarrow{\mathrm{tr}_{\mathcal{A}/X}} \mathcal{O}_X \longrightarrow 0 \tag{1.1}$$

is a split exact sequence of $\mathcal{O}_X$*-modules and* $\ker \mathrm{tr}_{\mathcal{A}/X}$ *is a locally free* $\mathcal{O}_X$*-modules of rank* $d-1$.

PROOF. (1) If B is a free A-algebra of rank d, then $\mathrm{tr}_{B/A}(1) = d$.

(2) The trace is surjective because d^{-1} has trace 1. The sequence splits because the structure homomorphism $\mathcal{O}_X \to \mathcal{A}$ followed by the multiplication by d^{-1} is a section of the trace.

From $\mathcal{A} = \mathcal{O}_X \oplus \ker \operatorname{tr}_{\mathcal{A}/X}$, one has that $\ker \operatorname{tr}_{\mathcal{A}/X}$ is quasi-coherent finitely presented and its stalks are free modules of rank $d-1$, hence it is locally free of rank $d-1$. □

1.3. Double covers

Definition 1.10. A *double cover* is a cover of degree 2.

Theorem 1.11 (Classification of double covers). *Let X be a scheme such that 2 is invertible in $\mathcal{O}_X(X)$. The following three categories are equivalent:*

(i) the category $\operatorname{Cov}_2(X)$ whose objects are double covers $Y \to X$ and whose arrows are isomorphisms over X;
(ii) the category $\operatorname{Alg}_2(X)$ whose objects are $\mathcal{O}_X$-algebras which are locally free of rank 2 and whose arrows are isomorphisms;
(iii) the category whose objects are pairs $(\mathcal{L}, m)$ where $\mathcal{L}$ is an invertible sheaf on X and $m\colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_X$ is a homomorphism of $\mathcal{O}_X$-modules and whose arrows are compatible isomorphisms.

Proof. The equivalence between (i) and (ii) is proved in Proposition 1.6. Denote by $\mathcal{C}$ the category in (iii) and consider the functor

$$\Phi\colon \operatorname{Alg}_2(X) \to \mathcal{C}$$

defined by

$$\mathcal{A} \mapsto \left(\ker \operatorname{tr}_{\mathcal{A}}, m_{\mathcal{A}}|_{(\ker \operatorname{tr}_{\mathcal{A}})^{\otimes 2}}\right),$$

where $m_{\mathcal{A}}\colon \mathcal{A}^{\otimes 2} \to \mathcal{A}$ is the multiplication. The sheaf $\ker \operatorname{tr}_{\mathcal{A}}$ is invertible by Proposition 1.9(2). To prove that Φ is well defined, one has to check that $m_{\mathcal{A}}$ maps $(\ker \operatorname{tr}_{\mathcal{A}})^{\otimes 2}$ into $\mathcal{O}_X$; this is a local problem, so we can work over the affine open subsets of X over which $\mathcal{A}$ and $\ker \operatorname{tr}_{\mathcal{A}}$ are free: since $\ker \operatorname{tr}_{\mathcal{A}}$ is free of rank 1, we are done because the square of a 2×2 matrix with null trace is scalar:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^2 = \begin{pmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{pmatrix}.$$

Let $(\mathcal{L}, m)$ be an object of $\mathcal{C}$; we are giving the $\mathcal{O}_X$-module $\mathcal{A}_{\mathcal{L}} = \mathcal{O}_X \oplus \mathcal{L}$ a structure of $\mathcal{O}_X$-algebra: the multiplication is given by

$$(s, l) \otimes (s', l') \mapsto (ss' + m(l \otimes l'), sl' + s'l),$$

where s, s' are sections of $\mathcal{O}_X$ and l, l' are sections of $\mathcal{L}$. It is quite easy to check that in this way $\mathcal{A}_{\mathcal{L}}$ is a sheaf of $\mathcal{O}_X$-algebras that is locally free of rank 2. The association $(\mathcal{L}, m) \mapsto \mathcal{A}_{\mathcal{L}}$ defines a functor $\Psi\colon \mathcal{C} \to \operatorname{Alg}_2(X)$. The functor $\Psi \circ \Phi$ is isomorphic to the identity of $\operatorname{Alg}_2(X)$ because for every $\mathcal{A} \in \operatorname{Alg}_2(X)$ the exact sequence (1.1) splits.

To prove that $\Phi \circ \Psi \simeq \mathrm{id}_{\mathcal{C}}$, one has to check that $\ker \mathrm{tr}_{\mathcal{A}_{\mathcal{L}}}$ is $\mathcal{L}$ for every $(\mathcal{L}, m)$ in $\mathcal{C}$. Since the problem is local, we can suppose that X is affine and $\mathcal{L}$ is free generated by a section l, hence $\{1, l\}$ is a basis for $\mathcal{A}_{\mathcal{L}}$; for every $a, b \in \mathcal{O}_X$, the multiplication by $a + bl$ is represented by the matrix

$$\begin{pmatrix} a & b \cdot m(l^2) \\ b & a \end{pmatrix},$$

hence $\mathrm{tr}_{\mathcal{A}_{\mathcal{L}}}(a + bl) = 2a$. Therefore, $a + bl$ has null trace if and only if $a = 0$, i.e. $a + bl \in \mathcal{L}$. □

EXAMPLE 1.12 (Double covers of $\mathbb{P}^1$). Let k be a field of characteristic $\neq 2$, let $P = \mathbb{P}^1_k$ be the projective line over k, let $d \in \mathbb{Z}$ be an integer, and let $F \in k[x_0, x_1]_{2d}$ be a homogeneous polynomial of degree $2d$ in the variables x_0, x_1. Since

$$\begin{aligned} k[x_0, x_1]_{2d} &= \mathrm{H}^0(P, \mathcal{O}_P(2d)) \\ &= \mathrm{H}^0(P, \mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_P(-2d), \mathcal{O}_P)) \\ &= \mathrm{Hom}_{\mathcal{O}_P}(\mathcal{O}_P(-d)^{\otimes 2}, \mathcal{O}_P), \end{aligned}$$

the form F corresponds to the homomorphism $m_F\colon \mathcal{O}_P(-d)^{\otimes 2} \to \mathcal{O}_P$ which is the multiplication by F. Let C_F be the double cover of P that corresponds to the pair $(\mathcal{O}_P(-d), m_F)$ according to Theorem 1.11, i.e. $C_F = \mathcal{S}pec_{\mathcal{O}_P}\mathcal{A}_F$, where $\mathcal{A}_F$ is the $\mathcal{O}_P$-algebra $\mathcal{O}_P \oplus \mathcal{O}_P(-d)$ with product given by

$$(a_1, l_1) \cdot (a_2, l_2) \mapsto (a_1 a_2 + m_F(l_1 \otimes l_2), a_1 l_2 + a_2 l_1).$$

We want to describe explicitly the cover $h\colon C_F \to P$. Let $\{U_0, U_1\}$ be the usual affine covering of P: $U_0 = \operatorname{Spec} k[x_1/x_0]$ and $U_1 = \operatorname{Spec} k[x_0/x_1]$. Since $\mathcal{O}_P(-d)|_{U_0} = x_0^{-d}\mathcal{O}_{U_0}$, $\mathcal{A}_F|_{U_0}$ is a free $\mathcal{O}_{U_0}$-module with basis $\{1, \xi = x_0^{-d}\}$ and the product of $\mathcal{A}_F|_{U_0}$ is given by

$$(a_1 + b_1\xi) \cdot (a_2 + b_2\xi) = a_1 a_2 + \frac{F}{x_0^{2d}} b_1 b_2 + (a_1 b_2 + a_2 b_1)\xi$$

where a_1, a_2, b_1, b_2 are sections of $\mathcal{O}_{U_0}$. So to obtain $\mathcal{A}_F|_{U_0}$ from $\mathcal{O}_{U_0}$ we have added an element ξ which satisfies $\xi^2 = F x_0^{-2d}$. Now it is easy to see that

$$h^{-1}(U_0) = \operatorname{Spec} k\left[\frac{x_1}{x_0}, y\right] \Big/ \left(y^2 - \frac{F}{x_0^{2d}}\right).$$

Analogously

$$h^{-1}(U_1) = \operatorname{Spec} k\left[\frac{x_0}{x_1}, z\right] \Big/ \left(z^2 - \frac{F}{x_1^{2d}}\right).$$

It is clear that C_F is a projective curve over k. When is it smooth over k?

Over the algebraic closure $\bar{k}$ the polynomial F factors as

$$F(x_0, x_1) = \prod_{i=1}^{2d} (b_i x_0 - a_i x_1)$$

for some $a_i, b_i \in \bar{k}$. Obviously a_i, b_i are not uniquely determined, but the points $[a_i, b_i] \in \mathbb{P}^1_{\bar{k}}(\bar{k})$, counted with multiplicities, are uniquely determined by F and are called the roots of F. We say that F is a *smooth form* if F has simple roots, i.e. the zero locus $\mathrm{V}(F)$ is made up of $2d$ distinct closed points in $\mathbb{P}^1_{\bar{k}}$.

With the jacobian criterion it is easy to see that C_F is smooth over k if and only if F is a smooth form. If this is case, C_F is a geometrically connected smooth curve over k of genus $d-1$. It is a hyperelliptic curve, as we will see in Chapter 4.

Now we will see that on a double cover there exists a canonical action of C_2, the cyclic group of order 2. We begin with the affine case.

LEMMA 1.13. *Let A be a ring in which 2 is invertible and let B be an A-algebra. Suppose that B is a finite free A-module of rank 2 and that $\ker \operatorname{tr}_{B/A}$ is a free A-module of rank 1. Then the map $\sigma\colon B \to B$, defined by*

$$\sigma(b) = \operatorname{tr}_{B/A}(b) - b$$

for all $b \in B$, is a homomorphism of A-algebras such that $\sigma \neq \mathrm{id}_B$, $\sigma^2 = \mathrm{id}_B$, and A coincides with the subring of elements which are invariant under the action of σ: $B^{\langle\sigma\rangle} = A$.

PROOF. Let $y \in \ker \operatorname{tr} \subseteq B$ be a generator of $\ker \operatorname{tr}$, then $\{1, y\}$ is a basis of B over A. Since y has zero trace, $y^2 \in A$. It is easy to see that the map σ is given by

$$\sigma(a_1 + a_2 y) = a_1 - a_2 y,$$

for all $a_1, a_2 \in A$. It is clear that σ is A-linear and $\sigma(1) = 1$. We prove that σ is multiplicative:

$$\begin{aligned} \sigma(a_1 + a_2 y) \cdot \sigma(a_3 + a_4 y) &= (a_1 - a_2 y)(a_3 - a_4 y) \\ &= a_1 a_3 + y^2 a_1 a_4 - y(a_1 a_4 + a_2 a_3) \\ &= \sigma(a_1 a_3 + y^2 a_2 a_4 + y(a_1 a_4 + a_2 a_3)). \end{aligned}$$

It is obvious that $\sigma^2 = \mathrm{id}$. An element $a_1 + a_2 y$ is invariant under the action of σ if and only if $2a_2 = 0$, i.e. $a_2 = 0$ because 2 is invertible in A. This proves that $\sigma \neq \mathrm{id}$ and the subring of σ-invariant elements is A. □

PROPOSITION 1.14. *If $f\colon X \to Y$ is a double cover and 2 is invertible in $\mathcal{O}_Y(Y)$, then there is a natural Y-automorphism σ of X of order*

2 *such that the quotient* $X/\langle\sigma\rangle$ *exists and* f *is isomorphic to natural projection* $X \to X/\langle\sigma\rangle$.

PROOF. The local construction in Lemma 1.13 may be easily globalized. □

1.4. Radicial and unramified morphisms

In this section we study radicial and unramified morphisms, in order to have a useful characterization of closed immersions (Theorem 1.30).

DEFINITION 1.15. A morphism $f\colon X \to Y$ of schemes is called *radicial* if, for every field K, the function $f_*\colon X(\operatorname{Spec} K) \to Y(\operatorname{Spec} K)$ is injective.

Sometimes, for brevity, we denote $X(\operatorname{Spec} R)$ by $X(R)$, for every scheme X and every ring R.

Recall that a field extension E/K is said *purely inseparable* if, for every field extension F/K, there exists at most one K-immersion of E into F. It is clear that a field extension E/K is purely inseparable if and only if the morphism $\operatorname{Spec} E \to \operatorname{Spec} K$ is radicial. A non-trivial algebraic field extension E/K is purely inseparable if and only if $\operatorname{char}(K) = p > 0$ and for each $\alpha \in E$ there exists an integer $n \geq 0$ such that $\alpha^{p^n} \in K$ (see [**Lan02**, V, §6]).

LEMMA 1.16. *Let* $f\colon X \to Y$ *be a morphism locally of finite type. Then* f *is surjective if and only if, for every algebraically closed field* Ω*, the function* $f_*\colon X(\operatorname{Spec}\Omega) \to Y(\operatorname{Spec}\Omega)$ *is surjective.*

PROOF. Suppose that f is surjective. Let $\alpha\colon \operatorname{Spec}\Omega \to Y$ be a morphism, where Ω is an algebraically closed field; α factors as the composition of $\operatorname{Spec}\Omega \to \operatorname{Spec} k(y) \to Y$, for some point $y \in Y$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Since f is locally of finite type, Hilbert's Nullstellensatz ([**AM69**, Proposition 7.9]) implies that $k(x)/k(y)$ is a finite field extension. Since Ω is algebraically closed extension of $k(y)$, there exists a $k(y)$-immersion of $k(x) \hookrightarrow \Omega$. The composition of $\operatorname{Spec}\Omega \to \operatorname{Spec} k(x)$ and $i_x\colon \operatorname{Spec} k(x) \to X$ is an element $\beta \in X(\Omega)$ such that $f_*(\beta) = \alpha$. This proves that $f_*\colon X(\Omega) \to Y(\Omega)$ is surjective.

Conversely, suppose that $f_*\colon X(\Omega) \to Y(\Omega)$ is surjective for every algebraically closed field Ω. We want to prove that f is surjective. Let $y \in Y$ be a point and let Ω be an algebraically closed extension of $k(y)$, this gives a morphism $\alpha\colon \operatorname{Spec}\Omega \to Y$ with image y. Since f_* is surjective, there exists a morphism $\beta\colon \operatorname{Spec}\Omega \to X$ such that $f \circ \beta = \alpha$. Let $x \in X$ the point which constitutes the image of the morphism β. It is clear that $f(x) = y$. □

PROPOSITION 1.17. *For a morphism of schemes* $f\colon X \to Y$*, the following conditions are equivalent:*

(i) *f is radicial;*

(ii) *for every algebraically closed field Ω, the function*

$$f_*\colon X(\operatorname{Spec}\Omega) \to Y(\operatorname{Spec}\Omega)$$

is injective;

(iii) *f is universally injective, i.e. for every morphism $Y' \to Y$ the base change morphism $X \times_Y Y' \to Y'$ is injective;*

(iv) *f is injective and, for every point $x \in X$, $k(x)/k(f(x))$ is a purely inseparable field extension;*

(v) *for every algebraically closed field Ω, the function*

$$(\Delta_f)_*\colon X(\operatorname{Spec}\Omega) \to (X \times_Y X)(\operatorname{Spec}\Omega)$$

is surjective;

(vi) *the diagonal morphism $\Delta_f\colon X \to X \times_Y X$ is surjective;*

(vii) *the projection $\mathrm{pr}_j\colon X \times_Y X \to X$ is injective, for $j = 1$ (resp. $j = 2$, or $j \in \{1, 2\}$).*

Proof. (i) ⇒ (ii): obvious.

(ii) ⇒ (iii): consider a cartesian diagram

(1.2)
$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

of morphisms of schemes. We want to prove that f' is injective. Let $x_1, x_2 \in X'$ such that $f'(x_1) = f'(x_2) = y$. Since $k(x_1) \supseteq k(y)$ and $k(x_2) \supseteq k(y)$, there exists an algebraically closed field Ω that contains both $k(x_1)$ and $k(x_2)$ (it suffices to take Ω as the algebraic closure of a residue field of the ring $k(x_1) \otimes_{k(y)} k(x_2)$ at a maximal ideal). For $j = 1, 2$, let $i_j\colon \operatorname{Spec} k(x_j) \to X'$ be the inclusion of the point x_j and let $\alpha_j\colon \operatorname{Spec}\Omega \to \operatorname{Spec} k(x_j)$ the morphism induced by the extension $\Omega \supseteq k(x_j)$.

Since $f'(x_1) = f'(x_2)$, $f'i_1\alpha_1 = f'i_2\alpha_2$, then $gf'i_1\alpha_1 = gf'i_2\alpha_2$, i.e. $fg'i_1\alpha_1 = fg'i_2\alpha_2$. By (ii), $g'i_1\alpha_1 = g'i_2\alpha_2$, that in addition to the equality $f'i_1\alpha_1 = f'i_2\alpha_2$ proves that $i_1\alpha_1 = i_2\alpha_2$ because the diagram (1.2) is cartesian. Then $x_1 = x_2$.

(iii) ⇒ (iv): since f is universally injective, f is injective. Pick $x \in X$, $y = f(x) \in Y$. We want to prove that $k(x)/k(y)$ is purely inseparable. Let K be a field extension of $k(y)$. Suppose that there are two $k(y)$-immersions $k(x) \hookrightarrow K$; they correspond to two morphisms $\alpha_1, \alpha_2\colon \operatorname{Spec} K \to \operatorname{Spec} k(x)$ such that the following diagrams commute for $j = 1, 2$:

$$\begin{array}{ccccc} & & \operatorname{Spec} k(x) & \xrightarrow{i_x} & X \\ & \nearrow^{\alpha_j} & \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} k(y) & \xrightarrow{i_y} & Y \end{array}$$

Consider the cartesian diagram

$$\begin{array}{ccc} X_K & \longrightarrow & X \\ \downarrow{\scriptstyle f_K} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} K & \longrightarrow & Y \end{array}$$

The morphisms $i_x\alpha_1$, $i_x\alpha_2$ induce two sections of f_K. Since f is universally injective, f_K is injective, then $i_x\alpha_1 = i_x\alpha_2$, therefore $\alpha_1 = \alpha_2$.

(iv) $\Rightarrow$ (i): let K be a field and let $\alpha_1, \alpha_2 \colon \operatorname{Spec} K \to X$ be two morphisms such that $f\alpha_1 = f\alpha_2$. Since f is injective, the images of α_1 and α_2 in X are the same point x. From the fact that $k(x)/k(f(x))$ is purely inseparable, one shows that $\alpha_1 = \alpha_2$.

(ii) $\Leftrightarrow$ (v): it holds in every category with fibred products and comes from contemplating the following diagram

$$\begin{array}{ccccc} & \nearrow & X & & \\ & & \downarrow{\scriptstyle \Delta_f} & & \\ \operatorname{Spec}\Omega & \longrightarrow & X \times_Y X & \xrightarrow{\mathrm{pr}_2} & X \\ & & {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow{f} & Y \end{array}$$

(v) $\Leftrightarrow$ (vi): it follows from Lemma 1.16 applied to the diagonal morphism Δ_f which is locally of finite type.

(vi) $\Leftrightarrow$ (vii): the composition

$$X \xrightarrow{\Delta_f} X \times_Y X \xrightarrow{\mathrm{pr}_j} X$$

is the identity of X, for $j = 1, 2$. □

PROPOSITION 1.18. *The class of radicial morphisms is stable under base change and local on the codomain with respect to the fpqc topology.*

PROOF. The stability under base change follows from the fact that radicial morphisms are exactly universally injective morphisms (Proposition 1.17).

Now we prove the local property. Let $f \colon X \to Y$ be a morphism of schemes and let $\{Y_i \to Y\}$ an fpqc covering such that for each i the projection $f_i \colon Y_i \times_Y X \to Y_i$ is radicial. Now we have a cartesian diagram

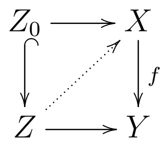
$$\begin{array}{ccc} \coprod_i (Y_i \times_Y X) & \longrightarrow & X \\ {\scriptstyle \coprod_i f_i}\downarrow & & \downarrow{\scriptstyle f} \\ \coprod_i Y_i & \xrightarrow{g} & Y \end{array}$$

The morphism $\coprod_i f_i$ is radicial and g is surjective, then f is radicial by [**EGA**, Proposition 2.6.1(v)]. □

DEFINITION 1.19. A morphism $f\colon X \to Y$ of schemes is called *formally unramified* if, for every Y-scheme Z and for every closed subscheme Z_0 of Z which is defined by a locally nilpotent ideal $\mathcal{I}$ of $\mathcal{O}_Z$, the function

$$\mathrm{Hom}_{(\mathrm{Sch}/Y)}(Z, X) \longrightarrow \mathrm{Hom}_{(\mathrm{Sch}/Y)}(Z_0, X)$$

induced by the closed immersion $Z_0 \hookrightarrow Z$ is injective.



According to [**EGA**, Remarques IV.17.1.2], in the definition above we may suppose that $Z = \operatorname{Spec} A$ is affine and the closed subscheme Z_0 is defined by an ideal I of A such that $I^2 = 0$.

A monomorphism of schemes is clearly formally unramified.

LEMMA 1.20. *Let A be a ring and let I a finitely generated ideal of A such that $I^2 = I$. Then there exist an idempotent element $e \in A$ and an isomorphism $A/I \to A[e^{-1}]$ of A-algebras.*

PROOF. By Nakayama's lemma ([**Mat89**, Theorem 2.2]), there exists $i \in I$ such that $(1 + i)I = 0$. In particular, $0 = (1 + i)i = i + i^2$ and $(1+i)^2 = 1+i+i+i^2 = 1+i$, so $e := 1+i$ is idempotent: $e^2 = e$.

Consider the localization map $\alpha\colon A \to A[e^{-1}]$. It is surjective because $a/e^n = ea/e^{n+1} = ea/e = a/1$, for every a in A. Every element x of I is in the kernel of α because $x/1 = ex/e = 0/e = 0$, since $eI = 0$; therefore α induces a well-defined surjective homomorphism $\beta\colon A/I \to A[e^{-1}]$ of A-algebras.

β is injective because, if $x \in A$ is such that $x/1 = 0$ in $A[e^{-1}]$, then $e^n x = 0$ for some $n \geq 0$; then $ex = 0$, i.e. $(1 + i)x = 0$, and then $x \in I$. □

LEMMA 1.21. *Let $A \to B$ be a ring homomorphism of finite type and let $\rho\colon B \otimes_A B \to B$ be the codiagonal map, i.e. the ring homomorphism defined by*

$$\rho\left(\sum_i b_i \otimes b_i\right) = \sum_i b_i b_i'.$$

If $\Omega^1_{B/A} = 0$, then there exists an idempotent $e \in B \otimes_A B$ such that ρ is isomorphic to the localization morphism $B \otimes_A B \to (B \otimes_A B)[e^{-1}]$.

PROOF. Suppose that $x_1, \ldots, x_n$ generate B as an A-algebra. The ideal $I := \ker \rho$ is generated by $x_i \otimes 1 - 1 \otimes x_i$, $i = 1, \ldots, n$, since [**Ray70**, Lemma III.1(1)]. Therefore I is a finitely generated ideal of $B \otimes_A B$. Since $\Omega^1_{B/A} = 0$, $I = I^2$. Apply Lemma 1.20. □

LEMMA 1.22. *Let K/k be a finite field extension. Then K/k is separable if and only if, for every (some) algebraically closed extension Ω of k, $\Omega \otimes_k K \simeq \Omega \times \cdots \times \Omega$ as Ω-algebras.*

PROOF. Suppose that K/k is separable. By primitive element theorem, there exists $\alpha \in K$ such that $K = k(\alpha)$. Let $p(t) \in k[t]$ be the minimal polynomial of α over k; it has distinct roots $\alpha_1, \ldots, \alpha_n$ in an algebraic closure of k. From $K \simeq k[t]/(p(t))$ we have isomorphisms of Ω-algebras:

$$\begin{aligned}\Omega \otimes_k K &\simeq \Omega \otimes_k k[t]/(p(t)) \\ &\simeq \Omega[t]/(p(t)) \\ &= \Omega[t]/((t-\alpha_1)\cdots(t-\alpha_n)) \\ &\simeq \prod_{i=1}^{n} \Omega[t]/(t-\alpha_i) \\ &\simeq \Omega \times \cdots \times \Omega.\end{aligned}$$

Conversely, suppose that there exists an isomorphism $\phi\colon \Omega \otimes_k K \to \Omega^{\oplus n}$ of Ω-algebras, where $n = [K : k]$. For $i = 1, \ldots, n$, consider the ith projection $\mathrm{pr}_i\colon \Omega^{\oplus n} \to \Omega$ and the homomorphism $\sigma_i\colon K \to \Omega$ defined by $\sigma_i(a) = (\mathrm{pr}_i \circ \phi)(1 \otimes a)$ for every $a \in K$. It is clear that $\sigma_1, \ldots, \sigma_n$ are k-immersions of K into Ω. We must show that they are distinct. If $\sigma_i = \sigma_j$, then the Ω-linear maps $\mathrm{pr}_i \circ \phi$, $\mathrm{pr}_j \circ \phi$ coincide on the subset $\{1 \otimes a \mid a \in K\}$, which generates $\Omega \otimes_k K$ as Ω-vector space; then $\mathrm{pr}_i \circ \phi = \mathrm{pr}_j \circ \phi$, hence $i = j$, because ϕ is an isomorphism. $\square$

PROPOSITION 1.23. *If $f\colon X \to Y$ is a morphism of schemes that is locally of finite type, then the following conditions are equivalent:*

(i) f is formally unramified;
(ii) $\Omega^1_{X/Y} = 0$;
(iii) for every point $y \in Y$, $\Omega^1_{X_y/k(y)} = 0$;
(iv) for every point $y \in Y$, the fibre X_y is a disjoint union of spectra of finite separable field extensions of $k(y)$;
(v) for every geometric point $\operatorname{Spec}\Omega \to Y$, the fibred product $X \times_Y \operatorname{Spec}\Omega$ is isomorphic to a disjoint union of copies of $\operatorname{Spec}\Omega$;
(vi) the diagonal morphism $\Delta_{X/Y}\colon X \to X \times_Y X$ is an open immersion.

PROOF. (i) $\Leftrightarrow$ (ii): this statement has a local nature and holds also without the assumption that f is locally of finite type. See [**EGA**, Proposition IV.17.2.1] or [**Ray70**, Théorème III.2].

(ii) $\Rightarrow$ (iii): obvious.

(iii) $\Rightarrow$ (ii): this is a local statement, hence we can suppose that X and Y are affine. Let B an A-algebra of finite type such that, for every prime ideal $\mathfrak{p}$ of A, $\Omega_{B\otimes_A k(\mathfrak{p})/k(\mathfrak{p})} = 0$. We want to prove that $\Omega_{B/A} = 0$. It suffices to show that $(\Omega_{B/A})_{\mathfrak{q}} = 0$ for every prime ideal $\mathfrak{q}$ of B. Since

B is of finite type over A, $\Omega_{B/A}$ is a finite B-module. By Nakayama's lemma it suffices to show that

$$(\Omega_{B/A})_{\mathfrak{q}} \otimes_{B_{\mathfrak{q}}} B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} = 0.$$

Let $\mathfrak{p} = \mathfrak{q} \cap A$ the prime ideal of A under $\mathfrak{q}$. Considering the commutative diagram of ring homomorphisms

$$\begin{array}{ccccc}
B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} & \longleftarrow & B_{\mathfrak{q}} & & \\
\uparrow & & \uparrow & & \\
B \otimes_A k(\mathfrak{p}) & \longleftarrow & B_{\mathfrak{p}} & \longleftarrow & B \\
\uparrow & & \uparrow & & \uparrow \\
k(\mathfrak{p}) & \longleftarrow & A_{\mathfrak{p}} & \longleftarrow & A
\end{array}$$

we have

$$\begin{aligned}
(\Omega_{B/A})_{\mathfrak{q}} \otimes_{B_{\mathfrak{q}}} B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} &= (\Omega_{B/A} \otimes_B B_{\mathfrak{q}}) \otimes_{B_{\mathfrak{q}}} B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \\
&= (\Omega_{B/A} \otimes_B (B \otimes_A k(\mathfrak{p})) \otimes_{B \otimes_A k(\mathfrak{p})} B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \\
&= \Omega_{B \otimes_A k(\mathfrak{p})/k(\mathfrak{p})} \otimes_{B \otimes_A k(\mathfrak{p})} B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \\
&= 0.
\end{aligned}$$

(iii) $\Leftrightarrow$ (iv): it follows from the following statement which is proved in [**Eis95**, Corollary 16.16]: if A is an algebra of finite type over a field k, then $\Omega_{A/k} = 0$ if and only if A is a finite direct product of fields, each finite and separable over k.

(iv) $\Leftrightarrow$ (v): apply Lemma 1.22 to an affine open cover of each fibre of f.

(vi) $\Rightarrow$ (ii): the sheaf $\Omega^1_{X/Y}$ is isomorphic to $\Delta^*_{X/Y}(\mathcal{I}/\mathcal{I}^2)$, where $\mathcal{I}$ is the sheaf of ideals that defines $\Delta_{X/Y}(X)$ as a closed subscheme of an open subscheme W of $X \times_Y X$. If (vi) holds, then we can take $W = \Delta_{X/Y}(X)$ and $\mathcal{I} = \mathcal{O}_W$.

(ii) $\Rightarrow$ (vi): since the diagonal morphism $\Delta_{X/Y}$ is a locally closed immersion, it suffices to show that it is a local isomorphism. But this follows from Lemma 1.21. $\square$

In the proposition below $k[\varepsilon]$ denotes $k[t]/(t^2)$, the ring of dual numbers over k.

Proposition 1.24. *Let k be an algebraically closed field and let X be a scheme locally of finite type over k. The following statements are equivalent:*

(i) $X \to \operatorname{Spec} k$ *is formally unramified;*
(ii) $\Omega^1_{X/\operatorname{Spec} k} = 0$*;*
(iii) X *is k-isomorphic to a disjoint union of copies of* $\operatorname{Spec} k$*;*
(iv) the diagonal morphism $\Delta\colon X \to X \times_k X$ *is an open immersion;*

(v) *for every k-rational point x of X, the tangent space T_xX is zero;*

(vi) *the function*

$$\mathrm{Hom}_{(\mathrm{Sch}/k)}(\operatorname{Spec} k[\varepsilon], X) \longrightarrow \mathrm{Hom}_{(\mathrm{Sch}/k)}(\operatorname{Spec} k, X)$$

induced by the closed immersion $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]$ *is bijective;*

(vii) $X \to \operatorname{Spec} k$ *is formally étale.*

Proof. The equivalence of (i), (ii), (iii) and (iv) follows from Proposition 1.23. (iii) ⇔ (v): obvious because X is locally of finite type over an algebraically closed field. (v) ⇔ (vi): obvious from the interpretation of the tangent space as the set of morphisms from the spectrum of the ring of dual numbers. (iii) ⇒ (vii): obvious. (v) ⇒ (i): obvious. □

Lemma 1.25. *If $f\colon X \to Y$ is a faithfully flat and quasi-compact morphism of schemes and $\mathcal{F} \in \mathrm{QCoh}(Y)$ is a quasi-coherent sheaf on Y such that $f^*\mathcal{F} = 0$, then $\mathcal{F} = 0$.*

Proof. We may suppose that Y is affine. Since f is quasi-compact, X is the union of a finite number of open affine subsets $X_1, \ldots, X_n$. Let $Z = X_1 \coprod \cdots \coprod X_n$ and $g\colon Z \to X$ be the natural morphism. It is clear that $f \circ g\colon Z \to Y$ is a faithfully flat morphism of affine schemes. Since $(f \circ g)^*\mathcal{F} = 0$, $\mathcal{F}(Y) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_Z(Z) = 0$, then $\mathcal{F}(Y) = 0$, hence $\mathcal{F} = 0$. □

Lemma 1.26. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of morphisms of schemes where g is faithfully flat and quasi-compact. If f' is formally unramified, then f is formally unramified.

Proof. Since f' is formally unramified, $0 = \Omega^1_{f'} = (g')^*\Omega^1_f$. By Lemma 1.25, $\Omega^1_f = 0$, so f is formally unramified. □

Proposition 1.27. *The class of formally unramified morphisms is stable under base change and is local on the codomain with respect to the fpqc topology.*

Proof. The stability under base change is quite easy. Now we prove the local property. Let $f\colon X \to Y$ be a morphism of schemes and let $\{Y_i \to Y\}$ be an fpqc covering such that for each i the projection $Y_i \times_Y X \to Y_i$ is formally unramified. The morphism $\phi\colon \coprod_i Y_i \to Y$ is fpqc, then there exist an open affine covering $\{V_j\}$ of Y and open quasi-compact subsets $W_j \subseteq \coprod_i Y_i$ such that $\phi(W_j) = V_j$ ([**Vis05**, Proposition 2.33(ii)]). For each j, it is clear that W_j is a finite disjoint union of open subschemes of Y_i's and the projection $W_j \times_Y X \to W_j$ is formally unramified. By Lemma 1.26, since $W_j \to V_j$ is faithfully flat and quasi-compact, the projection $V_j \times_Y X \to V_j$ is formally unramified. Now we conclude because $\{V_j\}$ is an open covering of Y and the class of formally unramified morphisms is obviously Zariski local on the codomain. □

PROPOSITION 1.28. *If $f\colon X \to Y$ is a morphism of schemes locally of finite type, then the following conditions are equivalent:*

(i) f is a monomorphism;
(ii) the diagonal morhism $\Delta_{X/Y}\colon X \to X \times_Y X$ is an isomorphism;
(iii) f is radicial and formally unramified;
(iv) for every point $y \in Y$, the fibre X_y is either empty or isomorphic to $\operatorname{Spec} k(y)$*;*
(v) for every algebraically closed field Ω and for every morphism $\operatorname{Spec}\Omega \to Y$*, the fibred product $X \times_Y \operatorname{Spec}\Omega$ is either empty or isomorphic to* $\operatorname{Spec}\Omega$.

PROOF. (i) ⇔ (ii): it holds in every category with fibred products.
(ii) ⇔ (iii): it follows from (i) ⇔ (vi) of Proposition 1.17 and from (i) ⇔ (vi) of Proposition 1.23.
(iii) ⇔ (iv): this follows from (i) ⇔ (iv) of Proposition 1.17 and from (i) ⇔ (iv) of Proposition 1.23.
(iv) ⇔ (v): obvious. □

PROPOSITION 1.29. *Let $f\colon X \to Y$ be a morphism of schemes which is locally of finite type, formally unramified and radicial. Suppose that there exists a morphism of schemes $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$.*
Then $g \circ f = \mathrm{id}_X$ and f is an isomorphism.

PROOF. By Proposition 1.28, f is a monomorphism of schemes. From $f \circ \mathrm{id}_X = f \circ (g \circ f)$ it follows that $\mathrm{id}_X = g \circ f$. □

THEOREM 1.30. *Let $f\colon X \to Y$ be a morphism of schemes. Suppose that:*

(1) f is proper;
(2) for every algebraically closed field Ω, the function $f_*\colon X(\Omega) \to Y(\Omega)$ is injective;
(3) f is formally unramified.

Then f is a closed immersion.

PROOF. By Proposition 1.28, f is a monomorphism of schemes and its fibres are discrete finite sets, i.e. f is quasi-finite. Since f is proper and quasi-finite, [**EGA**, Corollaire 18.12.4] implies that f is finite.

Now we want to prove that f is a closed immersion. Since f is finite, we may suppose that Y is affine and, consequently, X is affine. Let $\phi\colon A \to B$ a homomorphism of rings such that B is a finite A-module and the diagonal morphism $\operatorname{Spec} B \to \operatorname{Spec}(B \otimes_A B)$ is an isomorphism, i.e. the codiagonal homomorphism $\rho\colon B \otimes_A B \to B$ is an isomorphism. We shall prove that ϕ is surjective. Consider the finite A-module $M = \operatorname{coker} \phi$. Applying $- \otimes_A B$ to the exact sequence

$$A \xrightarrow{\phi} B \longrightarrow M \longrightarrow 0$$

we get the exact sequence

$$A \otimes_A B \xrightarrow{\phi\otimes 1} B \otimes_A B \longrightarrow M \otimes_A B \longrightarrow 0;$$

we see that $\rho \circ (\phi \otimes 1)\colon A \otimes_A B \to B$ is the natural isomorphism mapping $a \otimes b$ to $\phi(a)b$, hence $\phi \otimes 1$ is an isomorphisms, then $M \otimes_A B = 0$. Applying $M \otimes_A -$ to the surjection $B \twoheadrightarrow M$, we get a surjection $0 = M \otimes_A B \twoheadrightarrow M \otimes_A M$, hence $M \otimes_A M = 0$.

Fix a prime ideal $\mathfrak{p}$ of A. We have that $0 = (M \otimes_A M) \otimes_A k(\mathfrak{p}) = (M \otimes_A k(\mathfrak{p}))^{\otimes 2}$ is a $k(\mathfrak{p})$-vector space of dimension $(\dim_{k(\mathfrak{p})} M \otimes_A k(\mathfrak{p}))^2$, then $M \otimes_A k(\mathfrak{p}) = 0$, i.e. $M_{\mathfrak{p}} = \mathfrak{p} M_{\mathfrak{p}}$. $M_{\mathfrak{p}}$ is a finite module over the local ring $A_{\mathfrak{p}}$, then Nakayama's lemma implies that $M_{\mathfrak{p}} = 0$. Varying $\mathfrak{p}$ we get $M = 0$, then ϕ is surjective. $\square$

CHAPTER 2

Algebraic spaces and algebraic stacks

Algebraic spaces are a generalization of schemes that was introduced and studied by Michael Artin ([**Art69**]). Considering a scheme as the representable functor which it defines, we see that every scheme is a sheaf in the étale topology (Theorem A.6); algebraic spaces are defined as étale sheaves that are "locally" representable, i.e. algebraic spaces are obtained from affine schemes via glueing by étale morphisms. Algebraic spaces are the algebraic counterpart of Moishezon spaces, i.e. complex analytic spaces X that have $\dim X$ algebraically independent meromorphic functions.

Algebraic stacks, instead, are a generalization of schemes in another direction. They were introduced by David Mumford in the study of moduli problems ([**Mum65**], [**DM69**]): most of moduli functors are tautologically representable by algebraic stacks. Besides, with stacks one does not forget the automorphisms groups of the objects one wants to parametrize. Orbifolds are the differential topology counterpart of algebraic stacks.

The standard references for algebraic spaces and algebraic stacks are [**Knu71**] and [**LMB00**], respectively. A short introduction to this theory is provided in the appendix of [**Vis89**], while an encyclopaedic immense reference is [**St**].

In this chapter we recall the most important facts of the theory of algebraic spaces and algebraic stacks. In Section 2.5 we state the Keel-Mori theorem about the coarse moduli space of a separated Deligne-Mumford stack (Theorem 2.28), we determine the coarse moduli space of a quotient stack (Proposition 2.34), and we prove that a closed immersion of separated Deligne-Mumford stacks induces, in characteristic zero, a closed immersion on moduli spaces (Proposition 2.35). This last result will be useful to prove that, in characteristic zero, the coarse moduli space of hyperelliptic curves of genus g is a closed subscheme of the coarse moduli space of smooth curves of genus g (Proposition 4.28).

Convention 2.1. Throughout this chapter S stands for a fixed quasi-separated scheme.

2.1. Algebraic spaces

DEFINITION 2.2. An S-*space* is a sheaf

$$X\colon (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow (\mathrm{Set})$$

of sets on the étale site $(\mathrm{Sch}/S)_{\text{ét}}$.

Since the étale site is subcanonical (Theorem A.6), for every S-scheme X, the representable functor h_X is an S-space. According to Yoneda lemma, the category (Sch/S) of S-schemes is a full subcategory of the category of S-spaces. We will confuse each S-scheme with its representable functor, saying that an S-scheme is an S-space.

The category of S-spaces has fibred products and a terminal object.

DEFINITION 2.3. A morphism $f\colon X \to Y$ of S-spaces is said *schematic* if, for every S-scheme U and for every morphism $g\colon U \to Y$ of S-spaces, the fibred product $X \times_{f,Y,g} U$ is a scheme.

Schematic morphisms of S-spaces are stable by base change. Every morphism of S-schemes is schematic.

DEFINITION 2.4. If P is a property of morphisms of S-schemes which is stable under base change and étale-local on the codomain, then a schematic morphism $f\colon X \to Y$ of S-spaces is said to have the property P if, for every S-scheme U and for every morphism $g\colon U \to Y$ of S-spaces, the morphism $X \times_{f,Y,g} U \to U$ of S-schemes has the property P.

EXAMPLE 2.5. Examples of properties of morphisms of S-schemes which are stable under base change and fpqc local on the codomain are: separated, quasi-compact, locally of finite presentation, proper, affine, finite, flat, smooth, unramified, étale ([**Vis05**, Proposition 2.36]), radicial (Proposition 1.18), formally unramified (Proposition 1.27), surjective.

Therefore, since the étale topology is coarser than the fpqc topology, if P is one of the properties above, we have defined when a schematic morphism of S-spaces has the property P.

LEMMA 2.6. *Let X be an S-space. Then the following statements are equivalent:*

(i) the diagonal morphism $\Delta_X\colon X \to X \times_S X$ is schematic;
(ii) for all S-schemes U, V and for all morphisms $U \to X$, $V \to X$ of S-spaces, the fibred product $U \times_X V$ is a scheme;
(iii) for every S-scheme U, every morphism $U \to X$ of S-spaces is schematic.

PROOF. (i) $\Rightarrow$ (ii): let U, V be two S-schemes and let $U \to X$, $V \to X$ be two morphisms of S-spaces. This gives a morphism of

S-spaces $U \times_S V \to X \times_S X$; it is clear that the diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & U \times_S V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_X} & X \times_S X \end{array}$$

is cartesian. Since $U \times_S V$ is a scheme and Δ is schematic, $U \times_X V$ is a scheme.

(ii) $\Rightarrow$ (i): let U be an S-scheme and let $U \to X \times_S X$ be a morphism of S-spaces. This morphism factors as $(f \times g) \circ \Delta_U$, where $f, g \colon U \to X$ are morphisms of S-spaces. By (ii) the S-spaces $U \times_{f,X,g} U$ is a scheme. The three squares in the diagram

$$\begin{array}{ccc} X \times_{X \times_S X} U & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle \Delta_U} \\ U \times_{f,X,g} U & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow{\scriptstyle f \times g} \\ X & \xrightarrow{\Delta_X} & X \times_S X \end{array}$$

are cartesian. Then $X \times_{X \times_S X} U$ is a scheme because it is a fibred product of schemes.

(ii) $\Leftrightarrow$ (iii): obvious from the definitions. $\square$

Definition 2.7. An *algebraic space over S*, or simply an *algebraic S-space*, is an S-space X satisfying the following two conditions:

(1) the diagonal morphism $X \to X \times_S X$ is schematic and quasi-compact;
(2) there exists a scheme U and a morphism of S-spaces $U \to X$ which is schematic (automatically from (1) and Lemma 2.6), étale and surjective.

An S-scheme satisfying (2) is called an of X. It is clear that every quasi-separated S-scheme is an algebraic space over S.

Now we should define some properties of morphisms of algebraic spaces which are not schematic, but we omit this here not to bore the reader. For a property P of morphism of schemes which is stable under base change and is of local nature with respect to the étale topology, we say that a morphism of algebraic spaces $X \to Y$ has property P if, for any/some étale covering V of Y and for any/some étale covering U of $X \times_Y V$, the scheme morphism $U \to V$ has the property P. Analogously, for a property P of schemes which is local in the étale topology, we say that an algebraic space X has the property P if any/some presentation U of X has property P. For precise definitions we refer the reader to [**Knu71**].

2.2. Stacks over schemes

When we will say *a stack over* S, we will mean a stack over the étale site $(\mathrm{Sch}/S)_{\text{ét}}$. Obviously a stack with respect to the fpqc topology is a stack.

We denote by (St/S) the 2-category of stacks over S. We denote by (Gr/S) the 2-category of groupoids over (Sch/S). The sub-2-category (St/S) of (Gr/S) is stable for projective limits and fibred products.

The proposition below is very important because, if one wants to verify that a groupoid is a stack, one can check only particular coverings: Zariski coverings and faithfully flat morphisms of affine schemes.

Proposition 2.8. *Let* $\mathcal{F}$ *be a groupoid over* (Sch/S). *Suppose that the following conditions are satisfied.*

(i) $\mathcal{F}$ *is a stack with respect to the Zariski topology.*
(ii) Whenever $V \to U$ *is a flat surjective morphism of affine* S*-schemes, the restriction functor*

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V \to U)$$

is an equivalence of categories.

Then $\mathcal{F}$ *is a stack over* S *with respect to the fpqc topology.*

Proof. See [**Vis05**, Lemma 4.25]. □

Example 2.9. Since the fpqc site (Sch/S) is subcanonical (Theorem A.6), every S-scheme is a stack over S with respect to the fpqc topology. Every S-space is a stack over S. The category of S-spaces is a full subcategory of (St/S).

Another way to construct stacks over S is via ample sheaves as it is shown in the theorem below.

Theorem 2.10. *Let* P *be a class of flat proper morphism of finite presentation in* (Sch/S) *that is local in the fpqc topology.*

Suppose that for each morphism $\xi\colon X \to U$ *in* P *one has given an invertible sheaf* $\mathcal{L}_\xi$ *on* X *that is ample relative to the morphism* $X \to U$, *and for each cartesian diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\ U & \xrightarrow{\phi} & V \end{array}$$

an isomorphism $\rho_{f,\phi}\colon f^*\mathcal{L}_\eta \simeq \mathcal{L}_\xi$ *of invertible sheaves on* X. *These isomorphisms are required to satisfy the following compatibility condition:*

whenever we have a cartesian diagram of schemes

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \zeta} \\ U & \xrightarrow{\phi} & V & \xrightarrow{\psi} & W \end{array}$$

whose columns are in P*, we have the equality*

$$\rho_{gf,\psi\phi} = \rho_{f,\phi} \circ f^*\rho_{g,\psi} \colon (gf)^*\mathcal{L}_\zeta \longrightarrow \mathcal{L}_\xi.$$

Then $\mathrm{P}^{\mathrm{cart}}$ *is a stack in the fpqc topology.*

Proof. See [**Vis05**, Theorem 4.38]. □

Definition 2.11. An S-stack is called *representable* if it is isomorphic to an algebraic space over S. (See Caution A.23)

A morphism $F\colon \mathcal{X} \to \mathcal{Y}$ of S-stacks is called *representable* if, for every affine S-scheme U and every morphism $U \to \mathcal{Y}$, the fibred product $\mathcal{X} \times_{\mathcal{Y}} U$ is (isomorphic to) an algebraic space over S.

A morphism $F\colon \mathcal{X} \to \mathcal{Y}$ of S-stacks is called *schematic* if, for every affine S-scheme U and every morphism $U \to \mathcal{Y}$, the fibred product $\mathcal{X} \times_{\mathcal{Y}} Y$ is (isomorphic to) an S-scheme.

Remark 2.12. More concretely, an S-stack $\mathcal{X}$ is representable if, for every S-scheme U, the category $\mathcal{X}(U)$ is equivalent to a discrete category X_U and if the functor $(\mathrm{Sch}/S)^{\mathrm{op}} \to (\mathrm{Set})$ defined by

$$U \mapsto \mathrm{ob}\ X_U$$

is an algebraic space over S.

Remark 2.13. A morphism $F\colon \mathcal{X} \to \mathcal{Y}$ of S-stacks is representable if and only if the two following conditions hold:

- for each S-scheme U, the functor $F_U\colon \mathcal{X}(U) \to \mathcal{Y}(U)$ is faithful;
- for each S-scheme U and each object $y \in \mathcal{Y}(U)$, the functor

  $$(\mathrm{Sch}/U)^{\mathrm{op}} \to (\mathrm{Set})$$

  defined by

  $$(V \xrightarrow{h} U) \mapsto \{(x, \phi) \mid x \in \mathrm{ob}\ \mathcal{X}(V), \phi\colon h^*y \to F(x)\}/\simeq$$

  is an algebraic space over U.

Definition 2.14. Let P a property of morphisms of algebraic spaces over S which is stable under base change and étale-local on the codomain. A representable morphism $F\colon \mathcal{X} \to \mathcal{Y}$ of S-stacks is said to have the property P if, for every affine S-scheme U and every morphism $U \to \mathcal{Y}$ of S-stacks, the base change morphism $\mathcal{X} \times_{\mathcal{Y}} U \to U$ has the property P.

2.3. Algebraic stacks

Let us begin with discussing when the diagonal of a stack is representable.

LEMMA 2.15. *Let $\mathcal{X}$ be a stack over S. Then the following statements are equivalent:*

(i) the diagonal morphism $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable;
(ii) for all affine S-schemes U, V and for all morphisms $U \to \mathcal{X}$, $V \to \mathcal{X}$ over S, the fibred product $U \times_{\mathcal{X}} V$ is an algebraic space;
(iii) for every affine S-scheme U, every morphism $U \to \mathcal{X}$ over S is representable;
(iv) for all affine S-schemes U and objects $\xi, \eta \in \mathcal{X}(U)$, the sheaf $\mathbf{Isom}(\xi, \eta)$ is an algebraic U-space;
(v) for every algebraic S-space X, every morphism $X \to \mathcal{X}$ of S-stacks is representable.

PROOF. The proof is analogous to the proof of Lemma 2.6. See [**LMB00**, Corollaire 3.13]. □

DEFINITION 2.16. A stack $\mathcal{X}$ over the scheme S is called *algebraic* if it satisfies the following two axioms:

(1) the diagonal morphism $\Delta \colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, separated and quasi-compact;
(2) there exists an algebraic S-space X with an S-morphism $X \to \mathcal{X}$ which is representable, surjective, and smooth.

PROPOSITION 2.17. *If $\mathcal{X}$ is an algebraic stack over S, then the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is of finite type.*

PROOF. See [**LMB00**, Lemme 4.2]. □

DEFINITION 2.18. Let P be a property of algebraic S-spaces which is local on the domain with respect to smooth topology. We say that an algebraic stack $\mathcal{X}$ over S satisfies P if there exist an algebraic S-space satisfying P and a smooth surjective morphism $X \to \mathcal{X}$.

The following properties are local with respect to the smooth topology: locally noetherian, reduced, normal, Cohen-Macaulay, regular, Serre's conditions (R_n) and (S_n).

DEFINITION 2.19. Let P be a property of morphisms of algebraic S-spaces which is local with respect to the smooth topology. We say that a morphism $F \colon \mathcal{X} \to \mathcal{Y}$ of algebraic S-stacks satisfies P if there exist two algebraic S-spaces U and V and two surjective smooth morphisms $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ such that the morphism $V \to U$ of algebraic spaces satisfies P.

The following properties are local with respect to the smooth topology: surjective, universally open, locally of finite presentation, locally of finite type, flat, smooth.

2.4. Deligne-Mumford stacks

DEFINITION 2.20. A stack $\mathcal{X}$ over the scheme S is called *of Deligne-Mumford* if it satisfies the following two axioms:

(1) the diagonal morphism $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, separated, and quasi-compact;
(2) there exists an algebraic S-space X with an S-morphism $X \to \mathcal{X}$ which is representable, surjective, and étale.

Obviously a Deligne-Mumford stack is also algebraic, since an étale morphism is smooth. Every algebraic space is a Deligne-Mumford stack.

THEOREM 2.21. *Let $\mathcal{X}$ be an algebraic S-stack. The following conditions are equivalent:*

(1) $\mathcal{X}$ is Deligne-Mumford;
(2) the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified;
(3) the natural map $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is formally unramified.

If this is the case, the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is schematic, quasi-finite and quasi-affine.

PROOF. See [**LMB00**, Lemme 4.2, Théorème 8.1]. □

Recall that a morphism of (algebraic) S-stacks $F\colon \mathcal{X} \to \mathcal{Y}$ is a *monomorphism* if, for every S-scheme U, the functor $F_U\colon \mathcal{X}(U) \to \mathcal{Y}(U)$ is fully faithful.

PROPOSITION 2.22. *Let $\mathcal{X}$ be an algebraic S-stack and let $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ be the diagonal morphism. The following conditions are equivalent:*

(1) $\mathcal{X}$ is representable;
(2) $\mathcal{X}$ is Deligne-Mumford and Δ is a monomorphism;
(3) $\mathcal{X}$ is equivalent to a functor;
(4) Δ is a monomorphism;
(5) for each every S-scheme U and every object $\xi \in \mathcal{X}(U)$,

$$\mathrm{Aut}_{\mathcal{X}(U)}(\xi) = \{\mathrm{id}\};$$

(6) the natural map $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is an isomorphism.

PROOF. See [**LMB00**, Proposition 4.4, Corollaire 8.1.1]. □

PROPOSITION 2.23. *Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a fully faithful morphism of algebraic S-stacks. Then F is representable and formally unramified.*

Proof. Let U be an affine S-schemes. Since F is fully faithful, the algebraic stack $\mathcal{X} \times_{\mathcal{Y}} U$ is equivalent to a functor, then it is an algebraic space thanks to Proposition 2.22 and it is not difficult to show that $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is a monomorphism of algebraic spaces. This proves that F is represented by a monomorphism of algebraic spaces. □

Now we shall introduce an important class of algebraic stacks: quotients of schemes by group schemes.

Definition 2.24. Let X be an S-scheme and G be an S-group scheme that acts on X. The *quotient stack* of X by G, denoted by $[X/G]$, is the category fibred in groupoids over (Sch/S) defined as follows: for every S-scheme T, the fibre $[X/G](T)$ is the category of pairs (E, α), where $E \to T$ is a G-torsor and $\alpha \colon E \to X$ is a G-equivariant map.

There is a natural morphism $X \to [X/G]$ that corresponds to the trivial G-torsor $G \times_S X \to X$ and to the action $G \times_S X \to X$.

Definition 2.25. If G is an S-group scheme, the quotient stack $[S/G]$ is denoted by $\mathrm{B}G$ and is called the *classifying stack* of the group scheme G.

Proposition 2.26. *Let X be an S-scheme and let G be an S-group scheme. Then the groupoid $[X/G]$ is an S-stack. Moreover, if G is smooth (resp. étale) and separated over S, then $[X/G]$ is an algebraic (resp. Deligne-Mumford) S-stack.*

Proof. See [**LMB00**, 3.4.2, 4.6.1]. The natural morphism $X \to [X/G]$ is a smooth (resp. étale) presentation and $X \times_{[X/G]} X \simeq X \times_S G$, where the projections to X are the first projection and the action of G on X. □

We will prove in the following section that Deligne-Mumford stacks are quite near to be quotients of schemes by finite group actions.

2.5. Moduli spaces of stacks

Definition 2.27. Let $\mathcal{X}$ be an algebraic stack over S. A *coarse moduli space*, or simply a *moduli space*, for $\mathcal{X}$ is an algebraic S-space M with a morphism $\mathcal{X} \to M$ such that:

(1) for all algebraically closed fields Ω, the function

$$\mathcal{X}(\mathrm{Spec}\,\Omega)/\text{isomorphisms} \to M(\mathrm{Spec}\,\Omega)$$

is a bijection;

(2) whenever N is an algebraic S-space and $\mathcal{X} \to N$ is a morphism, then the morphism factors uniquely as $\mathcal{X} \to M \to N$.

By condition (2), such an M is unique up to unique isomorphism and is called *the* (coarse) moduli space of the stack $\mathcal{X}$. The theorem below asserts that coarse moduli spaces of algebraic stacks with finite diagonal exist. Recall that a Deligne-Mumford stack is separated if and only if its diagonal is finite, because the diagonal is quasi-finite (Theorem 2.21).

THEOREM 2.28 (Keel-Mori). *Let S be a notherian quasi-separated scheme and let $\mathcal{X}$ be an algebraic S-stack with finite diagonal. Then there exists an algebraic space M with a morphism $\mathcal{X} \to M$ such that:*

(1) $\mathcal{X} \to M$ is proper, quasi-finite and surjective;

(2) if Ω is an algebraically closed field, then the function

$$\mathcal{X}(\operatorname{Spec}\Omega)/\text{isomorphisms} \to M(\operatorname{Spec}\Omega)$$

is a bijection;

(3) whenever N is an algebraic S-space and $\mathcal{X} \to N$ is a morphism, then the morphism factors uniquely as $\mathcal{X} \to M \to N$; more generally

(4) whenever $S' \to S$ is a flat morphism of schemes, N is an algebraic S'-space, and $\mathcal{X} \times_S S' \to N$ is a morphism, then the morphism factors uniquely as $\mathcal{X} \times_S S' \to M \times_S S' \to N$.

PROOF. See the original source [**KM97**] or [**Con05**]. □

Conditions (2) and (3) say that M is the moduli space of $\mathcal{X}$. Condition (4) says that the formation of a coarse moduli space behaves well under flat base change.

Now we shall prove that separated Deligne-Mumford stacks are quite near to be quotients of schemes by finite group actions. More precisely, separated Deligne-Mumford are, étale locally on the moduli space, isomorphic to stack-theoretic quotients of an affine scheme by a symmetric group (Proposition 2.31).

LEMMA 2.29. *Let $f\colon X \to Y$ be a scheme morphism that is separated and locally of finite type. Let $y \in Y$ such that the fibre $f^{-1}(y)$ is finite and discrete.*

Then there exist an étale morphism $g\colon Y' \to Y$ and a point $y' \in Y'$ such that $g(y') = y$ and we have a decomposition $X \times_Y Y' = X_1 \coprod X_2$ such that, if $f'\colon X \times_Y Y' \to Y'$ is the base change morphism, $f'|_{X_1}\colon X_1 \to Y'$ is finite and $X_2 \cap (f')^{-1}(y') = \emptyset$.

PROOF. See [**EGA**, Corollaire IV.18.12.3]. □

LEMMA 2.30. *Let $f\colon X \to Y$ be a scheme morphism which is separated and quasi-finite.*

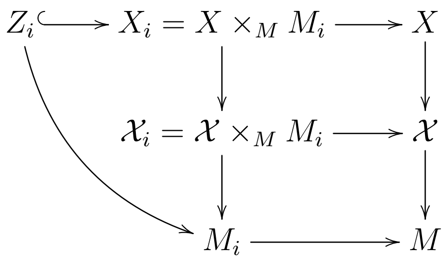
Then there exists an étale covering $\{Y_i \to Y\}$ and, for each i, an open and closed subscheme $Z_i \subseteq X \times_Y Y_i$ such that the induced morphism $Z_i \to Y_i$ is finite.

Proof. According to Lemma 2.29, for each point $y \in Y$, choose an étale morphism $g^y\colon Y^y \to Y$ and a closed open subscheme $Z^y \subseteq X \times_Y Y^y$ such that $y \in g^y(Y^y)$ and the induced morphism $Z^y \to Y^y$ is finite. Then $\{Y^y \to Y\}_{y \in Y}$ is the required étale covering. $\square$

Proposition 2.31 ([**AV02**, Lemma 2.3.3]). *Let $\mathcal{X}$ be a separated Deligne-Mumford stack over S and let M be its coarse moduli space.*

Then there exists an étale covering $\{M_i \to M\}$ such that for each i there are an affine scheme U_i and a finite group G_i acting on U_i, with the property that the pull-back $\mathcal{X} \times_M M_i$ is isomorphic to the stack-theoretic quotient $[U_i/G_i]$.

Proof. Since $\mathcal{X}$ is Deligne-Mumford, there exists a scheme X with a surjective étale morphism $X \to \mathcal{X}$. Then the composite $X \to M$ is a surjective separated quasi-finite morphism of schemes. Apply Lemma 2.30 to the morphism $X \to M$ and get an étale covering $\{M_i \to M\}$, open and closed subschemes $Z_i \subseteq X \times_M M_i$ such that $Z_i \to M_i$ is finite for each i. The situation is described in the diagram below.



Since $M_i \to M$ is flat, M_i is the coarse moduli space of the stack $\mathcal{X}_i = \mathcal{X} \times_M M_i$. Since $Z_i \to M_i$ is finite and $\mathcal{X}_i \to M_i$ is separated, the morphism $Z_i \to \mathcal{X}_i$ is finite. Therefore $Z_i \to \mathcal{X}_i$ is finite and étale.

Now, for each i, decompose $\mathcal{X}_i$ as $\coprod_{n \geq 0} \mathcal{Y}^i_n$ where $\mathcal{Y}^i_n$ is the closed open substack of $\mathcal{X}_i$ where the degree of $Z_i \to \mathcal{X}_i$ is equal to n. Since $\mathcal{X}_i \to M_i$ is a homeomorphism, the images of $\mathcal{Y}^i_n$ in M_i are closed open subspaces. Therefore, if we refine the étale covering $\{M_i \to M\}$, we may suppose that $Z_i \to \mathcal{X}_i$ is étale finite of constant degree n_i, for each i. Refining further if necessary, we may suppose that Z_i is affine for each i.

Let U_i be the S_{n_i}-torsor over $\mathcal{X}_i$ associated to the étale cover $Z_i \to \mathcal{X}_i$; in other words U_i is the space of the isomorphisms of the étale cover $Z_i \to \mathcal{X}_i$ with the trivial cover $\mathcal{X}_i \times \{1, \dots, n_i\} \to \mathcal{X}_i$. One sees that U_i is an open and closed subspace of $Z_i \times_{\mathcal{Y}_i} \cdots \times_{\mathcal{Y}_i} Z_i$ (n_i factors) and the first projection induces an étale cover $U_i \to Z_i$ of degree $(n_i - 1)!$. Then U_i is an affine scheme, because Z_i is affine and $U_i \to Z_i$ is finite. Finally $U_i \to \mathcal{X}_i$ is an S_{n_i}-torsor, then $\mathcal{X}_i$ is identified with $[U_i/S_{n_i}]$. $\square$

Now we shall determine the coarse moduli space of the stack-theoretic quotient of an affine scheme by the action of a finite group. We will see that it is the scheme-theoretic quotient, i.e. the spectrum of the invariant ring (Proposition 2.34). Before showing this, we need two commutative algebra lemmas.

LEMMA 2.32 ([**Bou72**, Chapter V, Theorem 2]). *Let A be a ring, G a finite group acting on A and A^G the ring of invariants. Let $\mathfrak{q} \in \operatorname{Spec} A$ be a prime ideal of A and $\mathfrak{p} = \mathfrak{q} \cap A^G$. Let D be the decomposition group of $\mathfrak{q}$, i.e. the subgroup of G made up of elements $\gamma \in G$ such that $\gamma(\mathfrak{q}) = \mathfrak{q}$. Then:*

(1) A is integral over A^G.
(2) If $\mathfrak{q}'$ is another prime ideal of A such that $\mathfrak{p} = \mathfrak{q}' \cap A^G$, then there exists an element $\gamma \in G$ such that $\mathfrak{q}' = \gamma(\mathfrak{q})$; in other words G operates transitively on the set of prime ideals of A lying over $\mathfrak{p}$.
(3) $k(\mathfrak{q})/k(\mathfrak{p})$ is an algebraic normal field extension.
(4) If $S = A^G \setminus \mathfrak{p}$, then G operates on $S^{-1}A$ and $(S^{-1}A)^G \simeq S^{-1}A^G = A_{\mathfrak{p}}$.
(5) The natural group homomorphism

$$D \to \operatorname{Aut}(k(\mathfrak{q})/k(\mathfrak{p}))$$

is surjective.

PROOF. (1) An element $a \in A$ is a root of the monic polynomial

$$f_a(x) = \prod_{\gamma \in G} (x - \gamma(a)) \in A^G[x].$$

(2) We prove that $\mathfrak{q}' \subseteq \cup_{\gamma \in G} \gamma(\mathfrak{q})$. Let $a \in \mathfrak{q}'$; then $b = \prod_{\gamma \in G} \gamma(a)$ is an invariant element and belongs to $\mathfrak{q}'$, then $b \in \mathfrak{q}' \cap A^G = \mathfrak{p} \subseteq \mathfrak{q}$. Since $\mathfrak{q}$ is prime, there exists $\gamma \in G$ such that $\gamma(a) \in \mathfrak{q}$, then $a \in \gamma^{-1}(\mathfrak{q})$.

The inclusion $\mathfrak{q}' \subseteq \cup_{\gamma \in G} \gamma(\mathfrak{q})$ implies, thanks to prime avoidance ([**AM69**, Proposition 1.11]), that there exists $\gamma \in G$ such that $\mathfrak{q}' \subseteq \gamma(\mathfrak{q})$. Now $\mathfrak{q}'$ and $\gamma(\mathfrak{q})$ are two prime ideals of A which lie over the same prime ideal $\mathfrak{p}$ of A^G. Then $\mathfrak{q}' = \gamma(\mathfrak{q})$ by [**AM69**, Corollary 5.9].

(3) The integral extension $A \supseteq A^G$ induces an integral extension $A/\mathfrak{q} \supseteq A^G/\mathfrak{p}$ of domains. Denote by S the multiplicative subset of $A^G/\mathfrak{p}$ made up of non-zero elements. Then $k(\mathfrak{p}) = S^{-1}(A^G/\mathfrak{p})$ is the quotient field of $A^G/\mathfrak{p}$. The ring $S^{-1}(A/\mathfrak{q})$ is a domain which is integral over the field $k(\mathfrak{q})$, then it is a field ([**AM69**, Proposition 5.7]). Since $S^{-1}(A/\mathfrak{q})$ is a field containing $A/\mathfrak{q}$ and contained in the quotient field of $A/\mathfrak{q}$, it is the quotient field of $A/\mathfrak{q}$: that is $k(\mathfrak{q}) = S^{-1}(A/\mathfrak{q})$. Then $k(\mathfrak{q})$ is an algebraic field extension over $k(\mathfrak{p})$, generated by the elements of $A/\mathfrak{q}$. This proves that $k(\mathfrak{q})$ is the splitting field over $k(\mathfrak{p})$ of the family of polynomials $\{\overline{f}_a(x)\}_{a \in A}$, where $\overline{f}_a$ is the polynomial in $(A^G/\mathfrak{p})[x]$ whose coefficients are the images of those of f_a under the canonical homomorphism $A^G \to A^G/\mathfrak{p}$.

(4) It is clear that $S^{-1}A^G \subseteq (S^{-1}A)^G$. We shall prove the opposite containment. Let $a/s \in (S^{-1}A)^G$, then $\gamma(a)/s = a/s$ for each $\gamma \in$

G. Then, for each $\gamma \in G$, there exists an element $u_\gamma \in S$ such that $su_\gamma \gamma(a) = su_\gamma a$. Let $u = \prod_{\gamma \in G} u_\gamma$. Therefore $\gamma(sua) = su \cdot \gamma(a) = sua$ for every $\gamma \in G$, then $sua \in A^G$. Hence $a/s = sua/s^2u \in S^{-1}A^G$.

(5) It is clear that D acts on $k(\mathfrak{q})$ with automorphisms of $k(\mathfrak{p})$-algebras. Thanks to (4), up to replace A^G with $(A^G)_\mathfrak{p}$ and A with $(A^G \setminus \mathfrak{p})^{-1}A$, we may assume that $\mathfrak{p}$ and $\mathfrak{q}$ are maximal ideals. For brevity we denote by k the field $k(\mathfrak{p}) = A^G/\mathfrak{p}$ and by $\overline{a}$ the element $a + \mathfrak{q} \in A/\mathfrak{q} = k(\mathfrak{q})$, for all $a \in A$.

Let L be the separable closure of k in $k(\mathfrak{q})$. It has been seen in the proof of (3) that every element of $k(\mathfrak{q})$ is a root of a polynomial in $k[x]$ of degree n, where n is the cardinality of the group G. Then $[k(\beta) : k] \leq n$ for all $\beta \in k(\mathfrak{q})$. We can therefore choose an element $\alpha \in L$ such that

$$[k(\alpha) : k] = \max_{\beta \in L} [k(\beta) : k].$$

If $L \supsetneqq k(\alpha)$, then there exists $\alpha' \in L$ such that $k(\alpha, \alpha') \supsetneqq k(\alpha)$ and by primitive element theorem ([**Lan02**, Theorem V.4.6]), there exists $\alpha'' \in L$ such that $k(\alpha'') = k(\alpha, \alpha')$, then $[k(\alpha'') : k] > [k(\alpha) : k]$ which is absurd. So we have proved that $L = k(\alpha)$.

The ideals $\gamma(\mathfrak{q})$ for $\gamma \in G \setminus D$ are maximal and distinct from $\mathfrak{q}$ by definition; by Chinese Remainder theorem there therefore exists $a \in A$ such that $\alpha = \overline{a} \in k(\mathfrak{q})$ and $a \in \gamma^{-1}(\mathfrak{q})$ for $\gamma \in G \setminus D$.

Let $\phi\colon k(\mathfrak{q}) \to k(\mathfrak{q})$ be an automorphism over k. Consider the polynomial

$$\overline{f_a}(x) = \prod_{\gamma \in G} (x - \overline{\gamma(a)}) \in k[x].$$

As $\overline{a}$ is a root of $\overline{f_a}$, $\phi(\overline{a})$ is a root of $\overline{f_a}$ and hence there exists $\sigma \in G$ such that $\phi(\overline{a}) = \overline{\sigma(a)}$. But $\phi(\overline{a}) \neq 0$ and, for $\gamma \in G \setminus D$, $\gamma(a) \in \mathfrak{q}$ and hence $\overline{\gamma(a)} = 0$; we conclude that necessarily $\sigma \in D$. But as ϕ and $\overline{\sigma}$ have the same value for the primitive element $\alpha = \overline{a}$ of L, they coincide on L; as $k(\mathfrak{q})$ is a purely inseparable extension of L, they coincide on $k(\mathfrak{q})$. Then $\phi = \overline{\sigma}$. □

Lemma 2.33 ([**Bou72**, Corollary p.333]). *Let A be a ring, G a finite group acting on A and A^G the ring of invariants. Let K be a field and let $f_1, f_2\colon A \to K$ be two ring homomorphisms with the same restriction to A^G. Then there exists $\gamma \in G$ such that $f_2 = f_1 \circ \gamma$.*

Proof. Let $\mathfrak{q}_i$ be the kernel of f_i ($i = 1, 2$) which is a prime ideal of A. By hypothesis $\mathfrak{q}_1 \cap A^G = \mathfrak{q}_2 \cap A^G = \mathfrak{p}$ is a prime ideal of A^G. There therefore exists $\tau \in G$ such that $\tau(\mathfrak{q}_2) = \mathfrak{q}_1$ (Lemma 2.32(2)); replacing f_1 by the homomorphism $f_1 \circ \tau$ we may then assume that $\mathfrak{q}_2 = \mathfrak{q}_1$ (an ideal which we shall denote by $\mathfrak{q}$). By taking the quotient we then derive from f_1 and f_2 two injective homomorphisms $f'_1, f'_2\colon A/\mathfrak{q} \to K$ which therefore extend to two injective $k(\mathfrak{p})$-homomorphisms $f''_1, f''_2\colon k(\mathfrak{q}) \to K$. As $k(\mathfrak{q})/k(\mathfrak{p})$ is a normal algebraic field extension (Lemma 2.32(3)), there exists an automorphism ϕ of $k(\mathfrak{q})$ over $k(\mathfrak{p})$ such that $f''_2 = f''_1 \circ \phi$. By Lemma 2.32(5), ϕ is of the form $\overline{\sigma}$, for some $\sigma \in G$. This proves that $f_2 = f_1 \circ \sigma$. $\square$

Proposition 2.34. *Let X be an affine scheme and let G be a finite group that acts on X. Then the coarse moduli space of the quotient stack $[X/G]$ is the quotient scheme X/G.*

Proof. Let $A = \mathrm{H}^0(X, \mathcal{O}_X)$; then $X = \operatorname{Spec} A$, $X/G = \operatorname{Spec} A^G$. The natural morphism $\pi\colon X \to X/G$ induced by the inclusion $A^G \subseteq A$ factors as

$$X \to [X/G] \to X/G, \tag{2.1}$$

where the morphism $X \to [X/G]$ corresponds to the trivial G-torsor $G \times_S X \to X$ and to the action $G \times_S X \to X$, and the morphism $[X/G] \to X/G$ maps the pair $(E \overset{G}{\to} T, \alpha\colon E \to X) \in [X/G](T)$ into the morphism $T \simeq E/G \to X/G$ induced by α.

Since the stack $[X/G]$ is Deligne-Mumford and separated, it has a moduli space M thanks to Keel-Mori theorem. The morphisms in (2.1) factor uniquely as

$$X \to [X/G] \to M \to X/G.$$

We want to prove that the morphism $M \to X/G$ is an isomorphism.

Since $X \to [X/G]$ is étale, finite and surjective and $[X/G] \to M$ is proper, quasi-finite and surjective (Theorem 2.28), the morphism of algebraic spaces $X \to M$ is finite and surjective. Since X is an affine scheme, Chevalley's theorem ([**Knu71**, Theorem III.4.1]) implies that M is an affine scheme.

Condition (2) in Definition 2.27 implies M is a quotient of X by the action of G in the category of schemes. Therefore $M \simeq X/G$. This concludes the proof of the assert.

Another proof would have been possible and would have avoided using Keel-Mori theorem. In fact one may prove directly that X/G is the coarse moduli space of $[X/G]$. For example Condition (1) in Definition 2.27 is satisfied because the morphism $\pi\colon X \to X/G$ is surjective and for every $x \in X$ the field extension $k(x)/k(\pi(x))$ is algebraic (Lemma 2.32(3)) and thanks to Lemma 2.33. $\square$

The proposition below shows that, in characteristic 0, a closed immersion of separated Deligne-Mumford stacks induces a closed immersions on moduli spaces.

Proposition 2.35. *Let S a scheme over $\mathbb{Q}$, let $\mathcal{X} \to \mathcal{Y}$ be a closed immersion of separated Deligne-Mumford S-stacks, and let M and N be the coarse moduli spaces of $\mathcal{X}$ and $\mathcal{Y}$. Then the induced morphism $M \to N$ is a closed immersion.*

Proof. Thanks to the Keel-Mori theorem M and N exist. Since $\mathcal{X} \to N$ is initial among morphisms from $\mathcal{X}$ to algebraic spaces, the composite $\mathcal{X} \to \mathcal{Y} \to N$ induces a unique morphism $M \to N$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array} \tag{2.2}$$

The vertical arrows in (2.2) are the morphisms from a stack to its moduli space. We want to prove that the morphism $M \to N$ is a closed immersion.

According to Proposition 2.31, there exists an étale covering $\{N_i \to N\}$ such that for each i there are an affine scheme U_i and a finite group G_i acting on U_i such that $\mathcal{Y} \times_N N_i \simeq [U_i/G_i]$.

For each i, the stack $\mathcal{X} \times_N N_i$ is a closed substack of $\mathcal{Y} \times_N N_i \simeq [U_i/G_i]$, hence there exists a closed subscheme V_i of U_i that is invariant under the action of G_i and such that $\mathcal{X} \times_N N_i \simeq [V_i/G_i]$. The situation is described in the diagram below

$$\begin{array}{ccccccc} [V_i/G_i] & & & \longrightarrow & & & [U_i/G_i] \\ & \searrow & & & & \swarrow & \\ \downarrow & & \mathcal{X} & \longrightarrow & \mathcal{Y} & & \downarrow \\ & & \downarrow & & \downarrow & & \\ & & M & \longrightarrow & N & & \\ & \nearrow & & & & \nwarrow & \\ M \times_N N_i & & & \longrightarrow & & & N_i \end{array} \tag{2.3}$$

where the four trapezoids around the small square are cartesian. Since $N_i \to N$ is flat, N_i is the coarse moduli space of $[U_i/G_i]$, then $N_i \simeq U_i/G_i$ by Proposition 2.34. Analogously $M \times_{N_i} N \simeq V_i/G_i$.

Since being a closed immersion is local with respect to étale topology on the codomain and $\{N_i \to N\}$ is an étale covering, to prove that $M \to N$ is a closed immersion it suffices to show that $M \times_{N_i} N \to N_i$ is a closed immersion for each i, i.e. $V_i/G_i \to U_i/G_i$ is a closed immersion.

Fix an i. Let $G = G_i$, $A = \mathrm{H}^0(U_i, \mathcal{O}_{U_i})$ and $I \subseteq A$ be the ideal that defines the closed subscheme V_i of U_i. We want to prove that the natural ring homomorphism $A^G \to (A/I)^G$ is surjective. Let $a \in A$ such that $\overline{a} = a + I \in A/I$ is invariant, i.e. $\overline{a} \in (A/I)^G$. Consider the element

$$b = \frac{1}{|G|} \sum_{\gamma \in G} \gamma(a) \in A^G \subseteq A.$$

(Here we are using that we are over $\mathbb{Q}$.) Then $\overline{b} = \overline{a}$. □

CHAPTER 3

Smooth curves

In this chapter we recall the properties of families of smooth curves of fixed genus and their quotients under the action of a finite group (Theorem 3.21), we introduce $\mathcal{M}_g$ the stack of smooth curves of genus g and we prove that it is a separated stack of Deligne-Mumford if $g \geq 2$ (Theorem 3.34). Our proof avoids to appeal to the theory of minimal surfaces, but uses elementary arguments about blowing-ups of surfaces over discrete valuation rings.

3.1. Smooth curves

DEFINITION 3.1. A *family of smooth curves of genus* g, or briefly *smooth curve of genus* g, is a proper smooth morphism of schemes whose geometric fibres are connected curves of genus g.

PROPOSITION 3.2. *Let* $f\colon X \to S$ *be a morphism of schemes. Then the following conditions are equivalent:*

(1) f *is a smooth curve of genus* g*;*
(2) f *is proper, flat, of finite presentation and for every* $s \in S$ *the fibre* X_s *is a projective smooth geometrically connected curve of genus* g *over* $k(s)$*.*

PROOF. The genus of a curve over a field is stable under base change ([**Liu02**, Corollary 5.2.27]) and a morphism of finite presentation is smooth if and only if is flat and has smooth fibres ([**EGA**, Théorème IV.17.5.1]). □

It is clear that a smooth curve of genus g is a proper Cohen-Macaulay morphism of relative dimension 1, hence we can apply the results of the Grothendieck duality in Appendix B. If f is a smooth curve of genus g, the dualizing sheaf ω_f is isomorphic to the sheaf Ω^1_f of 1-differential forms.

PROPOSITION 3.3. *The class of smooth curves of genus* g *is stable under base change and is local on the codomain in the fpqc topology.*

PROOF. The geometric fibres are the same in the two cases above. For a morphism of schemes, being proper or smooth is stable under base change. The local nature follows from [**Vis05**, Proposition 2.36]. □

EXAMPLE 3.4. For every ring A, $\mathbb{P}^1_A$ and

$$\operatorname{Proj} A[x_0, x_1, x_2]/(x_0x_1 - x_2^2)$$

are smooth curves of genus 0 over A. For every positive integer d,

$$\operatorname{Proj} \mathbb{Z}[1/d][x_0, x_1, x_2]/(x_0^d + x_1^d + x_2^d)$$

is a smooth curve of genus $(d-1)(d-2)/2$ over $\mathbb{Z}[1/d]$. If A is a ring in which 2 and 3 are invertible, then

$$\operatorname{Proj} A[a, b, (4a^3 + 27b^2)^{-1}][x_0, x_1, x_2]/(x_0x_2^2 - x_1^3 - ax_0^2x_1 - bx_0^3)$$

is a smooth curve of genus 1 over $A[a, b, (4a^3 + 27b^2)^{-1}]$.

The following lemma allows us to reduce the study of smooth curves to smooth curves over a noetherian base.

Lemma 3.5. *Let S be an affine scheme and $f\colon X \to S$ a smooth curve of genus g. Then there exist a noetherian affine scheme S_0 and a cartesian diagram*

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f_0} \\ S & \longrightarrow & S_0 \end{array}$$

where f_0 is a smooth curve of genus g. Moreover we can require that S_0 is the spectrum of a subring of $\mathcal{O}_S(S)$ which is of finite type over $\mathbb{Z}$ and the morphism $S \to S_0$ is induced by ring inclusion.

Proof. Since f is smooth and finitely presented, according to [**EGA**, Corollaire IV.17.7.9], there exists a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & X_\alpha \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f_\alpha} \\ S & \xrightarrow{u_\alpha} & S_\alpha \end{array}$$

where S_α is the spectrum of a subring of $\mathcal{O}_S(S)$ of finite typer over $\mathbb{Z}$ and f_α is smooth of finite type. Now consider the projective system $\{S_\lambda\}_{\lambda \geq \alpha}$ of affine schemes such that the rings $\mathcal{O}_{S_\lambda}(S_\lambda)$ are subrings of $\mathcal{O}_S(S)$ which are finitely generated extensions of $\mathcal{O}_{S_\alpha}(S_\alpha)$. Denote $f_\lambda\colon X_\lambda \to S_\lambda$ the base change of f_α to S_λ. It is clear that f is the projective limit of the f_λ's. Denote $u_\lambda\colon S \to S_\lambda$ and $u_{\lambda\mu}\colon S_\mu \to S_\lambda$, for $\mu \geq \lambda$, the morphisms induced by the inclusions.

Since f is proper and finitely presented, according to [**EGA**, Théorème IV.8.10.5(xii)], there exists an index $\beta \geq \alpha$ such that f_β is proper. Moreover f_β is smooth because it comes from f_α by base change.

It remains to verify that the fibres of some f_λ, $\lambda \geq \beta$, are geometrically connected curves of genus g. For every $\lambda \geq \beta$, consider the subset $E_\lambda \subseteq S_\lambda$ made up of points $s \in S_\lambda$ such that the fibre $f_\lambda^{-1}(s)$ satisfies the following three properties:

(1) $f_\lambda^{-1}(s)$ is geometrically connected over $k(s)$ (or equivalently geometrically integral because f_λ is smooth);
(2) $f_\lambda^{-1}(s)$ has dimension 1;

(3) the Euler characteristic of the structure sheaf of $f_\lambda^{-1}(s)$ over $k(s)$ is equal to $1-g$, i.e.

$$\chi_{k(s)}\left(f_\lambda^{-1}(s), \mathcal{O}_{f_\lambda^{-1}(s)}\right) = 1-g.$$

These three conditions do not depend on the base field, i.e. they hold over $k(s)$ if and only if they hold for $f_\lambda^{-1}(s) \otimes_{k(s)} K$ over some (every) field extension K of $k(s)$ ([**EGA**, Proposition IV.4.5.6, Corollaire IV.4.1.4], [**Liu02**, Lemma 5.2.26]); therefore by transitivity of fibres ([**EGA**, Proposition I.3.6.4]) we have the equalities $u_{\lambda\mu}^{-1}(E_\lambda) = E_\mu$, $u_\lambda^{-1}(E_\lambda) = S$, because f is a smooth curve of genus g.

According to [**EGA**, Théorème IV.12.2.4(viii)], condition (1) is open because f_λ is proper, flat and of finite presentation. According to [**EGA**, Proposition IV.9.2.6.1], condition (2) defines a locally constructible subset of S_λ. According to [**Mum70**, §5 Corollary 1], since f_λ is proper and flat and S_λ is noetherian, the Euler characteristic is locally constant, hence condition (3) is open. Therefore, for every $\lambda \geq \beta$, E_λ is a locally constructible subset of S_λ, in particular it is ind-constructible (see [**EGA**, Définition IV.1.9.4]).

Since the limit of the ind-constructible subsets E_λ, $\lambda \geq \beta$, coincides with the limit of S_λ's, according to [**EGA**, Corollaire IV.8.3.5], there exists an index $\gamma \geq \beta$ such that $E_\gamma = S_\gamma$. It is clear that $f_\gamma \colon X_\gamma \to S_\gamma$ is a smooth curve of genus g. □

PROPOSITION 3.6. *Let $f\colon X \to S$ be a smooth curve of genus $g \geq 2$. For every $n \geq 3$, the invertible sheaf $\omega_f^{\otimes n}$ is very ample relative to f and $f_*\omega_f^{\otimes n}$ is a locally free sheaf on S of rank $(2n-1)(g-1)$.*

PROOF. The sheaf $\Omega^1_{X/U} = \omega_f$ is an invertible sheaf on X because it is locally free (f is smooth, [**EGA**, Proposition IV.17.2.3]) and has rank 1 because the fibres of f are geometrically connected smooth curves.

Now we prove the thesis under the additional hypothesis that S is locally noetherian. Let $\mathcal{L} = \omega_f^{\otimes n}$. For every point $s \in S$ denote by $\mathcal{L}_s$ the restriction of $\mathcal{L}$ to the fibre X_s of f over s. We have:

$$\mathrm{H}^1(X_s, \mathcal{L}_s) = \mathrm{H}^1(X_s, \omega_{X_s/k(s)}^{\otimes n}) = \mathrm{H}^0(X_s, \omega_{X_s/k(s)}^{\otimes(1-n)}) = 0,$$

because

$$\deg \omega_{X_s/k(s)}^{\otimes(1-n)} = (2g-2)(1-n) < 0.$$

Besides, $\mathcal{L}_s$ is very ample relative to $\operatorname{Spec} k(s)$ because $\deg \mathcal{L}_s = n(2g-2) > 2g$ ([**Liu02**, Proposition 7.4.4(b)]). By Lemma B.1, Lemma B.4 and [**EGA**, Corollaire II.4.4.5], $\mathcal{L}$ is very ample relative to f and $f_*\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of rank

$$h^0(X_s, \mathcal{L}_s) = \deg \omega_{X_s/k(s)}^{\otimes n} + 1 - g = (2n-1)(g-1).$$

This concludes the proof when S is locally noetherian.

Now we prove the thesis without the additional hypothesis of noetherianity. Since f is quasi-compact, the statements are local on S, according to [**EGA**, Corollaire II.4.4.5]; therefore we can assume that S is affine. By Lemma 3.5, there exists a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{v} & X_0 \\ \downarrow f & & \downarrow f_0 \\ S & \xrightarrow{u} & S_0 \end{array}$$

where S_0 is a noetherian affine scheme and f_0 is a smooth curve of genus g. Let $\mathcal{L}_0 = \omega_{f_0}^{\otimes n}$. From the base change property of sheaves of differentials ([**Liu02**, Proposition 6.1.24]), we have $v^*\mathcal{L}_0 = \mathcal{L}$. From the noetherian case, $\mathcal{L}_0$ is very ample relative to f_0 and $(f_0)_*\mathcal{L}_0$ is locally free on S_0 of rank $(2n-1)(g-1)$. According to [**EGA**, Proposition II.4.4.10(iii)], $\mathcal{L} = v^*\mathcal{L}_0$ is very ample relative to f. From Lemma B.1 we have that $f_*\mathcal{L} = f_*v^*\mathcal{L}_0 \simeq u^*(f_0)_*\mathcal{L}_0$ is locally free, because $(f_0)_*\mathcal{L}_0$ is locally free. □

Remark 3.7. With the same techniques of the proof of Proposition 3.6 we could prove that, if f is a smooth curve of genus $g \geq 3$, the invertible sheaf $\omega_f^{\otimes 2}$ is very ample relative to f and $f_*\omega_f^{\otimes 2}$ is a locally free sheaf of rank $3g-3$.

Corollary 3.8. *A smooth curve of genus $g \geq 2$ is a projective morphism.*

Proof. Let f be a smooth curve of genus $g \geq 2$. By Proposition 3.6, the invertible sheaf $\omega_f^{\otimes 3}$ is very ample relative to f and $f_*\omega_f^{\otimes 3}$ is a quasi-coherent sheaf of finite type. Conclude with [**EGA**, Remarque II.5.5.4(i)] because f is proper. □

Lemma 3.9. *If $f\colon X \to S$ be a smooth curve of genus g, then $\mathrm{R}^1 f_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_S$-module of rank g.*

Proof. We may suppose that S is affine. Using Lemma 3.5 and Lemma B.3, we are reduced to proving the thesis in the noetherian case. The assert is true in the noetherian case, thanks to Lemma B.3, because $\mathrm{H}^2(X_s, \mathcal{O}_{X_s}) = 0$ and $\dim_{k(s)} \mathrm{H}^1(X_s, \mathcal{O}_{X_s}) = g$, for every point $s \in S$. □

3.2. Conic bundles

Definition 3.10. A *conic bundle* is a smooth curve of genus 0, i.e. a flat proper finitely presented morphism of schemes whose geometric fibres are isomorphic to the projective line $\mathbb{P}^1$.

A conic bundle is called also a *Brauer-Severi scheme* of relative dimension 1, or also a twisted $\mathbb{P}^1$.

Remark 3.11. According to [**Gro95c**, Corollaire 5.11 and §8], for every scheme S there exist natural bijections among the following three sets:

- the set of conic bundles over S;
- $\mathrm{H}^1(S, \mathrm{PGL}_2)$, the set of PGL_2-torsors over S;
- the set of Azumaya algebras over S of degree 2, i.e. sheaves of $\mathcal{O}_S$-algebras which are étale locally isomorphic to the sheaf of 2×2 matrices $\mathrm{M}_2(\mathcal{O}_S)$.

Example 3.12. For every ring A, $\mathbb{P}^1_A$ and

$$\operatorname{Proj} A[x_0, x_1, x_2]/(x_0x_1 - x_2^2)$$

are conic bundles over A. For every ring A in which 2 is invertible,

$$\operatorname{Proj} A[x_0, x_1, x_2]/(x_0^2 + x_1^2 + x_2^2)$$

is a a conic bundle over A. If $\mathcal{E}$ is a locally free sheaf of rank 2 on the scheme S, then $\mathbb{P}(\mathcal{E}) \to S$ is a conic bundle.

Proposition 3.13. *If $q\colon P \to S$ is a conic bundle and $n \geq 1$, then the invertible sheaf $\omega_q^{\otimes(-n)}$ is very ample relative to q and $q_*\omega_q^{\otimes(-n)}$ is a locally free sheaf on S of rank $2n+1$.*

Proof. The proof is completely analogous to that of Proposition 3.6 and is omitted. ☐

Corollary 3.14. *A conic bundle is a projective morphism*

Proof. See the proof of Corollary 3.8. ☐

Lemma 3.15. *If $q\colon P \to S$ is a conic bundle, then the natural map $\mathcal{O}_S \to q_*\mathcal{O}_P$ is an isomorphism.*

Proof. We may suppose that S is affine. Using Lemma 3.5 and Lemma B.1, we are reduced to proving the thesis in the noetherian case. The thesis is true in the noetherian case because, for every point $s \in S$, $\mathrm{H}^1(P_s, \mathcal{O}_{P_s}) = 0$. ☐

3.3. Finite quotients of curves

Lemma 3.16. *Let R be a ring, let M be an R-module, and let G be a finite group of R-automorphisms of M. Suppose that $|G|$ is invertible in R. Then:*

(1) if M is flat, then M^G is flat;
(2) for every R-module N, the natural map $M^G \otimes_R N \to (M \otimes_R N)^G$ is an isomorphism.

Proof. (1) Under our hypotheses, the exact sequence

$$0 \longrightarrow M^G \longrightarrow M \longrightarrow M/M^G \longrightarrow 0 \tag{3.1}$$

splits because a section $s\colon M/M^G \to M$ is given by

$$s(m + M^G) = m - \frac{1}{|G|}\sum_{g\in G} gm.$$

Hence M^G is a direct summand of M.

(2) The action of G on $M \otimes_R N$ is defined by $g(m\otimes n) = gm \otimes n$. Now we prove the injectivity of the map $\alpha\colon M^G \otimes_R N \to (M\otimes_R N)^G$ in the statement. Tensoring the split exact sequence (3.1) with N, we get that $M^G \otimes_R N$ is a submodule of $M \otimes_R N$, hence

$$M^G \otimes_R N \subseteq (M\otimes_R N)^G \subseteq M \otimes_R N.$$

Now we prove the surjectivity of α. Let $y = \sum_i m_i \otimes n_i \in (M \otimes_R N)^G$, with $m_i \in M$ and $n_i \in N$. Since y is G-invariant, $\sum_i gm_i \otimes n_i = \sum_i m_i \otimes n_i$, for every $g \in G$. Consider the element

$$x = \sum_i \left(\frac{1}{|G|}\sum_{g\in G} gm_i\right) \otimes n_i \in M^G \otimes_R N.$$

We have

$$\begin{aligned}\alpha(x) &= \frac{1}{|G|}\sum_{g\in G}\sum_i gm_i \otimes n_i\\ &= \frac{1}{|G|}\sum_{g\in G}\sum_i m_i \otimes n_i\\ &= \sum_i m_i \otimes n_i\\ &= y.\end{aligned}$$

□

Lemma 3.17. *Let R be a ring, let A be an R-algebra of finite presentation, let G be a finite group of automorphisms of R-algebras of A. If $|G|$ is invertible in R, then A^G is an R-algebra of finite presentation.*

Proof. Firstly we suppose that R is noetherian. Let $a_1, \ldots, a_m \in A$ be such that $A = R[a_1, \ldots, a_m]$. For every $i = 1, \ldots, m$, consider the polynomial

$$P_i(t) = \prod_{g\in G}(t - ga_i) = t^n + \sum_{j=1}^{n} b_{ij}t^{n-j}$$

and consider $B = R[\{b_{ij}\}_{i,j}]$ the subring of A generated by the b_{ij}'s over R. Since a_i is a root of the monic polynomial $P_i(t) \in B[t]$, we have that a_i is integral over B; then A is finite over B. Since the polynomials $P_i(t)$ are G-invariant, $b_{ij} \in A^G$, then $B \subseteq A^G \subseteq A$. Hilbert's basis theorem implies that B is a noetherian ring, then A^G is finite over B because it is a submodule of the finite B-module A. Since B is of finite type over R, A^G is of finite type over R and this concludes the proof of the noetherian case. (Notice that in this case we have not used that $|G|$ is invertible in R.)

Now we prove the general case. Since A is finitely presented over R, there exist a subring R_α of R and an R_α-algebra A_α of finite type such that R_α is of finite type over $\mathbb{Z}$ and $A \simeq R \otimes_{R_\alpha} A_\alpha$. Consider the filtrant inductive system $\{R_\lambda\}_{\lambda \geq \alpha}$ of subrings R_λ of R which are finitely generated extension of R_α. Set $A_\lambda = R_\lambda \otimes_{R_\alpha} A_\alpha$. We are in the situation of [**EGA**, Lemme IV.8.8.2.1], hence the R-automorphisms $g_1, \ldots, g_n$ of A come from R_λ-homomorphism $g_1^\lambda, \ldots, g_n^\lambda$ of A_λ, for $\lambda \gg \alpha$. The injectivity of the homomorphism (8.8.2.2) implies that, for $\lambda \gg \alpha$, the g_i^λ are automorphisms and satisfy the presentation of the group G. Besides, if G acts faithfully on A, we can require that it acts faithfully also on A_λ, for some $\lambda \gg \alpha$. Since $|G|$ is invertible in R, then it is invertible in R_λ, for some $\lambda \gg \alpha$; Lemma 3.16(2) implies that $A^G = (R \otimes_{R_\lambda} A_\lambda)^G \simeq R \otimes_{R_\lambda} A_\lambda^G$, then A^G is of finite presentation over R because A_λ^G is of finite type over the noetherian ring R_λ. □

Proposition 3.18. *Let $X \to S$ be a projective morphism of schemes. If G is a finite group of S-automorphisms of X, then the quotient $\pi\colon X \to Y = X/G$ exists in the category of schemes. Moreover:*

1. *(1) if V is an open affine subset of Y, then $\pi^{-1}(V)$ is an open affine G-invariant subset of X and $\mathcal{O}_Y(V) = \mathcal{O}_X(\pi^{-1}(V))^G$;*
2. *(2) if U is an open affine G-invariant subset of X, then $\pi(U)$ is an open affine subset of Y and $\mathcal{O}_Y(\pi(U)) = \mathcal{O}_X(U)^G$;*
3. *(3) the morphism π is surjective, open, closed, and finite.*

Finally, if $|G|$ is invertible in $\mathcal{O}_S(S)$, then:

- *(4) if X is flat over S, then Y is flat over S;*
- *(5) for every S-scheme S', the natural map $(S' \times_S X)/G \to S' \times_S (X/G)$ is an isomorphism;*
- *(6) Y is separated over S;*
- *(7) if X is of finite presentation over S, then Y is proper of finite presentation over S and the π is of finite presentation.*

Proof. It is a standard fact that the quotient of an affine scheme $\operatorname{Spec} A$ by a finite group of schemes automorphisms G is the affine scheme $\operatorname{Spec} A^G$, where A^G is the subring of A made up of elements which are G-invariant. Besides, it is easy to prove that A is integral over A^G.

Therefore, the existence of quotients by the action of finite groups depends on the possibility of covering X with affine open subsets which are G-invariant. Now we give an idea to construct the quotient X/G. Let $\{S_i\}$ be an affine cover of S and let X_i be the preimage of S_i in X. It is clear that, for every i, the open subset X_i is G-invariant and projective over S_i. Consider the orbit Gx of a point $x \in X_i$ under the action of G: Gx is a finite subset of X_i; then, according to [**Liu02**, Proposition 3.3.36(b)], there exists an affine open subset V of X_i such that $Gx \subseteq V$. The open subset $W = \cap_{g\in G} gV$ is a G-invariant open subset of X_i which cointains the point x; W is affine because X_i is separated. Therefore we have proved that we can cover X with G-invariant affine open subsets; from this it follows that the quotient $Y = X/G$ exists and (1) and (2) hold.

(3) The morphism π is surjective, open and closed because the underlying continuous function is the quotient map by the action of a finite group. The morphism π is finite because it is affine (from (1)), integral (because A is integral over A^G), and of finite type (because X is of finite type over S).

(4) follows from Lemma 3.16(1).

(5) follows from Lemma 3.16(2).

(6) Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times_S X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi\times_S\pi} \\ Y & \xrightarrow{\Delta_Y} & Y \times_S Y \end{array}$$

where Δ_X and Δ_Y are the diagonal morphisms. It suffices to show that $\Delta_Y(Y)$ is a close subset of $Y \times_S Y$, by [**Liu02**, Proposition 3.3.1]. $\Delta_Y(Y) = (\pi \times_S \pi)(\Delta_X(X))$ because π is surjective and the diagram above commutes. The subset $\Delta_X(X)$ is closed in $X \times_S X$ because X is separated over S. We conclude if we prove that $\pi \times_S \pi$ is a closed morphism. By the property of base change established in (5), we have

$$\begin{aligned}(X \times_S X)/(G \times G) &= ((X \times_S X)/(G \times \{1\}))/(\{1\} \times G) \\ &\simeq (X/G \times_S X)/(\{1\} \times G) \\ &\simeq X/G \times_S X/G \\ &= Y \times_S Y;\end{aligned}$$

therefore $\pi \times_S \pi$ is the quotient map of $X \times_S X$ respect with the action of the finite group $G \times G$. Hence $\pi \times_S \pi$ is a closed mapping.

(7) From Lemma 3.17 we have that Y is of finite presentation over S. The properness of $Y \to S$ follows from (6) and [**EGA**, Corollaire II.5.4.3(ii)]. The morphism π is of finite presentation by [**EGA**, Proposition IV.1.6.2(v)]. □

LEMMA 3.19. *Let k be a field, let X be a smooth geometrically connected projective curve over k, and let G be a finite group of k-automorphisms of X. If $|G|$ is invertible in k, then the quotient X/G is a smooth geometrically connected projective curve over k and the quotient morphism $X \to X/G$ is a cover (see Definition 1.3) of degree $|G|$.*

PROOF. Since X is irreducible of dimension 1 and the morphism $X \to X/G$ is finite and surjective, X/G is irreducible of dimension 1.

Now we want to prove that X/G is normal. This is a consequence of the following easy fact: if A is a normal domain and G is a finite group of ring automorphisms of A, then A^G is a normal domain. Hence the normality of X/G follows from X's.

In dimension 1 regularity is equivalent to normality, hence from the normality of $(\bar{k} \times_k X)/G \simeq \bar{k} \times_k (X/G)$ it follows that X/G is a smooth over k. Besides, X/G is projective over k because it is normal, proper over k, of dimension 1.

The quotient morphism is finite (Proposition 3.18), flat ([**Liu02**, Corollary 4.3.10]) and $[K(X) : K(X/G)] = |G|$. Conclude with Proposition 1.4. □

REMARK 3.20. In Lemma 3.19 the hypothesis that $|G|$ is invertible in k is unnecessary because if R is a field statement (2) of 3.16 holds also when $|G|$ is not invertible in R. In fact every module over a field is free.

THEOREM 3.21. *Let $X \to S$ be a smooth curve of genus $g \neq 1$ and let G be a finite group that acts on X with S-automorphisms, faithfully in every fibre of $X \to S$. If $|G|$ is invertible in $\mathcal{O}_S(S)$, then the quotient X/G exists, $X/G \to S$ is a proper smooth morphism whose fibres are geometrically connected smooth curves and $X \to X/G$ is a cover of degree $|G|$.*

PROOF. The morphism $X \to S$ is projective by Corollaries 3.8 and 3.14. According to (4) and (7) of Proposition 3.18, the morphism $X/G \to S$ is flat proper of finite presentation. Since Proposition 3.18(5) and Lemma 3.19, the geometric fibres of $X/G \to S$ are smooth connected projective curves.

It remains to study the morphism $h\colon X \to X/G$. It is finite surjective of finite presentation by Proposition 3.18 and, for every $s \in S$, the morphism $h_s\colon X_s \to (X/G)_s$ is a cover of degree $|G|$ by Lemma 3.19. Since $X \to S$ is flat of finite presentation and $X/G \to S$ is of finite presentation, the condition on h_s implies that h is flat by [**EGA**, Corollaire IV.11.3.11]. This proves that h is a cover of degree $|G|$. □

REMARK 3.22. In Theorem 3.21 the genus of the fibres of $X/G \to S$ may be non constant, but is always locally constant.

3.4. The stack of smooth curves: $\mathcal{M}_g$

DEFINITION 3.23. For every non-negative integer g, we denote by $\mathcal{M}_g$ the category defined as follows:

- Objects: morphisms of schemes $X \to S$ that are smooth curves of genus g;

- Arrows: an arrow from $X \to S$ to $Y \to T$ is a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S & \longrightarrow & T \end{array}$$

  of schemes. Composition of arrows is defined in the obvious way.

It is clear that $\mathcal{M}_g$ is a groupoid over the category (Sch) of schemes via the functor that maps an object $X \to S$ into S.

THEOREM 3.24. *If $g \neq 1$, then $\mathcal{M}_g$ is an algebraic stack.*

PROOF. Thanks to Proposition 3.6 and to Proposition 3.13, on every smooth curve of genus $g \neq 1$ there is a natural ample sheaf. By Theorem 2.10 $\mathcal{M}_g$ is a stack with respect to the fpqc topology, then is a stack with respect to the étale topology.

The diagonal $\mathcal{M}_g \to \mathcal{M}_g \times \mathcal{M}_g$ is representable, separated and quasi-compact because the functor of isomorphisms $\mathbf{Isom}_S(X_1, X_2)$ of two smooth curves $X_1 \to S$, $X_2 \to S$ of genus g is quasi-projective over S, thanks to the theory of Hilbert scheme.

Now we prove that $\mathcal{M}_g$ is algebraic in the case $g \geq 2$; the case $g = 0$ is completely analogous and is omitted. Fix an integer $\nu \geq 3$, consider the polynomial

$$p(t) = (2\nu t - 1)(g - 1) \in \mathbb{Q}[t]$$

and the number $n = p(1) - 1 = (2\nu - 1)(g - 1) - 1$. By Proposition 3.6 if f is a smooth curve of genus g then $\omega_f^{\otimes\nu}$ is very ample relative to f and $f_*\omega_f^{\otimes\nu}$ is a locally free sheaf of rank $n + 1$. Let Y be the groupoid over (Sch) whose fibre over the scheme S is the category

$$Y(S) = \left\{ (X \xrightarrow{f} S, \beta) \,\middle|\, (X \xrightarrow{f} S) \in \mathcal{M}_g(S),\ \beta \colon f_*\Omega_f^{\otimes\nu} \simeq \mathcal{O}_S^{\oplus(n+1)} \right\},$$

i.e. the objects of Y are smooth curves f plus a trivialization of $f_*\omega_f^{\otimes\nu}$. If $(f \colon X \to S, \beta) \in Y(S)$, then we have a closed immersion

$$X \hookrightarrow \mathbb{P}(f_*\Omega_f^{\otimes\nu}) \overset{\beta}{\simeq} \mathbb{P}^n_S$$

over S such that the fibres have the same Hilbert polynomial

$$\begin{aligned} \chi(\omega^{\otimes\nu t}) &= \deg \omega^{\otimes\nu t} + 1 - g \\ &= (2g - 2)\nu t + 1 - g \\ &= p(t). \end{aligned}$$

Hence Y is a subgroupoid of the groupoid represented by the Hilbert scheme $\mathrm{Hilb}^{p(t)}_{\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}}$ of closed subschemes of $\mathbb{P}^n_{\mathbb{Z}}$ with Hilbert polynomial $p(t)$ (for the definition of the Hilbert scheme see [**AK80**, Section 2], [**Nit05**], [**GW10**, Definition 14.136] or [**Gro95b**]). By [**MFK94**, Proposition 5.1] Y is (represented by) a quasi-projective scheme over $\mathbb{Z}$. The morphism $Y \to \mathcal{M}_g$ defined by

$$(X \to S, \beta) \mapsto (X \to S)$$

is representable smooth and surjective, because, for every scheme S and for every morphism $S \to \mathcal{M}_g$ (corresponding to the smooth curve $f\colon X \to S$), the fibred product

$$Y \times_{\mathcal{M}_g} S \to S$$

is the sheaf of bases of $f_*\omega_f^{\otimes \nu}$, which is a GL_{n+1}-torsor on S, hence a smooth surjective S-scheme. □

3.5. $\mathcal{M}_g$ is a Deligne-Mumford stack if $g \geq 2$

In this section we prove that $\mathcal{M}_g$ is a Deligne-Mumford stack if $g \geq 2$. The proof concerns the functor of isomorphisms defined in Proposition A.27 and is based on Proposition 3.33.

We begin studying the tangent space to the functor of automorphisms of a smooth curve of genus g.

PROPOSITION 3.25 ([**MO67**, Lemma 3.4]). *Let X be a proper scheme over the field k. Then there exists a natural isomorphism*

$$\mathrm{T}_{\mathrm{id}}\mathbf{Aut}_k(X) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X).$$

PROOF. The functor $\mathbf{Aut}_k(X)\colon (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Set})$ of automorphisms of X is representable by a group scheme which is locally of finite type over k ([**MO67**, Theorem 3.7]).

We will denote by $k[\varepsilon]$ the ring of dual numbers over k, i.e. $k[\varepsilon] = k[t]/(t^2)$ where t is an indeterminate over k. For simplicity of notation, we write $X \otimes_k k[\varepsilon]$ instead of $X \times_{\operatorname{Spec} k} \operatorname{Spec} k[\varepsilon]$.

The tangent space to $\mathbf{Aut}_k(X)$ at a k-rational point $p \in \mathbf{Aut}_k(X)$ is the set made up of elements $\phi \in \mathbf{Aut}_k(X)(\operatorname{Spec} k[\varepsilon])$ whose restriction to $\operatorname{Spec} k$ is p. In particular, the tangent space to $\mathbf{Aut}_k(X)$ at the identity $\mathrm{id}_X \in \mathbf{Aut}_k(X)(\operatorname{Spec} k)$ is

$$\mathrm{T}_{\mathrm{id}}\mathbf{Aut}_k(X) = \ker(\mathbf{Aut}_k(X)(\operatorname{Spec} k[\varepsilon]) \to \mathbf{Aut}_k(X)(\operatorname{Spec} k))$$

where the homomorphism $\mathbf{Aut}_k(X)(\operatorname{Spec} k[\varepsilon]) \to \mathbf{Aut}_k(X)(\operatorname{Spec} k)$ is induced by the closed immersion $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]$. Therefore we have

$$\begin{aligned}\mathrm{T}_{\mathrm{id}}\mathbf{Aut}_k(X) &\simeq \{\phi \in \mathrm{Hom}_k(\operatorname{Spec} k[\varepsilon], \mathbf{Aut}_k(X)) \mid \phi_k = \mathrm{id}_X\}\\ &\simeq \{\phi \in \mathrm{Aut}_{k[\varepsilon]}(X \otimes_k k[\varepsilon]) \mid \phi_k = \mathrm{id}_X\},\end{aligned}$$

where $\phi_k\colon X \to X$ is the morphism induced by $\phi\colon X \otimes_k k[\varepsilon] \to X \otimes_k k[\varepsilon]$ by base change with respect to the surjection $k[\varepsilon] \twoheadrightarrow k$ mapping ε to 0.

Now $X \otimes_k k[\varepsilon]$ is the relative spectrum $\mathcal{S}pec_X(\mathcal{O}_X[\varepsilon])$, where $\mathcal{O}_X[\varepsilon]$ is the $\mathcal{O}_X$-algebra $\mathcal{O}_X \oplus \mathcal{O}_X\varepsilon$ with $\varepsilon^2 = 0$. Denote by $\pi\colon \mathcal{O}_X[\varepsilon] \to \mathcal{O}_X$ the homomorphism of $\mathcal{O}_X$-algebras that maps ε to zero. Therefore

$\mathrm{T_{id}}\mathbf{Aut}_k(X)$ is in bijection with the set $\mathcal{S}$ of automorphisms of $k[\varepsilon]$-algebras $\phi\colon \mathcal{O}_X[\varepsilon] \to \mathcal{O}_X[\varepsilon]$ such that $\pi \circ \phi = \pi$. It remains to prove that $\mathcal{S}$ is in bijection with $\mathrm{Der}_k(\mathcal{O}_X)$.

Firstly we describe the map $\mathcal{S} \to \mathrm{Der}_k(\mathcal{O}_X)$. Let $\phi \in \mathcal{S}$. From $\pi \circ \phi = \pi$ one has $\phi(f) = f + D_\phi(f)\varepsilon$, for every $f \in \mathcal{O}_X$, for some map $D_\phi\colon \mathcal{O}_X \to \mathcal{O}_X$. D_ϕ is additive because ϕ is additive. $D_\phi(k) = 0$ because ϕ is k-linear. Since ϕ is multiplicative, for every $f, g \in \mathcal{O}_X$,

$$\begin{aligned} fg + D_\phi(fg)\varepsilon &= \phi(fg) \\ &= \phi(f)\phi(g) \\ &= (f + D_\phi(f)\varepsilon)(g + D_\phi(g)\varepsilon) \\ &= fg + (D_\phi(f)g + D_\phi(g)f)\varepsilon, \end{aligned}$$

i.e. $D_\phi(fg) = D_\phi(f)g + D_\phi(g)f$. Then D_ϕ is a derivation of $\mathcal{O}_X$ over k. So the map $\mathcal{S} \to \mathrm{Der}_k(\mathcal{O}_X)$ maps ϕ into D_ϕ.

Conversely, let D be a derivation of $\mathcal{O}_X$ over k. Consider the map $\phi\colon \mathcal{O}_X[\varepsilon] \to \mathcal{O}_X[\varepsilon]$ defined by $\phi(f + g\varepsilon) = f + D(f)\varepsilon + g\varepsilon$. It is not difficult to show that ϕ is a homomorphism of $k[\varepsilon]$-algebras such that $\pi \circ \phi = \pi$, i.e. $\phi \in \mathcal{S}$. □

We will use the theory of surfaces and valuations to prove that the functor of automorphisms of a smooth curve of genus $g \geq 2$ is proper.

LEMMA 3.26. *Let R be a valuation ring of a field K. Let t_0 be the closed point of $\operatorname{Spec} R$ and let t_1 be the generic point of $\operatorname{Spec} R$. Then, for every scheme X, the function*

$$X(\operatorname{Spec} R) \to \left\{ (x_0, x_1, i) \;\middle|\; \begin{array}{l} x_1 \in X,\ x_0 \in \overline{\{x_1\}} \subseteq X, \\ i\colon k(x_1) \to K \text{ is a ring homomorphism}, \\ R \text{ dominates } \mathcal{O}_{Z,x_0}, \text{ where } Z \text{ is the closed} \\ \text{subscheme } \overline{\{x_1\}} \text{ with the reduced structure} \end{array} \right\}$$

defined by

$$f \mapsto \big(f(t_0), f(t_1), f^\#\colon k(x_1) \to k(t_1)\big)$$

is bijective.

PROOF. See [**Har77**, Lemma II.4.4]. □

LEMMA 3.27. *Let X be an integral, locally noetherian scheme, let $x \in X$ be a point, and let $\pi\colon \tilde{X} \to X$ be the blowing-up along the closed subscheme $\overline{\{x\}}$ with the reduced structure. Then, for every point $y \in \pi^{-1}(x)$, the ideal $\mathfrak{m}_x\mathcal{O}_{\tilde{X},y}$ is a principal ideal of $\mathcal{O}_{\tilde{X},y}$.*

PROOF. We can suppose that $X = \operatorname{Spec} A$, where A is a noetherian domain with fraction field K. The point $x \in X$ corresponds to the prime ideal $\mathfrak{p}$ of A. Suppose that $\mathfrak{p}$ is generated by $a_1, \ldots, a_m \in A \setminus \{0\}$.

Then, by [**Liu02**, Lemma 8.1.4],

$$\tilde{X} = \bigcup_{i=1}^{m} \operatorname{Spec} A\left[\frac{a_1}{a_i}, \dots, \frac{a_m}{a_i}\right].$$

Suppose that $y \in \operatorname{Spec} B$, where

$$B = A\left[\frac{a_2}{a_1}, \dots, \frac{a_m}{a_1}\right] \subseteq K.$$

The point y corresponds to a prime ideal $\mathfrak{q}$ of B. From

$$a_j b = a_1 \frac{a_j}{a_1} b$$

for all $b \in B$, we have that $\mathfrak{p}B = a_1 B$. Then the ideal $\mathfrak{p}B_{\mathfrak{q}}$ is generated by a_1. □

THEOREM 3.28. *Let R be a Dedekind domain or a field. Let X and Y be integral normal schemes of dimension 2 that are projective and flat over* $\operatorname{Spec} R$. *Let $f\colon X \dashrightarrow Y$ be a birational map over* $\operatorname{Spec} R$.

Suppose that for every point $y \in Y$ of codimension 1 there exists a point $x \in X$ of codimension 1 such that the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ correspond to each other under the isomorphism $K(X) \simeq K(Y)$ induced by f.

Then f extends to a morphism $X \to Y$.

PROOF. See [**Liu02**, Theorem 8.3.20]. □

LEMMA 3.29. *Let R be a discrete valuation ring, let η be the generic point of* $\operatorname{Spec} R$, *and let $X \to \operatorname{Spec} R$, $Y \to \operatorname{Spec} R$ be two smooth curves of genus $g \geq 1$. Then the restriction function*

$$\operatorname{Isom}_{\operatorname{Spec} R}(X, Y) \to \operatorname{Isom}_{\operatorname{Spec} k(\eta)}(X_\eta, Y_\eta)$$

defined by $f \mapsto f_\eta$ is bijective.

PROOF OF LEMMA 3.29. Firstly we prove that the function is injective. Let $f, f'\colon X \to Y$ be two R-isomorphisms such that $f_\eta = f'_\eta\colon X_\eta \to Y_\eta$. Then f and f' are equal because they coincide on the open dense subset X_η of X, X is reduced and Y is separated over $\operatorname{Spec} R$.

To prove the surjectivity the most direct approach is to use the minimal regular models of arithmetic surfaces ([**Liu02**, Chapter 9], [**Lic68**], [**Sha66**]), but we do not use this approach here and we give an elementary proof. One easily sees that the thesis is implied by the following claim:

CLAIM 3.30. *Every isomorphism $X_\eta \to Y_\eta$ over the generic point η extends to a morphism $X \to Y$ over* $\operatorname{Spec} R$.

We prove this claim. Let $f\colon X_\eta \to Y_\eta$ be an isomorphism over $\operatorname{Spec} k(\eta)$. It defines a birational map $f\colon X \dashrightarrow Y$. We identify X_η and

Y_η via the isomorphism f and we identify the fields $K(X)$ and $K(Y)$ via f. Let s be the closed point of $\operatorname{Spec} R$ and let $v\colon K(X)^* = K(Y)^* \to \mathbb{Z}$ the discrete valuation corresponding to the generic point of the closed fibre Y_s (it is a discrete valuation because Y_s has codimension 1 in Y). Let $\mathcal{O}_v \subseteq K(X)$ the discrete valuation ring corresponding to v. Since the generic point of Y_s is over $s \in \operatorname{Spec} R$ and the generic point of Y is over $\eta \in \operatorname{Spec} R$, Lemma 3.26 gives a morphism $\operatorname{Spec} \mathcal{O}_v \to \operatorname{Spec} R$ that fits in a commutative diagram

(3.2)
$$\begin{array}{ccc} \operatorname{Spec} K(X) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_v & \longrightarrow & \operatorname{Spec} R \end{array}$$

where the top horizontal arrow is the immersion of the generic point of X and the left vertical arrow is induced by the inclusion $\mathcal{O}_v \subseteq K(X)$. Since X is proper over $\operatorname{Spec} R$, the valuative criterion of properness says that there exists a unique morphism $\operatorname{Spec} \mathcal{O}_v \to X$ that is a diagonal in the square (3.2). This morphism factors as $\operatorname{Spec} \mathcal{O}_v \to \operatorname{Spec} \mathcal{O}_{X,x} \to X$, where x is a point of X. The point x is called the *centre* of the valuation v on X, and it is the image of the closed point of $\operatorname{Spec} \mathcal{O}_v$.

If we prove that x is has codimension 1 we are done. In fact, if $\dim \mathcal{O}_{X,x} = 1$, then $\mathcal{O}_{X,x}$ is a discrete valuation ring which is dominated by the local ring $\mathcal{O}_v$, then $\mathcal{O}_{X,x} = \mathcal{O}_v$. At this point we can use Theorem 3.28. Recall that points of Y of codimension 1 are the closed points of the generic fibre Y_η (that make no problem because f is an isomorphism between X_η and Y_η) and the generic point of the closed fibre Y_s.

Suppose by contradiction that x has not codimension 1. Since x cannot be the generic point of X, x is a closed point. Now we consider the blowing-up $\pi_1\colon X_1 \to X$ with center the closed point x. Since π_1 is birational, we have a commutative diagram

(3.3)
$$\begin{array}{ccc} \operatorname{Spec} K(X) & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle \pi_1} \\ \operatorname{Spec} \mathcal{O}_v & \longrightarrow & X \end{array}$$

where the top horizontal arrow is the immersion of the generic point of X_1. Since π_1 is proper, the valuative criterion of properness implies that there exists a unique morphism $\operatorname{Spec} \mathcal{O}_v \to X_1$ that is a diagonal in the square (3.3). This morphism factors as $\operatorname{Spec} \mathcal{O}_v \to \operatorname{Spec} \mathcal{O}_{X_1,x_1} \to X_1$, where $x_1 \in X_1$ is the image of closed point of $\operatorname{Spec} \mathcal{O}_v$ in X_1 (the point x_1 is called the center of v on X_1). If x_1 has codimension 1, we stop. Otherwise, if x_1 is a closed point, we blow up X_1 along the closed point x_1. Let $X_2 \to X_1$ the blowing-up of X_1 along x_1, we find a centre $x_2 \in X_2$ of v over X_2. If x_2 has codimension 1 we stop, otherwise we continue in this way. Finally, we construct a (finite or infinite) sequence of scheme morphisms

$$\cdots \longrightarrow X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X \tag{3.4}$$

and a sequence of closed points $x_n \in X_n$, such that $\pi_{n+1} \colon X_{n+1} \to X_n$ is the blowing-up of X_n along x_n, where x_n is the centre of the valuation v on X_n. For brevity we denote the local ring $\mathcal{O}_{X_n,x_n}$ by $\mathcal{O}_n$. Let $\mathfrak{m}_n$ (resp. $\mathfrak{m}_v$) be the maximal ideal of $\mathcal{O}_n$ (resp. $\mathcal{O}_v$). Let k_n (resp. k_v) be the residue field of $\mathcal{O}_n$ (resp. $\mathcal{O}_v$). It is clear that:

(1) $\pi_{n+1}(x_{n+1}) = x_n$;
(2) $\mathcal{O}_n \subseteq \mathcal{O}_{n+1} \subseteq \mathcal{O}_v$ are local homomorphisms of local domains.

We want to prove that the sequence (3.4) is finite.

CLAIM 3.31. *If the sequence* (3.4) *does not stop, then* $\mathcal{O}_v = \cup_{n\geq 0}\mathcal{O}_n$.

PROOF OF CLAIM 3.31. Let $f \in \mathcal{O}_v$, $f \neq 0$. Consider the sequence of natural numbers $\{r_n\}$ defined as below:

$$r_n = \min\{v(a) \mid a \in \mathcal{O}_n, af \in \mathcal{O}_n\} \in \mathbb{N} \cup \{\infty\}.$$

Since $\mathcal{O}_n \subseteq \mathcal{O}_{n+1}$, $r_n \geq r_{n+1}$.

We prove that $r_n < \infty$ for all n. Since $K(X)$ is the quotient field of $\mathcal{O}_n$, there exist $f_1, f_2 \in \mathcal{O}_n$, $f_2 \neq 0$ such that $f = f_1/f_2$; we see that $f_2 f \in \mathcal{O}_n$, then $r_n \leq v(f_2)$.

Now we prove that $r_n > r_{n+1}$ if $r_n > 0$. Let $a \in \mathcal{O}_n$ be such that $af \in \mathcal{O}_n$ and $r_n = v(a)$. Since $r_n > 0$, $v(a) > 0$ then $a \in \mathfrak{m}_v \cap \mathcal{O}_n = \mathfrak{m}_n$. Analogously $v(af) = v(a) + v(f) \geq v(a) > 0$ implies that $af \in \mathfrak{m}_v \cap \mathcal{O}_n = \mathfrak{m}_n$. By Lemma 3.27 the ideal $\mathfrak{m}_n\mathcal{O}_{n+1}$ is principal and call $t \in \mathcal{O}_{n+1}$ a generator. From $a \in \mathfrak{m}_n\mathcal{O}_{n+1}$ and $af \in \mathfrak{m}_n\mathcal{O}_{n+1}$ we have that $a = bt$ and $af = ct$ for some $b, c \in \mathcal{O}_{n+1}$. Then $bf = c \in \mathcal{O}_{n+1}$ and $r_{n+1} \leq v(b) = v(a) - v(t) < v(a) = r_n$ because $t \in \mathfrak{m}_v$.

Since the sequence (3.4) is infinite, there exists n such that $r_n = 0$. Then there exists $a \in \mathcal{O}_n$ such that $a \in \mathcal{O}_v^*$ and $af \in \mathcal{O}_n$. It is clear that a is invertible in $\mathcal{O}_n$, then $f \in \mathcal{O}_n$. This concludes the proof of Claim 3.31. □

CLAIM 3.32. *If the sequence* (3.4) *does not stop, then the field extension* $k_v/k(x)$ *is algebraic.*

PROOF OF CLAIM 3.32. For each $n \geq 0$, $\mathcal{O}_n$ is a noetherian local domain of dimension 2. The dimension inequality ([**Mat89**, Theorem 15.5]) applied to the extension $\mathcal{O}_n \subseteq \mathcal{O}_{n+1}$ shows that $\dim \mathcal{O}_{n+1} + \mathrm{tr.deg}_{k_n} k_{n+1} \leq \dim \mathcal{O}_n$, hence the field extension k_{n+1}/k_n is algebraic.

From Claim 3.31 we deduce that $k_v = \cup_{n\geq 0} k_n$; therefore k_v/k_0 is an algebraic field extension. This concludes the proof of Claim 3.32. □

Since x is a closed point of the curve X_s, $k(x)/k(s)$ is a finite field extension. Since Y_s is a curve over $k(s)$, $\mathrm{tr.deg}_{k(s)} k_v = 1$. Therefore the field extension $k_v/k(x)$ cannot be algebraic.

We have shown that from the assumption that the sequence (3.4) is infinite we derive an absurd, hence the sequence is finite. Call $\tilde{X}$ the leftmost scheme in the sequence and call $\pi\colon \tilde{X} \to X$ the composition of the successive blowing-ups. Denote by $\tilde{x}$ the centre of v on $\tilde{X}$; it has codimension 1, then is a generic point of the closed fibre $\tilde{X}_s$. Since π is a composition of a finite sequence of blowing-ups along closed points, the fibre $\tilde{X}_s$ is union of X_s and projective lines. The point $\tilde{x}$ cannot be the generic point of X_s, then it is the generic point of one of these projective lines. The closure $\Gamma = \overline{\{\tilde{x}\}}$ is isomorphic over $k(s)$ to $\mathbb{P}^1_K$ for some finite field extension K of $k(s)$. Since the discrete valuation ring $\mathcal{O}_{\tilde{X},\tilde{x}}$ is dominated by the local ring $\mathcal{O}_v$, we have $\mathcal{O}_{\tilde{X},\tilde{x}} = \mathcal{O}_v$, then we have a birational map over $k(s)$ between the two curves Γ and Y_s. This map is an isomorphism because these curves are complete, then

$$\begin{aligned}
1 - g &= \chi_{k(s)}(Y_s, \mathcal{O}_{Y_s}) \\
&= \chi_{k(s)}(\Gamma, \mathcal{O}_\Gamma) \\
&= [K : k(s)] \cdot \chi_K(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}) \\
&= [K : k(s)] \\
&\geq 1
\end{aligned}$$

which is absurd because $g \geq 1$.

So we have found an absurd if we suppose that x has not codimension 1. Therefore x has codimension 1 and we have proved above that in this case f extends to a morphism $X \to Y$. This concludes the proof of Claim 3.30 and Lemma 3.29. □

PROPOSITION 3.33. *If $X \to S$ be a smooth curve of genus $g \geq 2$, then the scheme $\mathbf{Aut}_S(X)$ of S-automorphism of X is finite, of finite presentation, and formally unramified over S.*

PROOF. With the theory of Hilbert scheme one can prove that the functor $\mathbf{Aut}_S(X)$ is represented by a quasi-projective scheme over S (see [**ACG11**, Chapter IX, §7] or [**Gro95b**, §4]). Reducing to the noetherian case, one can prove that $\mathbf{Aut}_S(X)$ is locally of finite presentation over S.

To prove that $\mathbf{Aut}_S(X)$ is formally unramified over S we use Proposition 1.23. Taking geometric fibres, we may assume that S is the spectrum of an algebraically closed field k. By Proposition 3.25 the tangent space of $\mathbf{Aut}_k(X)$ at the identity is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{H}^0(X, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)) = \mathrm{H}^0(X, \omega_X^\vee),$$

which is zero because $\omega_X^\vee$ has negative degree ($\deg \omega_X^\vee = 2 - 2g$). Since $\mathbf{Aut}_k(X)$ is a group scheme over k, it is homogeneous, then the tangent space of $\mathbf{Aut}_k(X)$ at each point is zero. This proves that $\mathbf{Aut}_k(X)$ is isomorphic to the disjoint union of a finite number of copies of $\operatorname{Spec} k$. This concludes the proof that $\mathbf{Aut}_S(X)$ is formally unramified over S.

The discussion above shows also that $\mathbf{Aut}_S(X)$ is quasi-finite over S. To conclude the proof of the proposition, it suffices to prove that $\mathbf{Aut}_S(X) \to S$ is proper and apply [**EGA**, Théorème IV.8.11.1]. We use the valuative criterion for properness. Let R be a DVR, let K be its fraction field and let $\operatorname{Spec} R \to S$ be a morphism. We want to prove that the function

$$\operatorname{Hom}_S(\operatorname{Spec} R, \mathbf{Aut}_S(X)) \to \operatorname{Hom}_S(\operatorname{Spec} K, \mathbf{Aut}_S(X)) \tag{3.5}$$

induced by the inclusion of the generic point $\operatorname{Spec} K \hookrightarrow \operatorname{Spec} R$ is bijective. The function in (3.5) is equal to the function

$$\begin{array}{c}\operatorname{Hom}_{\operatorname{Spec} R}(\operatorname{Spec} R, \mathbf{Aut}_S(X) \times_S \operatorname{Spec} R) \\ \Big\downarrow \\ \operatorname{Hom}_{\operatorname{Spec} R}(\operatorname{Spec} K, \mathbf{Aut}_S(X) \times_S \operatorname{Spec} R);\end{array}$$

it is clear that $\mathbf{Aut}_S(X) \times_S \operatorname{Spec} R \simeq \mathbf{Aut}_{\operatorname{Spec} R}(X \times_S \operatorname{Spec} R)$. Therefore we may assume that $\operatorname{Spec} R \to S$ is the identity. Now we are done because the valuative criterion holds by Lemma 3.29. □

THEOREM 3.34. *If $g \geq 2$, then $\mathcal{M}_g$ is a Deligne-Mumford stack with finite diagonal.*

PROOF. The stack $\mathcal{M}_g$ is algebraic (Theorem 3.24) and the functor of isomorphisms is finite and unramified by the proposition above. □

Suppose $g \geq 2$. For any scheme S, we denote by $\mathcal{M}_{g,S}$ the stack $\mathcal{M}_g \times_{\operatorname{Spec} \mathbb{Z}} S$. Since $\mathcal{M}_{g,S}$ is a separated Deligne-Mumford stack (Theorem 3.34), it has a coarse moduli space $M_{g,S}$ thanks to the Keel-Mori theorem (Theorem 2.28). For brevity we denote by M_g the coarse moduli space $M_{g,\mathbb{Z}}$ of $\mathcal{M}_g$.

There is a natural morphism

$$M_{g,S} \to M_g \times_{\operatorname{Spec} \mathbb{Z}} S,$$

which is an isomorphism when S is flat over $\operatorname{Spec} \mathbb{Z}$, for example if S is the spectrum of a field of characteristic 0. M_g is quasi-projective over any open subset of $\operatorname{Spec} \mathbb{Z}$ which is different from the whole $\operatorname{Spec} \mathbb{Z}$ ([**MFK94**, Corollary 7.11, p. 143]) and, for any algebraically closed field k, $M_{g,\operatorname{Spec} k}$ is a quasi-projective normal scheme over k of dimension $3g - 3$.

CHAPTER 4

Hyperelliptic curves

In this chapter we present the classical facts about hyperelliptic curves over an algebraically closed field and we study families of hyperelliptic curves. Then we define the stack $\mathcal{H}_g$ of hyperelliptic curves of genus g and we prove that it is a closed substack of the stack $\mathcal{M}_g$ of smooth curves of genus g (Theorem 4.26). Finally we see that this induces a closed immersion on moduli spaces in characteristic zero (Proposition 4.28).

Convention 4.1. In this chapter all schemes are over $\mathbb{Z}[1/2]$, i.e. 2 is invertible in the structure sheaf of all schemes. We denote by $\mathcal{M}_g$ the stack $\mathcal{M}_g \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[1/2]$ without risk of confusion.

4.1. Hyperelliptic curves over algebraically closed fields

Convention 4.2. In this section, k denotes an algebraically closed field of characteristic $\operatorname{char}(k) \neq 2$. Every morphism between two k-schemes will be a k-morphism.

Definition 4.3. A *hyperelliptic curve* over k is a projective smooth connected curve X over k with genus $g \geq 2$ such that there exists a double cover $X \to \mathbb{P}^1_k$.

Since $\operatorname{char}(k) \neq 2$, a double cover $X \to \mathbb{P}^1_k$ is a Galois cover of degree 2.

Definition 4.4. Let X be a hyperelliptic curve over k. A *hyperelliptic involution* of X is a k-automorphism $\sigma\colon X \to X$ which generates the Galois group of some double cover $X \to \mathbb{P}^1_k$ over k.

In the example below we will see the prototype of hyperelliptic curves.

Example 4.5. Let $g \geq 2$ be an integer, let $\varepsilon \in \{0,1\}$, and let $f \in k[x]$ be a polynomial of degree $2g+1+\varepsilon$ with distinct roots. The Jacobian criterion implies that the affine scheme $U = \operatorname{Spec} k[x,y]/(y^2 - f(x))$ is smooth over k. The projection $\pi\colon U \to \mathbb{A}^1_k$ onto the x-axis is a double cover, ramified at $2g+1+\varepsilon$ points (corresponding to the roots of f).

Let X be the normalization of $\mathbb{P}^1_k$ in the field $K(U)$ of rational functions of U, hence X is a projective smooth integral curve over k and we have a double cover $h\colon X \to \mathbb{P}^1_k$. Riemann-Hurwitz formula implies that the ramification divisor R of h has even degree. Since π is finite, $h^{-1}(\mathbb{A}^1_k) = U$, hence R already contains $2g+1+\varepsilon$ points in U. If $\varepsilon = 1$ then h is not ramified at ∞, otherwise h is ramified at ∞. In both cases $\deg R = 2g+2$, hence X is a hyperelliptic curve of genus g.

LEMMA 4.6. *Let A be a UFD with fraction field K such that 2 is invertible in A. If $K(y)$ is a field extension of K of degree 2 such that $y^2 \in A$ is square-free, then $A[y]$ is the integral closure of A in $K(y)$.*

PROOF. Let $f, g \in K$ and let $\alpha = f + gy$ be an element which is integral over A. Let $\sigma\colon K(y) \to K(y)$ be the K-homomorphism which maps y to $-y$. The element $\sigma(\alpha) = f - gy$ is integral over A, hence $\alpha + \sigma(\alpha) = 2f$ is integral over A, then $f \in A$. The element $\alpha \cdot \sigma(\alpha) = f^2 - 4y^2g^2$ is integral over A, hence $y^2g^2 \in A$. Since y^2 is square-free, $g \in A$. □

The following proposition shows that every hyperelliptic curve can be constructed as in Example 4.5.

PROPOSITION 4.7. *Let X be a hyperelliptic curve over k of genus $g \geq 2$ and let $h\colon X \to \mathbb{P}^1_k$ be a double cover.*

(a) *The field of the rational functions of X is $K(X) = k(t)[y]$ with the relation $y^2 = p(t)$, where $p \in k[t]$ is a square-free polynomial with degree $2g+1 \leq \deg p \leq 2g+2$.*

(b) *The curve X is the union of two affine open subschemes*
$$U = \operatorname{Spec} k[t, Y]/(Y^2 - p(t)),$$
$$V = \operatorname{Spec} k[s, Z]/(Z^2 - q(s)),$$
where $q(s) = p(1/s)s^{2g+2}$, and U and V glue along $U_t \simeq V_s$ with relations $t = 1/s$ and $y = t^{g+1}z$.

(c) *The morphism h is given by the glueing of $U \to \mathbb{A}^1_k, (t, y) \mapsto t$ and $V \to \mathbb{A}^1_k, (s, z) \mapsto s$.*

(d) *a k-basis of $\mathrm{H}^0(X, \Omega^1_{X/k})$ is made up of*
$$\omega_i = \frac{t^i dt}{2y},$$
for $i = 0, \ldots, g-1$.

(e) *The hyperelliptic involution σ associated to the double cover h is defined by $\sigma|_U\colon (t, y) \mapsto (t, -y)$ and $\sigma|_V\colon (s, z) \mapsto (s, -z)$.*

(f) *The ramification points of h are precisely those fixed by σ;*

(g) *The invertible sheaf $\ker \operatorname{tr}_h$ has degree $-g-1$ on $\mathbb{P}^1_k$, i.e. it is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(-g-1)$.*

PROOF. Let x_0, x_1 be the homogeneous coordinates of $\mathbb{P}^1_k$, let $U_0 = (\mathbb{P}^1_k)_{x_0}$ and $U_1 = (\mathbb{P}^1_k)_{x_1}$ be the standard affine charts, and let $t = x_1/x_0$, $s = x_0/x_1$ be the affine coordinates of U_0 and U_1, respectively. The double cover h induces a Galois extension $K(X)/k(t)$ of degree 2, hence

$K(X) = k(t)[y]$ with $y^2 \in k(t)$. Multiplying by an element of $k(t)^*$ if necessary, we may suppose that $y^2 = p(t)$ is a square-free polynomial of degree d.

Now pick $U = h^{-1}(U_0)$ and $V = h^{-1}(U_1)$. Since h is finite and X is normal, U is affine and its coordinate ring $\mathcal{O}_X(U)$ is the integral closure of $\mathcal{O}_{\mathbb{P}^1_k}(U_0) = k[t]$ in the field $K(X)$, hence $\mathcal{O}_X(U) = k[t, y]$ from Lemma 4.6.

Let $\varepsilon = 0$ if d is even and let $\varepsilon = 1$ if d is odd, and let $r = [(d+1)/2] = (d+\varepsilon)/2$. We have that

$$\left(\frac{y}{t^r}\right)^2 = \frac{p(t)}{t^d} \cdot t^{-\varepsilon} = p\left(\frac{1}{s}\right) \cdot s^{d+\varepsilon} = q(s)$$

is a polynomial in the indeterminate $s = t^{-1}$. A simple calculation shows that $\deg q = d + \varepsilon - 1$ if $p(0) = 0$, $\deg q = d + \varepsilon$ if $p(0) \neq 0$, and q is square-free. Define $z = s^r y \in K(X)$: it is clear that $z^2 = q(s)$. Since h is finite and X is normal, V is affine and its coordinate ring $\mathcal{O}_X(V)$ is the integral closure of $\mathcal{O}_{\mathbb{P}^1_k}(U_1) = k[s]$ in the field $K(X)$, hence $\mathcal{O}_X(V) = k[s, z]$ from Lemma 4.6.

To conclude the proof of (a), (b), and (c), we have to show that $2g + 1 \leq d \leq 2g + 2$ and $r = g + 1$. Since every ramification of f is tame, the ramification divisor R of f is made up of simple points that corresponds to the roots of p and q, hence $\deg R = d + \varepsilon$. But Riemann-Hurwitz formula implies $\deg R = 2g+2$, hence $d+\varepsilon = 2g+2$.

(d): Since $h^0(\omega_{X/k}) = p_a(X) = g$, it suffices to prove that the ω_i's are globally defined and linearly independent over k. The differential form ω_i is defined over U_y, but from

$$\omega_i = \frac{p'(t)t^i dt}{2yp'(t)} = \frac{t^i 2y dy}{2yp'(t)} = \frac{t^i dy}{p'(t)}$$
$$\omega_i = \frac{s^{g+1} d(s^{-1})}{2s^i z} = -\frac{s^{g-1-i} ds}{2z}$$
$$\omega_i = \frac{s^{g-1-i} q'(s) ds}{2zq'(s)} = -\frac{s^{g-1-i} 2z dz}{2zq'(s)} = -\frac{s^{g-1-i} dz}{q'(s)}$$

we see that ω_i can be defined over all X, because $p'(t)$ and y cannot vanish at the same time and analogously for $q'(s)$ and z. It is easy to prove that the differentials $\omega_0, \ldots, \omega_{g-1}$ are linearly independent over k.

(e), (f): Obvious.

(g) The sheaf $\ker \operatorname{tr}_h$ is trivialized by y on U_0 and by z on U_1. The relation $y = t^{g+1} z$ gives the result. □

Corollary 4.8. *Every hyperelliptic involution of a hyperelliptic curve X over k acts as $-\mathrm{id}$ on $\mathrm{H}^0(X, \Omega^1_{X/k})$.*

Proof. It follows immediately from Proposition 4.7(d),(e); but may be proved also directly. □

Proposition 4.9. *A hyperelliptic curve X over k has a unique hyperelliptic involution.*

Proof. Let $h\colon X \to \mathbb{P}^1_k$ be a double cover and let σ be the associated hyperelliptic involution. We use the notations of Proposition 4.7. Let τ be another hyperelliptic involution, then from Corollary 4.8 we have

$$-\tau(t)\omega_0 = \tau(t)\tau(\omega_0) = \tau(t\omega_0) = \tau(\omega_1) = -\omega_1 = -t\omega_0,$$

hence $\tau(t) = t$, i.e. τ fixes the subfield $k(t)$ of $K(X)$. This means that τ is an element of $\mathrm{Gal}(K(X)/k(t)) = \{1, \sigma\}$, hence $\tau = \sigma$. □

Corollary 4.10. *If X is a hyperelliptic curve over k, then there exists a unique double cover $X \to \mathbb{P}^1_k$ up to automorphisms of $\mathbb{P}^1_k$.*

Proof. Let $h, h'\colon X \to \mathbb{P}^1_k$ be two double covers. They induces on the generic point ξ of X two field embeddings $h^{\#}_{\xi}, h'^{\#}_{\xi}\colon K(\mathbb{P}^1_k) \to K(X)$. The images of $h^{\#}_{\xi}$ and $h'^{\#}_{\xi}$ coincides because they coincide with the subfield of $K(X)$ which is fixed by *the* hyperelliptic involution of X. Therefore we can find a k-automorphism ϕ of $K(\mathbb{P}^1_k)$ such that $h'^{\#}_{\xi} = h^{\#}_{\xi} \circ \phi$. Since X and $\mathbb{P}^1_k$ are smooth, the field automorphism ϕ comes from an automorphism f of $\mathbb{P}^1_k$ such that $h' = f \circ h$. □

Corollary 4.11. *The hyperelliptic involution of a hyperelliptic curve X over k is in the center of the group $\mathrm{Aut}_k(X)$ of automorphisms of X.*

Proof. Let σ be the hyperelliptic involution, let τ be an automorphism, and let $h\colon X \to \mathbb{P}^1_k$ be a double cover. The hyperelliptic involution associated to the double cover $h \circ \tau$ is $\tau\sigma\tau^{-1}$, then $\tau\sigma\tau^{-1} = \sigma$, by the uniqueness of the hyperelliptic involution. □

Remark 4.12. Let D be an effective divisor on the projective smooth connected curve X over k and let $\mathcal{L}$ be an invertible sheaf on X, then

$$h^0(\mathcal{L}) - \deg D \leq h^0(\mathcal{L}(-D)) \leq h^0(\mathcal{L}).$$

These inequalities can be proved by looking at the cohomology long exact sequence of $\mathcal{L} \otimes_{\mathcal{O}_X} (4.1)$, where

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0 \tag{4.1}$$

is the short exact sequence defined by the closed subscheme $D \subseteq X$.

Lemma 4.13. *Let X be a projective smooth connected curve over k of genus $g \geq 1$. If $\mathcal{L}$ is an invertible sheaf on X such that $\deg \mathcal{L} > 0$, then $h^0(\mathcal{L}) \leq \deg \mathcal{L}$.*

Proof. We may assume $h^0(\mathcal{L}) > 0$. Then $\mathcal{L} \simeq \mathcal{O}_X(D)$ for some effective divisor D. From Remark 4.12 we see that $h^0(\mathcal{L}) \leq h^0(\mathcal{O}_X) + \deg \mathcal{L} = 1 + \deg \mathcal{L}$. We proceed by induction on $\deg D = \deg \mathcal{L}$.

We prove the base step of induction: $\deg \mathcal{L} = \deg D = 1$. Suppose by contradiction that $h^0(\mathcal{L}) = 2$. For every closed point p of X, the divisor $D - p$ has degree 0 and the corresponding invertible sheaf has non-zero global sections because $h^0(\mathcal{L}(-p)) \geq h^0(\mathcal{L}) - 1 = 1$; hence, for every closed point p of X, $D - p$ is a principal divisor, i.e. $\mathcal{L}(-p) \simeq \mathcal{O}_X$, and $h^0(\mathcal{L}(-p)) = h^0(\mathcal{O}_X) = 1 < 2 = h^0(\mathcal{L})$. This proves that the linear system given by $\mathcal{L}$ has no base points, i.e. $\mathcal{L}$ is generated by its global sections. Therefore, the sheaf $\mathcal{L}$ induces a morphism $X \to \mathbb{P}^1_k$ of degree $\deg \mathcal{L} = 1$, hence X is isomorphic to $\mathbb{P}^1_k$, which is absurd because $g \geq 1$.

Now we are proving the inductive step. We can write $D = D' + p$ for some effective divisor D' and some closed point p. From Remark 4.12 we have $h^0(\mathcal{L}) \leq h^0(\mathcal{O}_X(D')) + 1 \leq \deg D' + 1 = \deg D$. □

THEOREM 4.14. *Let X be a projective smooth connected curve over k of genus $g \geq 2$. The following conditions are equivalent:*

(1) X is hyperelliptic;
(2) the canonical map $X \to \mathbb{P}^{g-1}_k$ decomposes into a double cover $X \to \mathbb{P}^1_k$ followed by the Veronese $(g-1)$-uple embedding $\mathbb{P}^1_k \to \mathbb{P}^{g-1}_k$;
(3) the canonical map $X \to \mathbb{P}^{g-1}_k$ is not a closed immersion;
(4) the canonical sheaf $\omega_{X/k}$ is not very ample;
(5) there exist two closed points $p, q \in X$, possibly equal, such that $h^0(\mathcal{O}_X(p+q)) = 2$;
(6) there exists an invertible sheaf $\mathcal{L}$ on X such that $\deg \mathcal{L} = 2$ and $h^0(\mathcal{L}) = 2$.

PROOF. (1) ⇒ (2): choose a double cover $h \colon X \to \mathbb{P}^1_k$ and use notations of Proposition 4.7. The sheaf

$$\mathcal{L} = \left(\frac{dt}{2y}\right)^{-1} \Omega^1_{X/k}$$

is invertible on X, is contained in the sheaf of rational functions of X, and is isomorphic to the canonical sheaf $\Omega^1_{X/k}$. The canonical map $X \to \mathbb{P}^{g-1}_k$ is defined by the sheaf $\mathcal{L}$ with sections $1, t, \dots, t^{g-1}$ (Proposition 4.7(d)). On the other hand, it is clear that

$$\mathcal{L} \simeq h^* \mathcal{O}_{\mathbb{P}^1_k}(g-1) \simeq h^* v^*_{1,g-1} \mathcal{O}_{\mathbb{P}^{g-1}_k}(1),$$

where $v_{1,g-1} \colon \mathbb{P}^1_k \to \mathbb{P}^{g-1}_k$ is the Veronese $(g-1)$-embedding. Therefore the canonical map $X \to \mathbb{P}^{g-1}_k$ coincides with $v_{1,g-1} \circ h$.

(2) ⇒ (3): obvious.

(3) ⇔ (4): by definition.

(4) ⇔ (5): [**Har77**, Proposition IV.3.1(b)] says that an invertible sheaf $\mathcal{L}$ is very ample if and only if $h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2$ for every two closed points $p, q \in X$, possibly equal. This shows that (4) is equivalent with saying that there exist two closed points $p, q \in X$ such that $h^0(\omega_X(-p-q)) \neq g - 2$. Riemann-Roch formula applied to the

divisor $p+q$ shows that $h^0(\omega_X(-p-q)) = h^0(\mathcal{O}_X(p+q)) - 3 + g$, then (4) is equivalent with $h^0(\mathcal{O}_X(p+q)) \neq 1$ for some points $p, q \in X$.

On the other hand, for every two closed points $p, q \in X$ we have that $h^0(\mathcal{O}_X(p+q)) \geq 1$, because $p+q$ is effective, and $h^0(\mathcal{O}_X(p+q)) \leq 2$ from Lemma 4.13.

(5) $\Rightarrow$ (6): pick $\mathcal{L} = \mathcal{O}_X(p+q)$.

(6) $\Rightarrow$ (1): it suffices to show that $\mathcal{L}$ is generated by its global sections, i.e. $h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1 = 1$ for every closed point $p \in X$ ([**Har77**, Proposition IV.3.1(a)]). From Remark 4.12 we have that $1 \leq h^0(\mathcal{L}(-p)) \leq 2$ and $h^0(\mathcal{L}(-p)) \leq \deg \mathcal{L}(-p) = 1$ from Lemma 4.13. $\square$

COROLLARY 4.15. *Every projective smooth connected curve of genus 2 over k is hyperelliptic.*

PROOF. The canonical sheaf $\omega_{X/k}$ has degree 2 and $h^0(\omega_{X/k}) = 2$. Use condition (6) in the theorem above. $\square$

4.2. The groupoid of hyperelliptic curves: $\mathcal{H}_g$

DEFINITION 4.16. A *family of hyperelliptic curves of genus g* is a commutative triangle of morphisms of schemes

$$\begin{array}{ccc} X & \longrightarrow & P \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{4.2}$$

where $P \to S$ is a conic bundle, $X \to S$ is a double cover, and $X \to S$ is a family of smooth curves of genus g.

Often we will denote by $(X \to P \to S)$ the triangle (4.2).

DEFINITION 4.17. The category $\mathcal{H}_g$ of hyperelliptic curves of genus g is the category defined as follows:

- the objects of $\mathcal{H}_g$ are families
$$(X \to P \to S)$$
of hyperelliptic curves of genus g (Definition 4.16);
- an arrow from $(X' \to P' \to S')$ to $(X \to P \to S)$ is a triple (a, b, c), where $a\colon X' \to X$, $b\colon P' \to P$, and $c\colon S' \to S$ are morphisms of schemes such that the following squares are cartesian:
$$\begin{array}{ccccc} X' & \longrightarrow & P' & \longrightarrow & S' \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ X & \longrightarrow & P & \longrightarrow & S \end{array}$$
Composition of arrows is defined in the obvious way.

It is clear that $\mathcal{H}_g$ is a fibred category in groupoids over $(\mathrm{Sch}/\mathbb{Z}[1/2])$, according to Convention 4.1.

Now we present two other equivalent descriptions of $\mathcal{H}_g$ that come from constructing the conic bundle P from the curve X (Proposition 4.18) or from constructing the curve X from the conic bundle P (Proposition 4.19). Necessary tools for these descriptions are the theory of finite quotients of curves (Section 3.3) and the classification of double covers (Section 1.3).

PROPOSITION 4.18. *The groupoid $\mathcal{H}_g$ of hyperelliptic curves is equivalent to the groupoid over $(\mathrm{Sch}/\mathbb{Z}[1/2])$ defined as follows:*

- *objects are pairs*
$$(X \to S, \sigma),$$
where $X \to S$ is a smooth curve of genus g and σ is an S-automorphism of X such that $\sigma^2 = \mathrm{id}_X$, $\sigma \neq \mathrm{id}_X$, $X/\langle\sigma\rangle \to S$ is a conic bundle;
- *an arrow from $(p' \colon X' \to S', \sigma')$ to $(p \colon X \to S, \sigma)$ is a pair (a, c) of morphisms of schemes $a \colon X' \to X$, $c \colon S' \to S$ such that the diagram*
$$\begin{CD} X' @>{p'}>> S' \\ @V{a}VV @VV{c}V \\ X @>{p}>> S \end{CD}$$
is commutative and cartesian and $\sigma \circ a = a \circ \sigma'$. Composition of arrows is defined in the obvious way.

PROOF. Denote by $\mathcal{A}$ the groupoid defined above. Consider the functor $\mathcal{H}_g \to \mathcal{A}$ defined by
$$(X \to P \to S) \mapsto (X \to S, \sigma),$$
where $\sigma \colon X \to X$ is the natural automorphism of order 2 constructed in Proposition 1.14. Since $P \simeq X/\langle\sigma\rangle$, the morphism $X/\langle\sigma\rangle \to S$ is a conic bundle and then $(X \to S, \sigma)$ is an object of $\mathcal{A}$.

Now consider the functor $\mathcal{A} \to \mathcal{H}_g$ defined by
$$(X \to S, \sigma) \mapsto (X \to X/\langle\sigma\rangle \to S).$$
This morphism is well defined because $X \to X/\langle\sigma\rangle$ is a double cover (Theorem 3.21). It is easy to check that this functor is a quasi-inverse to the previous functor. □

PROPOSITION 4.19. *The groupoid $\mathcal{H}_g$ of hyperelliptic curves is equivalent to the groupoid over $(\mathrm{Sch}/\mathbb{Z}[1/2])$ defined as follows:*

- *objects are triplets*
$$(P \to S, \mathcal{L}, m),$$

where $P \to S$ is a conic bundle, $\mathcal{L} \in \mathrm{Pic}(P)$ is an invertible sheaf on P and $m\colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_P$ is a homomorphism of $\mathcal{O}_P$-modules such that for every geometric point $s\colon \mathrm{Spec}\,\Omega \to S$ the restriction $\mathcal{L}_s$ to the geometric fibre $P_s \simeq \mathbb{P}^1_\Omega$ has degree $-g-1$ and the homomorphism $m_s\colon \mathcal{L}_s^{\otimes 2} \simeq \mathcal{O}_{P_s}(-2g-2) \to \mathcal{O}_{P_s}$ corresponds to a smooth form of degree $2g+2$;

- *an arrow from $(P' \to S', \mathcal{L}', m')$ to $(P \to S, \mathcal{L}, m)$ is a triple (b, c, α) where $b\colon P' \to P$, $c\colon S' \to S$ are morphisms of schemes such that the diagram*

$$\begin{array}{ccc} P' & \longrightarrow & S' \\ \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle c} \\ P & \longrightarrow & S \end{array}$$

is commutative and cartesian and $\alpha\colon b^*\mathcal{L} \to \mathcal{L}'$ is an isomorphism such that $m' \circ \alpha^{\otimes 2} = m$. Composition of arrows is defined in the obvious way.

Proof. Recall that a homogeneous polynomial $F \in k[x_0, x_1]$ of degree d over an algebraically closed field k is called a smooth form if the zero locus $\mathrm{V}(F) \subseteq \mathbb{P}^1_k$ consists of distinct d closed points.

Denote by $\mathcal{B}$ the groupoid defined above. Consider the functor $\mathcal{B} \to \mathcal{H}_g$ defined by

$$(P \to S, \mathcal{L}, m) \mapsto (\mathcal{S}pec_{\mathcal{O}_P}(\mathcal{O}_P \oplus \mathcal{L}) \to P \to S)$$

where the product in the $\mathcal{O}_P$-algebra $\mathcal{O}_P \oplus \mathcal{L}$ is given by

$$(a_1 \oplus b_1) \cdot (a_2 \oplus b_2) = (a_1 a_2 + m(b_1 b_2) \oplus a_1 b_2 + a_2 b_1),$$

where a_1, a_2 are sections of $\mathcal{O}_P$ and b_1, b_2 are sections of $\mathcal{L}$. We want to prove that this functor is well defined, i.e that $(X \to P \to S)$ is an object of $\mathcal{H}_g(S)$, where $X = \mathcal{S}pec_{\mathcal{O}_P}(\mathcal{O}_P \oplus \mathcal{L})$. Clearly $X \to P$ is a double cover, hence it suffices to show that $X \to S$ is a smooth curve of genus g. The morphism $X \to S$ is proper, flat and finitely presented because it is the composition of two proper flat morphisms of finite presentation: the double cover $X \to P$ and the conic bundle $P \to S$. We shall investigate the geometric fibres of $X \to S$; choose a geometric point $s\colon \mathrm{Spec}\,\Omega \to S$ of S, where Ω is an algebraically closed field. Since $\mathcal{L}_s$ has degree $-g-1$ over P_s and m_s corresponds to a smooth form of degree $2g+2$, Example 1.12 shows that $X_s = \mathcal{S}pec_{\mathcal{O}_{P_s}}(\mathcal{O}_{P_s} \oplus \mathcal{L}_s)$ is a smooth connected curve of genus g over the field Ω.

Now we consider the functor $\mathcal{H}_g \to \mathcal{B}$ defined by

$$(X \xrightarrow{h} P \to S) \mapsto (P \to S, \ker \mathrm{tr}_h, \mu|_{(\ker \mathrm{tr}_h)^{\otimes 2}}),$$

where $\mu\colon (h_*\mathcal{O}_X)^{\otimes 2} \to h_*\mathcal{O}_X$ is the product of the $\mathcal{O}_P$-algebra $h_*\mathcal{O}_X$. Thanks to Theorem 1.11 and to Example 1.12, this functor is well defined and is a quasi-inverse of the functor defined above. $\square$

Now we prove that the conic bundle of a hyperelliptic curve is uniquely determined.

LEMMA 4.20 ([**LK79**, Proposition 3.8]). *Let $f\colon X \to S$ be a smooth curve of genus g, and let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is isomorphic to ω_f if and only if the following two conditions hold:*

(i) there exists an isomorphism $\mathrm{R}^1 f_*\mathcal{L} \simeq \mathcal{O}_S$;

(ii) for every point $s \in S$, the restriction $\mathcal{L}_s$ of $\mathcal{L}$ to the fibre X_s has degree $\deg \mathcal{L}_s = 2g-2$.

PROOF. The two conditions are necessary because the degree of the canonical sheaf on a complete smooth curve of genus g is $2g-2$ and the isomorphism between $\mathrm{R}^1 f_*(\omega_f)$ and $\mathcal{O}_S$ is provided by the trace map γ_f since the fibres of f are geometrically integral (Proposition B.5(6)).

Now we prove that the two conditions are sufficient. Since the fibres of f have dimension 1, the direct image functors $\mathrm{R}^i f_*$ are null for $i \geq 2$. Suppose that $\mathrm{R}^1 f_*\mathcal{L} \simeq \mathcal{O}_S$. Then, for every $i > 0$, the sheaf $\mathrm{R}^i f_*(\mathcal{L})$ is locally free of finite rank. By Theorem B.6 with $\mathcal{F} = \mathcal{L}$, $\mathcal{G} = \mathcal{O}_S$, $n = 1$, $m = 0$ and $i = 1$, we have natural isomorphisms that commute with base change:

$$f_*(\mathcal{L}^\vee \otimes \omega_f) \simeq \mathcal{H}om_{\mathcal{O}_S}(\mathrm{R}^1 f_*(\mathcal{L}), \mathcal{O}_S) \simeq \mathcal{H}om_{\mathcal{O}_S}(\mathcal{O}_S, \mathcal{O}_S) \simeq \mathcal{O}_S. \tag{4.3}$$

By applying f^*, we obtain a homomorphism $\phi\colon \mathcal{O}_X \to \mathcal{L}^\vee \otimes \omega_f$ of $\mathcal{O}_X$-modules. We shall prove that ϕ is an isomorphism. Since the isomorphisms in (4.3) are compatible with base change, for every point s of S the restriction $(\mathcal{L}^\vee \otimes \omega_f)|_{X_s}$ is an invertible sheaf on the fibre X_s of degree 0 such that $\mathrm{H}^0(X_s, (\mathcal{L}^\vee \otimes \omega_f)|_{X_s}) \simeq k(s)$, hence it is the trivial line bundle $\mathcal{O}_{X_s}$; this proves the bijectivity of the restriction of ϕ to each fibre of f, i.e. for every point $s \in S$ the induced homomorphism

$$\phi \otimes 1\colon \mathcal{O}_X \otimes_{\mathcal{O}_S} k(s) \longrightarrow (\mathcal{L}^\vee \otimes \omega_f) \otimes_{\mathcal{O}_S} k(s)$$

is an isomorphism of sheaves on the fibre X_s. Since f is finitely presented, the local flatness criterion ([**EGA**, Proposition IV.11.3.7]) implies that ϕ is injective and $\mathcal{Q} = \operatorname{coker} \phi$ is flat over S. Since $\mathcal{O}_X$ and $\mathcal{L}^\vee \otimes \omega_f$ are invertible sheaves on X, $\mathcal{Q}$ is finitely presented over X. If we apply [**EGA**, Proposition IV.11.3.7] to the homomorphism $\mathcal{Q} \to 0$ we obtain that $\mathcal{Q} = 0$. This proves that ϕ is an isomorphism, therefore $\mathcal{L} \simeq \omega_f$. □

PROPOSITION 4.21 ([**LK79**, Lemma 5.7]). *If $(X \xrightarrow{h} P \xrightarrow{q} S) \in \mathcal{H}_g(S)$ is a family of hyperelliptic curves of genus g and $f = q \circ h$, then there exists a closed immersion $i\colon P \hookrightarrow \mathbb{P}(f_*\omega_f)$ such that $i \circ h$ is the canonical morphism of f.*

PROOF. Since h is a double cover and 2 is invertible in $\mathcal{O}_P$, $\mathcal{L} = \ker \operatorname{tr}_h$ and $\mathcal{M} = \mathcal{L}^\vee \otimes_{\mathcal{O}_P} \omega_q$ are invertible sheaves on P and $h_*\mathcal{O}_X \simeq \mathcal{O}_P \oplus \mathcal{L}$ (Proposition 1.9). We shall prove that $h^*\mathcal{M}$ is isomorphic to ω_f by means of Lemma 4.20.

For every geometric point $s\colon \operatorname{Spec}\Omega \to S$ of S, the restriction $\mathcal{L}_s$ to the fibre $P_s \simeq \mathbb{P}^1_\Omega$ has degree $-g-1<0$ by Proposition 4.7(g); hence for every point $s \in S$ we have $\mathrm{H}^0(P_s, \mathcal{L}_s) = 0$. By Lemma B.2, $q_*\mathcal{L} = 0$. (Actually Lemma B.2 is stated for locally noetherian schemes, but it is sufficient to our case up to use some easy arguments of noetherian approximation.)

Note that we have $\mathrm{R}^j f_* = (\mathrm{R}^j q_*) \circ h_*$, for $j \geq 0$, since h is affine. Therefore

$$\begin{aligned}\mathrm{R}^1 f_*(h^*\mathcal{M}) &\simeq \mathrm{R}^1 q_*(h_*(h^*\mathcal{M}))\\ &\simeq \mathrm{R}^1 q_*(\mathcal{M} \otimes h_*\mathcal{O}_X)\\ &\simeq \mathrm{R}^1 q_*((\mathcal{L}^\vee \otimes \omega_q) \otimes (\mathcal{O}_P \oplus \mathcal{L}))\\ &\simeq \mathrm{R}^1 q_*((\mathcal{O}_P \oplus \mathcal{L})^\vee \otimes \omega_q)\\ &\simeq \mathcal{H}om_{\mathcal{O}_S}(q_*(\mathcal{O}_P \oplus \mathcal{L}), \mathcal{O}_S),\end{aligned}$$

where the last isomorphism holds for Grothendieck duality (Theorem B.6 with $\mathcal{F} = \mathcal{O}_P \oplus \mathcal{L}$, $\mathcal{G} = \mathcal{O}_S$, $n=1$, $i=m=0$), because $\mathrm{R}^1 q_*(\mathcal{O}_P \oplus \mathcal{L}) = \mathrm{R}^1 q_*(h_*\mathcal{O}_X) = \mathrm{R}^1 f_*(\mathcal{O}_X)$ is locally free of rank g by Lemma 3.9. Now $q_*\mathcal{O}_P \simeq \mathcal{O}_S$, by Lemma 3.15, and $q_*\mathcal{L} = 0$, then $\mathrm{R}^1 f_*(h^*\mathcal{M}) \simeq \mathcal{O}_S$. This is the first condition in Lemma 4.20.

Now we see that the restriction $\mathcal{M}_s$ of $\mathcal{M}$ to the fibres of q has degree $g-1$, because $\mathcal{L}_s$ has degree $-g-1$ and $(\omega_q)_s$ has degree -2. Since h has degree 2, the degree of $h^*\mathcal{M}$ at the fibres of f has degree $2(g-1)$. This is the second condition in Lemma 4.20. Therefore we have proved that $h^*\mathcal{M} \simeq \omega_f$.

For every point $s \in S$, the restriction $\mathcal{M}_s$ to the fibre P_s is very ample relative to $\operatorname{Spec} k(s)$ because $\deg \mathcal{M}_s \geq 1$ and

$$h^1(P_s, \mathcal{M}_s) = h^0(P_s, \omega_{P_s} \otimes \mathcal{M}_s^\vee) = h^0(P_s, \mathcal{L}_s) = 0$$

because $\deg \mathcal{L}_s < 0$. By Lemma B.4, the sheaf $\mathcal{M}$ is very ample relative to q, hence it induces a S-closed immersion $i\colon P \hookrightarrow \mathbb{P}(q_*\mathcal{M})$. (Actually Lemma B.4 is stated for locally noetherian schemes, but it is sufficient up to use some easy arguments of noetherian approximation.)

Since h is affine, $f_*h^*\mathcal{M} \simeq q_*\mathcal{M}$. Then $h^*\mathcal{M} \simeq \omega_f$ implies that $f_*\omega_f \simeq q_*\mathcal{M}$. Besides h^* applied to the surjection $q^*q_*\mathcal{M} \to \mathcal{M}$ yields the natural surjection $f^*f_*\omega_f \to \omega_f$. This proves that $i \circ h$ is the canonical map of X over S. □

PROPOSITION 4.22. *Let $f\colon X \to S$ be a smooth curve of genus $g \geq 2$. Then the following conditions are equivalent:*

- *(a) there exists a conic bundle $q\colon P \to S$ and a double cover $h\colon X \to P$ such that $f = q \circ h$;*
- *(b) there exists an S-morphism $\sigma\colon X \to X$ such that $\sigma^2 = \mathrm{id}_X$ and $X/\langle\sigma\rangle \to S$ is a conic bundle;*
- *(c) the image of the canonical morphism $X \to \mathbb{P}(f_*\omega_f)$ of f is a conic bundle over S.*

PROOF. It follows from the propositions above. □

4.3. $\mathcal{H}_g$ is an algebraic stack

Let V be the space of homogeneous polynomials of degree $2g+2$ in the variables x_0, x_1 over $\mathbb{Z}[1/2]$, i.e. linear combinations of x_0^{2g+2}, $x_0^{2g+1}x_1$, $x_0^{2g}x_1^2$, $\dots$, x_1^{2g+2}; it is an affine space of dimension $2g+3$ over $\mathbb{Z}[1/2]$. Let $V_{\mathrm{sm}} \subseteq V$ be the open subscheme corresponding to smooth forms, i.e. forms F such that for every algebraically closed field k of characteristic $\neq 2$ the zero locus $\operatorname{Proj} k[x_0, x_1]/(\overline{F}(x_0, x_1)) \subseteq \mathbb{P}^1_k$ consists of distinct $2g+2$ closed points. One can prove that $V \setminus V_{\mathrm{sm}}$ is an irreducible hypersurface (see [**GKZ94**]).

There is a natural action of $\mathrm{GL}_2 = \mathrm{GL}_{2,\mathbb{Z}[1/2]}$ on V_{sm}, defined in functorial notation by $A \cdot f(x) = f(A^{-1}x)$. The subgroup scheme $\mu_{g+1} = \mu_{g+1,\mathbb{Z}[1/2]} \subseteq \mathrm{GL}_2$, embedded by sending a $(g+1)^{\mathrm{th}}$ root of 1 α into the diagonal matrix αI_2, acts trivially on V_{sm}, so this induces an action of the quotient GL_2/μ_{g+1} on V_{sm}.

THEOREM 4.23 ([**AV04**, Corollary 4.2]). *The fibred category $\mathcal{H}_g$ is isomorphic to the quotient stack $[V_{\mathrm{sm}}/(\mathrm{GL}_2/\mu_{g+1})]$ by the action described above.*

PROOF. We identify $\mathcal{H}_g$ with the groupoid defined in Proposition 4.19, hence an object of $\mathcal{H}_g(S)$ is a triplet

$$(P \to S, \mathcal{L}, m),$$

where $P \to S$ is a conic bundle, $\mathcal{L} \in \operatorname{Pic}(P)$ is an invertible sheaf on P and $m\colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_P$ is a homomorphism of $\mathcal{O}_P$-modules such that for every geometric point $s\colon \operatorname{Spec}\Omega \to S$ the restriction $\mathcal{L}_s$ to the geometric fibre $P_s \simeq \mathbb{P}^1_\Omega$ has degree $-g-1$ and the homomorphism $m_s\colon \mathcal{L}_s^{\otimes 2} \simeq \mathcal{O}_{P_s}(-2g-2) \to \mathcal{O}_{P_s}$ corresponds to a smooth form of degree $2g+2$.

Consider the auxiliary groupoid $\mathcal{F}$, whose objects over a base scheme S are

$$(P \to S, \mathcal{L}, m, \phi)$$

where $(P \to S, \mathcal{L}, m)$ is an object of $\mathcal{H}_g(S)$ and $\phi\colon (P, \mathcal{L}) \simeq (\mathbb{P}^1_S, \mathcal{O}(-g-1))$ is an isomorphism over S (by this we mean the pair consisting of an isomorphism of S-schemes $\phi_0\colon P \simeq \mathbb{P}^1_S$, plus an isomorphism $\phi_1\colon \mathcal{L} \simeq \phi_0^*\mathcal{O}(-g-1)$). The arrows in $\mathcal{F}$ are arrows in $\mathcal{H}_g$ preserving the isomorphisms ϕ. No object of $\mathcal{F}$ has a nontrivial automorphism mapping to identity in the category of schemes, so $\mathcal{F}$ is equivalent to a functor.

For any object $(P \to S, \mathcal{L}, m, \phi)$ of $\mathcal{F}(S)$ take the composition

$$\phi \circ m \circ (\phi^{-1})^{\otimes 2}\colon \mathcal{O}_{\mathbb{P}^1_S}(-2g-2) \to \mathcal{O}_{\mathbb{P}^1_S}$$

corresponding to a section of $\mathcal{O}_{\mathbb{P}^1_S}(2g+2)$ that is smooth on any geometric fibre of $\mathbb{P}^1_S \to S$, that is, to an element of $V_{\mathrm{sm}}(S)$. This defines a functor $\mathcal{F} \to V_{\mathrm{sm}}$ of groupoids over $(\mathrm{Sch}/\mathbb{Z}[1/2])$. There is also a functor in the other direction, by sending a section $f \in \mathcal{O}_{\mathbb{P}^1_S}(2g+2)$, interpreted as a homomorphism $f\colon \mathcal{O}_{\mathbb{P}^1_S}(-2g-2) \to \mathcal{O}_{\mathbb{P}^1_S}$, into the object

$$\Bigl(\mathbb{P}^1_S \to S, \mathcal{O}_{\mathbb{P}^1_S}(-g-1), f\colon \mathcal{O}_{\mathbb{P}^1_S}(-g-1)^{\otimes 2} \to \mathcal{O}_{\mathbb{P}^1_S}, \mathrm{id}\Bigr)$$

of $\mathcal{F}(S)$. It is not difficult to check that this gives a quasi-inverse to the previous functor. So we get an equivalence between $\mathcal{F}$ and V_{sm}.

Now, for each integer $e \in \mathbb{Z}$ consider the functor $\mathbf{Aut}(\mathbb{P}^1_{\mathbb{Z}[1/2]}, \mathcal{O}(e))$ from the category of schemes over $\mathbb{Z}[1/2]$ to the category of groups that sends each schemes S into the group of automorphisms of the pair $(\mathbb{P}^1_S, \mathcal{O}(e))$ over the identity of S. This is a sheaf in the fpqc topology because it is clearly sheaf in the Zariski topology and quasi-compact faithfully flat morphisms are morphisms of descent with respect to the fibred category of morphisms of schemes ([**SGA1**, Exposé VIII, Théorème 5.2]) and to the fibred category of quasi-coherent sheaves ([**SGA1**, Exposé VIII, Théorème 1.1]). The sheaf $\mathbf{Aut}(\mathbb{P}^1_{\mathbb{Z}[1/2]}, \mathcal{O}(1))$ can be identified with $\mathrm{GL}_2 = \mathrm{GL}_{2,\mathbb{Z}[1/2]}$ because an isomorphism of the pair $(\mathbb{P}^1_S, \mathcal{O}(1))$ gives via $\pi\colon \mathbb{P}^1_S \to S$ an automorphism of $\pi_*\mathcal{O}(1) = \mathcal{O}_S^{\oplus 2}$ as an $\mathcal{O}_S$-module, and conversely. There is a natural homomorphism of sheaves of groups

$$\mathbf{Aut}(\mathbb{P}^1_{\mathbb{Z}[1/2]}, \mathcal{O}(1)) \longrightarrow \mathbf{Aut}(\mathbb{P}^1_{\mathbb{Z}[1/2]}, \mathcal{O}(e)) \tag{4.4}$$

sending each automorphism $(\phi_0, \phi_1)\colon (\mathbb{P}^1_S, \mathcal{O}(1)) \simeq (\mathbb{P}^1_S, \mathcal{O}(1))$ into

$$(\phi_0, \phi_1^{\otimes e})\colon (\mathbb{P}^1_S, \mathcal{O}(e)) \simeq (\mathbb{P}^1_S, \mathcal{O}(e)).$$

The homomorphism (4.4) is a surjective homomorphism of fppf sheaves and, if we identify $\mathbf{Aut}(\mathbb{P}^1_{\mathbb{Z}[1/2]}, \mathcal{O}(1))$ with $\mathrm{GL}_{2,\mathbb{Z}[1/2]}$, the kernel of (4.4) is the subgroup $\mu_{|e|,\mathbb{Z}[1/2]}$ embedded diagonally. So we get an isomorphism

$$\mathbf{Aut}\left(\mathbb{P}^1_{\mathbb{Z}[1/2]}, \mathcal{O}(-g-1)\right) \simeq \mathrm{GL}_{2,\mathbb{Z}[1/2]}/\mu_{g+1,\mathbb{Z}[1/2]}.$$

There is a left action of $\mathbf{Aut}(\mathbb{P}^1_{\mathbb{Z}[1/2]}, \mathcal{O}(-g-1))$ on $\mathcal{F}$: if

$$(P \to S, \mathcal{L}, m, \phi\colon (P, \mathcal{L}) \simeq (\mathbb{P}^1_S, \mathcal{O}(-g-1))$$

is an object of $\mathcal{F}(S)$ and $\alpha\colon (\mathbb{P}^1_S, \mathcal{O}(-g-1)) \simeq (\mathbb{P}^1_S, \mathcal{O}(-g-1))$ is an element of $\mathbf{Aut}(\mathbb{P}^1_{\mathbb{Z}[1/2]}, \mathcal{O}(-g-1))(S)$, we associate with these the object

$$\left(P \to S, \mathcal{L}, m, \alpha \circ \phi\colon (P, \mathcal{L}) \simeq (\mathbb{P}^1_S, \mathcal{O}(-g-1))\right).$$

Furthermore, given an invertible sheaf $\mathcal{L}$ on the conic bundle $P \to S$ whose degree is $-g-1$ on every geometric fibre, there is an fppf covering $S' \to S$ such that the pullback of the pair $(P, \mathcal{L})$ to S' is isomorphic to $(\mathbb{P}^1_{S'}, \mathcal{O}(-g-1))$; this fact implies that that the forgetful morphism $\mathcal{F} \to \mathcal{H}_g$

$$(P \to S, \mathcal{L}, m, \phi) \mapsto (P \to S, \mathcal{L}, m)$$

is a torsor with group $\mathbf{Aut}(\mathbb{P}^1_{\mathbb{Z}[1/2]}, \mathcal{O}(-g-1)) = \mathrm{GL}_{2,\mathbb{Z}[1/2]}/\mu_{g+1,\mathbb{Z}[1/2]}$.

If we identify $\mathcal{F}$ with V_{sm}, we obtain that $\mathcal{H}_g$ is isomorphic to the quotient stack $[V_{\mathrm{sm}}/(\mathrm{GL}_{2,\mathbb{Z}[1/2]}/\mu_{g+1,\mathbb{Z}[1/2]})]$, where the action is described above. □

COROLLARY 4.24. *$\mathcal{H}_g$ is an irreducible smooth algebraic stack of finite type over $\mathbb{Z}[1/2]$, of relative dimension $2g-1$.*

PROOF. It follows from Theorem 4.23 because V_{sm} has relative dimension $2g+3$ and GL_2/μ_{g+1} has relative dimension 4. □

4.4. $\mathcal{H}_g$ is a closed substack of $\mathcal{M}_g$ if $g \geq 2$

The purpose of this section is to prove that the $\mathcal{H}_g$ is a closed substack of $\mathcal{M}_g$ via the forgetful functor $\mathcal{H}_g \to \mathcal{M}_g$ defined by

$$(X \to P \to S) \mapsto (X \to S).$$

Our method consists of factorizing the morphism $\mathcal{H}_g \to \mathcal{M}_g$ as $\mathcal{H}_g \to \mathcal{I}_{\mathcal{M}_g} \to \mathcal{M}_g$, where $\mathcal{I}_{\mathcal{M}_g}$ is the inertia stack of $\mathcal{M}_g$ defined in Section A.5, the morphism $\mathcal{H}_g \to \mathcal{I}_{\mathcal{M}_g}$ maps a hyperelliptic curve into the canonical involution, and the morphism $\mathcal{I}_{\mathcal{M}_g} \to \mathcal{M}_g$ is the natural forgetful morphism.

PROPOSITION 4.25. *The morphism $\mathcal{H}_g \to \mathcal{I}_{\mathcal{M}_g}$ defined by*

$$(X \to P \to S) \mapsto (X \to S, \sigma)$$

where $\sigma\colon X \to X$ is the canonical involution of X over S is representable, proper and formally unramified.

PROOF. We use the description of $\mathcal{H}_g$ give in Proposition 4.18; hence an object of $\mathcal{H}_g(S)$ is a pair $(X \to S, \sigma)$ such that $X \to S$ is a smooth curve of genus g and $\sigma\colon X \to X$ is an automorphism over S such that $\sigma^2 = \mathrm{id}_X$ and $X/\langle\sigma\rangle \to S$ is a conic bundle. On the other hand, an object of $\mathcal{I}_{\mathcal{M}_g}(S)$ is a pair $(X \to S, \sigma)$ such that $X \to S$ is a smooth curve of genus g and $\sigma\colon X \to X$ is an automorphism over S. Then $\mathcal{H}_g$ is a subgropoid of $\mathcal{I}_{\mathcal{M}_g}$, i.e. $\mathcal{H}_g(S)$ is a full subcategory of $\mathcal{I}_{\mathcal{M}_g}(S)$ for every scheme S. Therefore the morphism $\mathcal{H}_g \to \mathcal{I}_{\mathcal{M}_g}$ is a representable and formally unramified (Proposition 2.23).

Since $\mathcal{H}_g$ is of finite type over $\mathcal{I}_{\mathcal{M}_g}$ (Corollary 4.24), we use the valuative criterion to show that the morphism $\mathcal{H}_g \to \mathcal{I}_{\mathcal{M}_g}$ is proper. Suppose we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathcal{H}_g \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \xrightarrow{\alpha} & \mathcal{I}_{\mathcal{M}_g} \end{array} \tag{4.5}$$

where R is discrete valuation ring and K is its quotient field. The morphism α is given by a pair $(X \to \operatorname{Spec} R, \sigma)$, where $X \to \operatorname{Spec} R$

is a smooth curve of genus g and $\sigma\colon X \to X$ is an automorphism over $\operatorname{Spec} R$. Let η be the generic point of $\operatorname{Spec} R$. The commutativity of the diagram above implies that the pull-back $(X_\eta \to \operatorname{Spec} K, \sigma_\eta)$ of α to the generic point η is an element of $\mathcal{H}_g(\operatorname{Spec} K)$, i.e. $\sigma_\eta^2 = \mathrm{id}_{X_\eta}$ and $X_\eta/\langle\sigma_\eta\rangle \to \operatorname{Spec} K$ is smooth curve of genus 0. Since σ^2 and id_X coincide over the generic fibre X_η and X is reduced and separated over $\operatorname{Spec} R$, $\sigma^2 = \mathrm{id}_X$. Then, by Theorem 3.21, $X/\langle\sigma\rangle \to S$ is a proper smooth morphism whose fibres are geometrically connected curves. Since $(X/\langle\sigma\rangle)_\eta = X_\eta/\langle\sigma_\eta\rangle$ has genus 0 and the Euler characteristic of the fibres is locally constant, the closed fibre of $X/\langle\sigma\rangle \to \operatorname{Spec} R$ has genus 0. Then we have proved that $(X \to S, \sigma)$ is an object of $\mathcal{H}_g(S)$; this object corresponds to a morphism $\operatorname{Spec} R \to \mathcal{H}_g$ that fits as a diagonal in the square (4.5). □

THEOREM 4.26. *If $g \geq 2$, the morphism $\mathcal{H}_g \to \mathcal{M}_g$ defined by*

$$(X \to P \to S) \mapsto (X \to S)$$

is a closed immersion.

PROOF. It is representable, proper and formally unramified because it is the composition of two representable, proper and formally unramified morphisms: $\mathcal{H}_g \to \mathcal{I}_{\mathcal{M}_g}$ and $\mathcal{I}_{\mathcal{M}_g} \to \mathcal{M}_g$. (Recall that $\mathcal{I}_{\mathcal{M}_g} \to \mathcal{M}_g$ is representable, finite and unramified because $\mathcal{M}_g$ is Deligne-Mumford (Theorem 3.34)).

To apply Theorem 1.30, it suffices to show that $\mathcal{H}_g \to \mathcal{M}_g$ is injective on geometric points. But this follows from the uniqueness of the g^1_2 for a smooth connected curve of genus g over an algebraically closed field (Corollary 4.10). □

COROLLARY 4.27. *If $g \geq 2$, $\mathcal{H}_g$ is a Deligne-Mumford stack with finite diagonal.*

For any scheme S over $\mathbb{Z}[1/2]$, let $\mathcal{H}_{g,S}$ be the stack $\mathcal{H}_g \times_{\operatorname{Spec}\mathbb{Z}[1/2]} S$. Since $\mathcal{H}_{g,S}$ is a separated Deligne-Mumford stack, it has a coarse moduli space $H_{g,S}$ thanks to the Keel-Mori theorem (Theorem 2.28). For brevity we denote by H_g the coarse moduli space $H_{g,\mathbb{Z}[1/2]}$ of $\mathcal{H}_g$.

There is a natural morphism

$$H_{g,S} \to H_g \times_{\operatorname{Spec}\mathbb{Z}[1/2]} S,$$

which is an isomorphism when S is flat over $\operatorname{Spec}\mathbb{Z}[1/2]$, for example if S the spectrum of a field of characteristic 0.

The closed immersion $\mathcal{H}_g \to \mathcal{M}_{g,\mathbb{Z}[1/2]}$ induces a morphism

$$H_{g,S} \to M_{g,S}$$

for every scheme S over $\mathbb{Z}[1/2]$.

PROPOSITION 4.28. *If $g \geq 2$ and k is a field of characteristic 0, then the induced morphism $H_{g,\operatorname{Spec} k} \to M_{g,\operatorname{Spec} k}$ is a closed immersion.*

PROOF. It follows immediately from Proposition 2.35 and Theorem 4.26. □

APPENDIX A

Basic notions of descent theory

In this chapter we will recall, without proofs, the rudiments of descent theory, following [**Vis05**]. We will define Grothendieck topologies and sheaves on a site. Then we will introduce categories fibred in groupoids over a fixed category $\mathcal{C}$, which arise as a generalization of contravariant functors from $\mathcal{C}$ to the category of sets. Finally we will define stacks over a site, which are morally "sheaves of categories".

For a full treatment of these notions, in addition to the already mentioned [**Vis05**], we refer the reader to [**St**] and to the original sources [**SGA1**, exposés VI, VIII] and [**Gro95a**].

A.1. Grothendieck topologies and sheaves

Definition A.1. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is the assignment to each object U of $\mathcal{C}$ of a collection of sets of arrows $\{U_i \to U\}$, called *coverings* of U, so that the following conditions are satisfied.

(i) If $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering.
(ii) If $\{U_i \to U\}$ is a covering and $V \to U$ is an arrow, then the fibred products $\{U_i \times_U V\}$ exist and the collection of projections $\{U_i \times_U V \to V\}$ is a covering.
(iii) If $\{U_i \to U\}$ is a covering and for each index i we have a covering $\{V_{ij} \to U_i\}$, then the collection of composites $\{V_{ij} \to U\}$ is a covering of U.

A category with a Grothendieck topology is called a *site*.

Definition A.2. Let S be a scheme. The *Zariski topology* on (Sch/S) is the Grothendieck topology on (Sch/S) whose coverings are collections $\{f_i \colon U_i \to U\}$ of open embeddings such that $U = \cup_i f(U_i)$.

The *étale topology* on (Sch/S) is the Grothendieck topology on (Sch/S) whose coverings are collections $\{f_i \colon U_i \to U\}$ of étale S-morphisms such that $U = \cup_i f(U_i)$.

The *fppf topology* on (Sch/S) is the Grothendieck topology on the category (Sch/S) whose coverings are collections $\{f_i \colon U_i \to U\}$ of flat S-morphisms locally of finite presentation such that $U = \cup_i f(U_i)$.

The *fpqc topology* on (Sch/S) is the Grothendieck topology on (Sch/S) whose coverings are collections $\{U_i \to U\}$ of S-morphisms such that the induced morphism $\coprod_i U_i \to U$ is fpqc.

Remark A.3. An fpqc morphism is a faithfully flat morphism $X \to Y$ such that every quasi-compact open subset of Y is the image of a quasi-compact open subset of X. For other characterizations of fpqc morphisms, see [**Vis05**, Proposition 2.33].

The fpqc topology is finer than the fppf topology, which is finer than the étale topology, which is in turn finer than the Zariski topology.

Definition A.4. Let $\mathcal{C}$ be a site. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is called a *sheaf* if the following condition is satisfied.

Suppose that we are given a covering $\{U_i \to U\}$ in $\mathcal{C}$, and a set of elements $a_i \in F(U_i)$. Denote by $\mathrm{pr}_1\colon U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2\colon U_i \times_U U_j \to U_j$ the first and the second projection respectively, and assume that $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j \in F(U_i \times_U U_j)$ for all i and j. Then there is a unique section $a \in F(U)$ whose pullback to $F(U_i)$ is a_i for all i.

If F and G are two sheaves on the site $\mathcal{C}$, a *morphism of sheaves* from F to G is a natural transformation $F \to G$. We denote by $\mathrm{Sh}(\mathcal{C})$ the category of sheaves on the site $\mathcal{C}$.

Definition A.5. A site $\mathcal{C}$ is called *subcanonical* if, for every object X of $\mathcal{C}$, the representable functor $\mathrm{h}_X\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is a sheaf.

Theorem A.6 (Grothendieck). *For every scheme S, the category (Sch/S) equipped with the fpqc topology is a subcanonical site.*

Proof. See [**Vis05**, Theorem 2.55]. □

Therefore the Zariski site $(\mathrm{Sch}/S)_{\mathrm{Zar}}$, the étale site $(\mathrm{Sch}/S)_{\text{ét}}$, and the fppf site $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ are subcanonical.

A.2. Groupoids

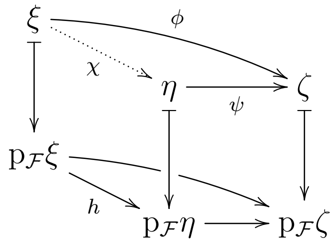
Definition A.7. Let $\mathcal{C}$ be a category. A *category fibred in groupoids* (or briefly a *groupoid*) *over* $\mathcal{C}$ is a category $\mathcal{F}$ with a functor $\mathrm{p}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$ such that the following two conditions are satisfied.

(1) If $f\colon U \to V$ is an arrow in $\mathcal{C}$ and η is an object of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}(\eta) = V$, then there exists an arrow $\phi\colon \xi \to \eta$ in $\mathcal{F}$ such that $\mathrm{p}_{\mathcal{F}}(\phi) = f$.

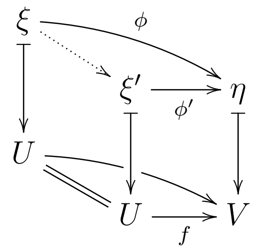
$$\begin{array}{ccc} \xi & \overset{\phi}{\dashrightarrow} & \eta \\ \vdots & & \downarrow \\ U & \xrightarrow{f} & V \end{array}$$

(2) If $\phi\colon \xi \to \zeta$ and $\psi\colon \eta \to \zeta$ are arrows in $\mathcal{F}$, and $h\colon \mathrm{p}_{\mathcal{F}}(\xi) \to \mathrm{p}_{\mathcal{F}}(\eta)$ is such that $\mathrm{p}_{\mathcal{F}}(\psi) \circ h = \mathrm{p}_{\mathcal{F}}(\phi)$, then there is a unique

arrow $\chi \colon \xi \to \eta$ such that $\psi \circ \chi = \phi$ and $\mathrm{p}_{\mathcal{F}}(\xi) = h$.



Axiom (2) implies that an arrow ϕ in $\mathcal{F}$ is an isomorphism if $\mathrm{p}_{\mathcal{F}}(\phi)$ is an isomorphism. As a consequence of (2), the object ξ in (1) is unique up to a canonical isomorphism, and should be thought of as the *pull-back* of η along f. We denote it with $f^*\eta$.



EXAMPLE A.8. Let $\mathcal{C}$ be a category with fibred products. Let P be a class of morphisms of $\mathcal{C}$ which is closed by base change and by composing with isomorphisms. One can consider the category $\mathrm{P}^{\mathrm{cart}}$ whose objects are morphisms belonging to P and whose arrows are cartesian squares. The functor that maps a morphism to its target gives $\mathrm{P}^{\mathrm{cart}}$ the structure of a groupoid over $\mathcal{C}$.

DEFINITION A.9. Let $\mathcal{F}$ be a groupoid over the category $\mathcal{C}$. The *fibre* of an object U of $\mathcal{C}$ is the subcategory $\mathcal{F}(U)$ of $\mathcal{F}$ whose objects are ξ such that $\mathrm{p}_{\mathcal{F}}(\xi) = U$ and whose arrows are arrows of $\mathcal{F}$ which are sent to id_U by the structural functor $\mathrm{p}_{\mathcal{F}}$.

The fibre is a groupoid, i.e. all arrows in $\mathcal{F}(U)$ are isomorphisms.

DEFINITION A.10. If $\mathcal{F}$ and $\mathcal{G}$ are groupoids over $\mathcal{C}$, then a *morphism* from $\mathcal{F}$ to $\mathcal{G}$ is a functor $F \colon \mathcal{F} \to \mathcal{G}$ such that $\mathrm{p}_{\mathcal{F}} = \mathrm{p}_{\mathcal{G}} \circ F$.

A morphism $\mathcal{F} \to \mathcal{G}$ of groupoids over $\mathcal{C}$ induces by restriction a functor $F_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ for every object U of $\mathcal{C}$.

PROPOSITION A.11. *A morphism $F \colon \mathcal{F} \to \mathcal{G}$ of groupoids over $\mathcal{C}$ is fully faithful if and only if every restriction $F_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful.*

PROOF. If F is fully faithful, then every restriction F_U is fully faithful.

Conversely, we need to show that, given two objects ξ' and η' of $\mathcal{F}$ and an arrow $\phi \colon F\xi' \to F\eta'$ in $\mathcal{G}$, there is a unique arrow $\phi' \colon \xi' \to \eta'$ in

$\mathcal{F}$ with $F\phi' = \phi$. Set $\xi = F\xi'$, $\eta = F\eta'$, and $U = p_{\mathcal{F}}\xi'$. Let $\eta_1' \to \eta'$ be a pull-back of η' to U, $\eta_1 = F\eta_1'$. Let $\eta_1 \to \eta$ be the image of $\eta_1' \to \eta'$, so every morphism $\xi \to \eta$ factors uniquely as $\xi \to \eta_1 \to \eta$, where the arrow $\xi \to \eta_1$ is in $\mathcal{G}(U)$. Analogously all arrows $\xi' \to \eta'$ factor uniquely through η_1'; since every arrow $\xi \to \eta_1$ in $\mathcal{G}(U)$ lifts uniquely to an arrow $\xi' \to \eta_1'$ in $\mathcal{F}(U)$, we have proved the proposition. □

Definition A.12. Let $\mathcal{F}$ and $\mathcal{G}$ be groupoids over $\mathcal{C}$ and let $F, G\colon \mathcal{F} \to \mathcal{G}$ be two morphisms. A *base-preserving natural transformation* from F to G is a natural transformation $\alpha\colon F \to G$ such that, for any object ξ of $\mathcal{F}$, the arrow $\alpha_\xi\colon F\xi \to G\xi$ is in $\mathcal{G}(U)$, where $U = \mathrm{p}_{\mathcal{F}}\xi = \mathrm{p}_{\mathcal{G}}(F\xi) = \mathrm{p}_{\mathcal{G}}(G\xi)$.

An *isomorphism* of F with G is a base-preserving natural transformation $F \to G$ which is an isomorphism of functors.

It is immediate to check that the inverse of a base-preserving isomorphism is also base-preserving.

In this way, groupoids over a category $\mathcal{C}$ form a 2-category $(\mathrm{Gr}/\mathcal{C})$: the objects are groupoids over $\mathcal{C}$, 1-arrows are morphism according to Definition A.10, 2-arrows are base-preserving natural transformations between functors which define 1-arrows.

Example A.13 (Disjoint sum of groupoids). Let $\{\mathcal{F}_i\}_{i\in I}$ be a family of groupoids over the category $\mathcal{C}$. Consider the category $\mathcal{F}$ defined as follows: the objects of $\mathcal{F}$ are pairs (i, ξ) where $i \in I$ and $\xi \in \mathcal{F}_i$; for two such objects (i, ξ), (j, η), $\mathrm{Hom}_{\mathcal{F}}((i,\xi),(j,\eta))$ is by definition $\mathrm{Hom}_{\mathcal{F}_i}(\xi, \eta)$ if $i = j$, and it is empty if $i \neq j$. The category $\mathcal{F}$ is a groupoid over $\mathcal{C}$ in an obvious way, it is denoted by $\coprod_{i\in I} \mathcal{F}_i$, and it is called the *disjoint sum* of the family $\{\mathcal{F}_i\}_{i\in I}$. It is a coproduct in the category $(\mathrm{Gr}/\mathcal{C})$ in a natural way.

Example A.14 (Fibred product of groupoids). Let

$$\begin{array}{ccc} \mathcal{F} & & \mathcal{G} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathcal{H} & \end{array}$$

be two morphism of groupoids over $\mathcal{C}$. The category $\mathcal{F} \times_{F,\mathcal{H},G} \mathcal{G}$, also denoted by $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ when there is no risk of confusion, is defined as follows:

- objects are triplets (ξ, η, h), where ξ is an object of $\mathcal{F}$, η is an object of $\mathcal{G}$ and $h\colon F\xi \to G\eta$ is an isomorphism in $\mathcal{H}$, such that $\mathrm{p}_{\mathcal{F}}(\xi) = \mathrm{p}_{\mathcal{G}}(\eta) = U$ and $\mathrm{p}_{\mathcal{H}}(h) = \mathrm{id}_U$;
- an arrow from a triple (ξ_1, η_1, h_1) to a triple (ξ_2, η_2, h_2) is a pair (f, g), where $f\colon \xi_1 \to \xi_2$ is an arrow in $\mathcal{F}$ and $g\colon \eta_1 \to \eta_2$

is an arrow in $\mathcal{G}$ such that the diagram

$$\begin{array}{ccc} F\xi_1 & \xrightarrow{Ff} & F\xi_2 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} \\ G\eta_1 & \xrightarrow{Gg} & G\eta_2 \end{array}$$

commutes in $\mathcal{H}$. Composition of arrows is defined in the obvious way.

The mapping $(\xi, \eta, h) \mapsto \mathrm{p}_{\mathcal{F}}(\xi) = \mathrm{p}_{\mathcal{G}}(\eta)$ defines a functor $\mathcal{F} \times_{\mathcal{H}} \mathcal{G} \to \mathcal{C}$ that is easily checked to make $\mathcal{F} \times_H \mathcal{G}$ a groupoid over $\mathcal{C}$.

EXAMPLE A.15. The category $\mathcal{C}$ with the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is a groupoid over $\mathcal{C}$. It is a terminal object of $(\mathrm{Gr}/\mathcal{C})$.

Examples A.13, A.14, and A.15 show that the category $(\mathrm{Gr}/\mathcal{C})$ has coproducts, fibred products, and finite products, although we should have been more precise to use these notions with 2-categories.

EXAMPLE A.16 (Diagonal morphism). Let $F \colon \mathcal{F} \to \mathcal{G}$ be a morphism of groupoids over the category $\mathcal{C}$. The *diagonal morphism* of F is the morphism

$$\Delta_F \colon \mathcal{F} \longrightarrow \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$$

which maps the object ξ of $\mathcal{F}$ into the object $(\xi, \xi, \mathrm{id}_{F\xi})$.

DEFINITION A.17. Let $\mathcal{F}$ and $\mathcal{G}$ be two groupoids over $\mathcal{C}$. An *equivalence* of $\mathcal{F}$ with $\mathcal{G}$ is a morphism $F \colon \mathcal{F} \to \mathcal{G}$, such that there exists another morphism $G \colon \mathcal{G} \to \mathcal{F}$, together with isomorphisms of $G \circ F$ with $\mathrm{id}_{\mathcal{F}}$ and of $F \circ G$ with $\mathrm{id}_{\mathcal{G}}$.

We call G simply an *inverse* to F.

PROPOSITION A.18. *Let $F \colon \mathcal{F} \to \mathcal{G}$ be a morphism of groupoids over $\mathcal{C}$. Then F is an equivalence of groupoids if and only if the restriction $F_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories for any object U of $\mathcal{C}$.*

PROOF. Suppose that $G \colon \mathcal{G} \to \mathcal{F}$ is an inverse to F; the two isomorphisms $F \circ G \simeq \mathrm{id}_{\mathcal{G}}$ and $G \circ F \simeq \mathrm{id}_{\mathcal{F}}$ restrict to isomorphisms $F_U \circ G_U \simeq \mathrm{id}_{\mathcal{G}(U)}$ and $G_U \circ F_U \simeq \mathrm{id}_{\mathcal{F}(U)}$, so G_U is an inverse to F_U.

Conversely, we assume that $F_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories for any object U of $\mathcal{C}$, and construct an inverse $G \colon \mathcal{G} \to \mathcal{F}$. For any object ξ of $\mathcal{G}$ pick an object $G\xi$ of $\mathcal{F}(U)$, where $U = p_{\mathcal{G}}\xi$, together with an isomorphism $\alpha_\xi \colon \xi \simeq F(G\xi)$ in $\mathcal{G}(U)$; these $G\xi$ and α_ξ exist because $F_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories.

Now, if $\phi \colon \xi \to \eta$ is an arrow in $\mathcal{G}$, by Proposition A.11 there is a unique arrow $G\phi \colon G\xi \to G\eta$ such that $F(G\phi) = \alpha_\eta \circ \phi \circ \alpha_\xi^{-1}$.

These operations define a functor $G \colon \mathcal{G} \to \mathcal{F}$. It is immediate to check that by sending each object ξ to the isomorphism $\alpha_\xi \colon \xi \simeq F(G\xi)$ we define an isomorphism of functors $\mathrm{id}_{\mathcal{F}} \simeq F \circ G \colon \mathcal{G} \to \mathcal{G}$.

We only have left to check that $G \circ F : \mathcal{F} \to \mathcal{F}$ is isomorphic to the identity $\mathrm{id}_{\mathcal{F}}$.

Fix an object ξ' of $\mathcal{F}$ over an object U of $\mathcal{C}$; we have a canonical isomorphism $\alpha_{F\xi'} : F\xi' \simeq F(G(F\xi'))$ in $\mathcal{G}(U)$. Since F_U is fully faithful there is a unique isomorphism $\beta_{\xi'} : \xi' \simeq G(F\xi')$ in $\mathcal{F}(U)$ such that $F\beta_{\xi'} = \alpha_{F\xi'}$; one checks easily that this defines an isomorphism of functors $\beta : G \circ F \simeq \mathrm{id}_{\mathcal{G}}$. □

A.3. Representable groupoids

In this section $\mathcal{C}$ is a fixed category. We begin with an example.

EXAMPLE A.19. Let $\Phi : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ be a functor. Let $\mathcal{F}_\Phi$ be the category defined as follows: the objects of $\mathcal{F}_\Phi$ are pairs (U, ξ) where U is an object of $\mathcal{C}$ and $\xi \in \Phi U$, i.e. $\mathrm{ob}\mathcal{F}_\Phi = \coprod_{U \in \mathrm{ob}\mathcal{C}} \Phi U$; an arrow from (U, ξ) to (V, η) is an arrow $f : U \to V$ of $\mathcal{C}$ with the property that $(\Phi f)(\eta) = \xi$. The assignment $(U, \xi) \mapsto U$ gives $\mathcal{F}_\Phi$ the structure of a groupoid over $\mathcal{C}$.

The costruction of Example A.19 is functorial: with each natural transformation of functors $\phi : \Phi \to \Phi'$ we may associate a morphism $F_\phi : \mathcal{F}_\Phi \to \mathcal{F}_{\Phi'}$ of groupoids over $\mathcal{C}$. This produces a functor

$$\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})) \longrightarrow (\mathrm{Gr}/\mathcal{C}),$$

which can be checked to be fully faithful ([**Vis05**, Section 3.4]). We identify a functor $\Phi : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ with the corresponding groupoid $\mathcal{F}_\Phi$ over $\mathcal{C}$.

In particular, given an object X of $\mathcal{C}$, we have the representable functor $\mathrm{h}_X : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$, defined on the objects of $\mathcal{C}$ by the rule $\mathrm{h}_X U = \mathrm{Hom}_{\mathcal{C}}(U, X)$. The groupoid over $\mathcal{C}$ associated with this functor is the comma category $(\mathcal{C}/X)$ of X-objects, and the functor $(\mathcal{C}/X) \to \mathcal{C}$ is the functor that forgets the arrow into X.

So the situation is the following. From Yoneda's lemma we see that the category $\mathcal{C}$ is embedded into the category $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set}))$ of functors $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$, while the category of functors is embedded into the category of groupoids:

$$\mathcal{C} \hookrightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})) \hookrightarrow (\mathrm{Gr}/\mathcal{C}). \tag{A.1}$$

We will identify an object of $\mathcal{C}$ with the corresponding groupoid over $\mathcal{C}$.

DEFINITION A.20. A groupoid over $\mathcal{C}$ is *representable* if it is equivalent to a category of the form $(\mathcal{C}/X)$ for some object X of $\mathcal{C}$.

PROPOSITION A.21. *A groupoid $\mathcal{F}$ over $\mathcal{C}$ is representable if and only if it has a terminal object.*

PROOF. See [**Vis05**, Section 3.6]. □

REMARK A.22. Let $\mathcal{F}$ be a groupoid over $\mathcal{C}$ and let X be an object of $\mathcal{C}$. If we think of X as the groupoid $(\mathcal{C}/X)$ over $\mathcal{C}$, we see that giving a morphism $X \to \mathcal{F}$ of groupoids is equivalent to giving an object of $\mathcal{F}(X)$: with the morphism $F\colon X = (\mathcal{C}/X) \to \mathcal{F}$ we associate the object $F(\mathrm{id}_X) \in \mathcal{F}(X)$. This defines a functor

$$\mathrm{Hom}_{\mathcal{C}}(X, \mathcal{F}) \longrightarrow \mathcal{F}(X)$$

which can be checked to be an equivalence of categories (see [**Vis05**, Section 3.6]). This is a sort of Yoneda lemma.

CAUTION A.23. The definition of representable algebraic stacks is different from the definition of representable groupoid. In fact, an algebraic stack is said to be representable if it is equivalent to an algebraic space, i.e. an object which is more general than a scheme.

A.4. Stacks

DEFINITION A.24. Let $\mathcal{C}$ be a site, $\mathcal{F}$ be a groupoid over $\mathcal{C}$, and $\mathcal{U} = \{U_i \to U\}$ be a covering. An *object with descent data* $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$ is a collection of objects $\xi_i \in \mathcal{F}(U_i)$, together with isomorphisms $\phi_{ij}\colon \mathrm{pr}_2^*\xi_j \simeq \mathrm{pr}_1^*\xi_i$ in $\mathcal{F}(U_i \times_U U_j)$, such that the following cocycle condition is satisfied: for any triple of indices i, j and k, we have the equality

$$\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^*\phi_{ij} \circ \mathrm{pr}_{23}^*\phi_{jk}\colon \mathrm{pr}_3^*\xi_k \to \mathrm{pr}_1^*\xi_i,$$

where the pr_{ab} and pr_a are projections on the a^{th} and b^{th} factor, or the a^{th} factor respectively.

An *arrow between objects with descent data*

$$\{\alpha_i\}\colon (\{\xi_i\}, \{\phi_{ij}\}) \longrightarrow (\{\eta_i\}, \{\psi_{ij}\})$$

on the covering $\mathcal{U}$ is a collection of arrows $\alpha_i\colon \xi_i \to \eta_i$ in $\mathcal{F}(U_i)$, with the property that for each pair of indices i, j, we have

$$\mathrm{pr}_1^*\alpha_j \circ \phi_{ij} = \psi_{ij} \circ \mathrm{pr}_2^*\alpha_j.$$

We denote with $\mathcal{F}(\mathcal{U})$ the *category of objects with descent data* in the covering $\mathcal{U}$.

REMARK A.25. Pull-backs in a groupoid are not unique and fibred products in a category are not unique. Hence, in the definitions above we made a fixed choice of fibred products and pull-backs. If one had made another choice, the resulting category $\mathcal{F}(\mathcal{U})$ would be equivalent. From now on, we will forget this annoying considerations.

If a groupoid over a site is thought as a presheaf, then objects with descent data should be thought as local sections that are locally compatible.

A global section can be restricted to produce local sections that are compatible. Hence, if $\mathcal{F}$ is a groupoid over the site $\mathcal{C}$ and $\mathcal{U}$ is a covering of the object U, then one has a functor

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(\mathcal{U}) \tag{A.2}$$

that sends a global section to its pull-backs.

Definition A.26. A groupoid $\mathcal{F}$ over a site $\mathcal{C}$ is called a *stack* if, for every object U of $\mathcal{C}$ and every covering $\mathcal{U}$ of U, the functor in (A.2) is an equivalence of categories.

The condition that the functor of (A.2) is an equivalence means that every object with descent data (i.e. locally compatible sections) comes from a "unique" global section. Therefore a stack over $\mathcal{C}$ is, morally, a "sheaf of categories".

Proposition A.27. *If $\mathcal{F}$ is a groupoid over the site $\mathcal{C}$, then the following two conditions are equivalent.*

(a) For any object U in $\mathcal{C}$ and any two objects ξ and η in $\mathcal{F}(U)$, the functor

$$\mathbf{Isom}_U(\xi,\eta)\colon (\mathcal{C}/U)^{\mathrm{op}} \to (\mathrm{Set}),$$

which associates to a morphism $f\colon V \to U$ the set of isomorphisms in $\mathcal{F}(V)$ between $f^*\xi$ and $f^*\eta$, is a sheaf.

(b) For any object U in $\mathcal{C}$ and any covering $\mathcal{U}$ of U, the restriction functor $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is fully faithful.

Proof. See [**Vis05**, Proposition 4.7]. □

A groupoid satisfying the conditions of the proposition above is called a *prestack*. Prestacks should be thought as separated presheaves of categories.

Proposition A.28. *Let $\mathcal{F}$ be a prestack over the site $\mathcal{C}$. Then $\mathcal{F}$ is a stack if and only if the following condition is satisfied.*

- *Let $\{U_i \to U\}$ be a covering of U in $\mathcal{C}$. Let $\xi_i \in \mathcal{F}(U_i)$ and let*

$$\phi_{ij}\colon \xi_j|_{U_i\times_U U_j} \longrightarrow \xi_i|_{U_i\times_U U_j}$$

be isomorphisms in $\mathcal{F}(U_i \times_U U_j)$ satisfying the cocycle condition. Then there is $\xi \in \mathcal{F}(U)$ with isomorphisms $\psi_i\colon \xi|_{U_i} \to \xi_i$ such that

$$\phi_{ij} = (\psi_i|_{U_i\times_X U_j}) \circ (\psi_j|_{U_i\times_U U_j})^{-1}.$$

Proof. One already knows that the restriction functor is fully faithful. The condition in the statement means that it is essentially surjective. □

Definition A.29. If $\mathcal{C}$ is a site, we will denote by $(\mathrm{St}/\mathcal{C})$ the full 2-subcategory of $(\mathrm{Gr}/\mathcal{C})$ whose objects are stacks over $\mathcal{C}$.

Note that $(\mathrm{St}/\mathcal{C})$ depends on the Grothendieck topology of $\mathcal{C}$, while $(\mathrm{Gr}/\mathcal{C})$ does not. There are obvious notions of morphism of stacks, equivalence of stacks, representable stack, representable morphism of stacks.

PROPOSITION A.30. *If $\mathcal{C}$ be a site, then a functor $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is a stack over $\mathcal{C}$ if and only if it is a sheaf.*

PROOF. See [**Vis05**, Proposition 4.9(ii)]. □

COROLLARY A.31. *If $\mathcal{C}$ is a subcanonical site, every object X of $\mathcal{C}$ is a stack over $\mathcal{C}$.*

If $\mathcal{C}$ is a subcanonical site, we can enlarge the system of embeddings (A.1) to the diagram

$$\begin{array}{ccccc} \mathcal{C} & \hookrightarrow & \mathrm{Sh}(\mathcal{C}) & \hookrightarrow & (\mathrm{St}/\mathcal{C}) \\ \| & & \downarrow & & \downarrow \\ \mathcal{C} & \hookrightarrow & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})) & \hookrightarrow & (\mathrm{Gr}/\mathcal{C}) \end{array}$$

PROPOSITION A.32. *Let $\mathcal{C}$ be a subcanonical site with fibred products, P be a class of arrows of $\mathcal{C}$ which is closed by base change and by composing with isomorphisms. Then the groupoid $\mathrm{P}^{\mathrm{cart}}$ (see Example A.8) is a prestack.*

PROOF. The proof is quite easy and can be found in [**Vis05**, Proposition 4.31]. □

DEFINITION A.33. Let $\mathcal{C}$ be a site with fibred products. A class of arrows P in $\mathcal{C}$ is said *local* if it satisfies the following two conditions:

(1) P is closed by base change and by composing with isomorphisms;
(2) if $\{U_i \to U\}$ is a covering in $\mathcal{C}$ and $X \to U$ is an arrow such that the projections $U_i \times_U X \to U_i$ are in P for all i, then $X \to U$ is also in P.

It is not true that if P is a local class then $\mathrm{P}^{\mathrm{cart}}$ is a stack.

A.5. Stacks of inertia

In this section we define the inertia groupoid of a groupoid.

DEFINITION A.34. Let $F\colon \mathcal{F} \to \mathcal{G}$ be a morphism of groupoids over the category $\mathcal{C}$. The *inertia groupoid of the morphism F* is the category $\mathcal{I}_F$ defined as follows:

- objects are pairs (ξ, α), where ξ is an object of $\mathcal{F}$ and $\alpha\colon \xi \to \xi$ is an arrow such that $F\alpha = \mathrm{id}_{F\xi}$;

- an arrow from (ξ, α) to (η, β) is an arrow $f\colon \xi \to \eta$ in $\mathcal{F}$ such that the diagram

$$\begin{array}{ccc} \xi & \xrightarrow{f} & \eta \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \xi & \xrightarrow{f} & \eta \end{array}$$

commutes. Composition of arrows is defined in the obvious way.

Proposition A.35. *If $F\colon \mathcal{F} \to \mathcal{G}$ is a morphism of groupoids over the category $\mathcal{C}$, then the functor*

$$\mathcal{I}_F \longrightarrow \mathcal{F} \times_{\Delta_F,\ \mathcal{F}\times_{\mathcal{G}}\mathcal{F},\ \Delta_F} \mathcal{F} \tag{A.3}$$

defined by

$$(\xi, \alpha) \mapsto (\xi, \xi, (\alpha, \mathrm{id}_\xi))$$

is an equivalence of categories over $\mathcal{C}$.

Proof. We denote by $\mathcal{A}$ the category on the right in (A.3). According to the definition of fibred product of groupoids (Example A.14), the objects of $\mathcal{A}$ are terns (ξ, η, h), where ξ and η are objects of $\mathcal{F}$ and $h\colon \Delta_F\xi \to \Delta_F\eta$ is an arrow in $\mathcal{F} \times_{\mathcal{G}} \mathcal{F}$, such that $\mathrm{p}_{\mathcal{F}}(\xi) = \mathrm{p}_{\mathcal{F}}(\eta) = U \in \mathrm{ob}\,\mathcal{C}$ and $\mathrm{p}_{\mathcal{F}\times_{\mathcal{G}}\mathcal{F}}(h) = \mathrm{id}_U$. Recall that $\Delta_F\xi = (\xi, \xi, \mathrm{id}_{F\xi})$, and analogously for η. An arrow $h\colon \Delta_F h \to \Delta_F\eta$ is a pair (f, g), where $f\colon \xi \to \eta$ and $g\colon \xi \to \eta$ are arrows in $\mathcal{F}$ such that the diagram

$$\begin{array}{ccc} F\xi & \xrightarrow{Ff} & F\eta \\ {\scriptstyle F\mathrm{id}_\xi}\downarrow & & \downarrow{\scriptstyle F\mathrm{id}_\eta} \\ F\xi & \xrightarrow{Fg} & F\eta \end{array}$$

commutes, that is $Ff = Fg$.

In $\mathcal{A}$ an arrow $(\xi_1, \eta_1, h_1 = (f_1, g_1)) \to (\xi_2, \eta_2, h_2 = (f_2, g_2))$ is a pair (ϕ, ψ) where $\phi\colon \xi_1 \to \xi_2$ and $\psi\colon \eta_1 \to \eta_2$ are arrows in $\mathcal{F}$, such that the diagram

$$\begin{array}{ccc} \Delta_F\xi_1 & \xrightarrow{\Delta_F\phi} & \Delta_F\xi_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ \Delta_F\eta_1 & \xrightarrow{\Delta_F\psi} & \Delta_F\eta_2 \end{array}$$

commutes in $\mathcal{F} \times_{\mathcal{G}} \mathcal{F}$, i.e. $g_2\phi = \psi g_1$ and $f_2\phi = \psi f_1$.

Now it is immediate to see that the category $\mathcal{A}$ is equivalent over $\mathcal{C}$ to the category $\mathcal{B}$ defined as follows:

- objects of $\mathcal{B}$ are (ξ, η, f, g), where ξ and η are objects of $\mathcal{F}$ and $f, g\colon \xi \to \eta$ are arrows of $\mathcal{F}$, such that $\mathrm{p}_{\mathcal{F}}(\xi) = \mathrm{p}_{\mathcal{F}}(\eta) = U$, $Ff = Fg$ and $\mathrm{p}_{\mathcal{F}}(f) = \mathrm{p}_{\mathcal{F}}(g) = \mathrm{id}_U$;

- an arrow from $(\xi_1, \eta_1, f_1, g_1)$ to $(\xi_2, \eta_2, f_2, g_2)$ in $\mathcal{B}$ is a pair (ϕ, ψ) where $\phi\colon \xi_1 \to \xi_2$ and $\psi\colon \eta_1 \to \eta_2$ are arrows in $\mathcal{F}$ such that $g_2\phi = \psi g_1$ and $f_2\phi = \psi f_1$.

The functor in (A.3) corresponds to the functor $\mathcal{I}_F \to \mathcal{B}$ defined by

$$\begin{aligned} (\xi, \alpha) \quad &\mapsto \quad (\xi, \xi, \alpha, \mathrm{id}_\xi), \\ (\xi, \alpha) \xrightarrow{f} (\eta, \beta) \quad &\mapsto \quad (\xi, \xi, \alpha, \mathrm{id}_\xi) \xrightarrow{(f,f)} (\eta, \eta, \beta, \mathrm{id}_\eta). \end{aligned}$$

A quasi-inverse of this functor is the functor $\mathcal{B} \to \mathcal{I}_F$ defined by

$$\begin{aligned} (\xi, \eta, f, g) \quad &\mapsto \quad (\xi, g^{-1}f), \\ (\xi_1, \eta_1, f_1, g_1) \xrightarrow{(\phi,\psi)} (\xi_2, \eta_2, f_2, g_2) \quad &\mapsto \quad (\xi_1, g_1^{-1}f_1) \xrightarrow{\phi} (\xi_2, g_2^{-1}f_2), \end{aligned}$$

because the composition $\mathcal{I}_F \to \mathcal{B} \to \mathcal{I}_F$ is the identity of $\mathcal{I}_\mathcal{F}$ and the composite $\mathcal{B} \to \mathcal{I}_F \to \mathcal{B}$, defined by

$$\begin{aligned} (\xi, \eta, f, g) &\mapsto (\xi, \xi, g^{-1}f, \mathrm{id}_\xi), \\ (\phi, \psi) &\mapsto (\phi, \phi), \end{aligned}$$

is isomorphic to the identity of $\mathcal{B}$ via the natural transformations

$$\begin{aligned} (g^{-1}f, g^{-1}fg^{-1})\colon (\xi, \eta, f, g) &\longrightarrow (\xi, \xi, g^{-1}f, \mathrm{id}_\xi), \\ (f^{-1}g, gf^{-1}g)\colon (\xi, \xi, g^{-1}f, \mathrm{id}_\xi) &\longrightarrow (\xi, \eta, f, g). \end{aligned}$$ □

COROLLARY A.36. *The inertia groupoid of a morphism of groupoids over the category $\mathcal{C}$ is a groupoid over $\mathcal{C}$.*

APPENDIX B

A quick review of Grothendieck duality

Here we will show some facts of base change theory of a flat coherent sheaf with respect to a proper morphism and we will recall the basics of Grothendieck duality theory.

LEMMA B.1. *Let $f\colon X \to Y$ be a proper morphism of locally noetherian schemes, let $\mathcal{F}$ be coherent sheaf on X such that $\mathcal{F}$ is flat over Y and, for every point $y \in Y$, $\mathrm{H}^1(X_y, \mathcal{F}_y) = 0$, where $\mathcal{F}_y$ is the restriction of $\mathcal{F}$ to the fibre X_y.*

Then $f_*\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module of rank $\dim_{k(y)} \mathrm{H}^0(X_y, \mathcal{F}_y)$ at $y \in Y$ and for every morphism $g\colon Y' \to Y$ and every cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

the natural homomorphism

$$g^* f_* \mathcal{F} \longrightarrow (f')_*(g')^*\mathcal{F} \tag{B.1}$$

is an isomorphism.

PROOF. We can suppose that $Y = \operatorname{Spec} A$ is affine and noetherian, hence we can define the functors $T^i\colon (\mathrm{Mod}_A) \to (\mathrm{Mod}_A)$

$$T^i(M) = \mathrm{H}^i(X, \mathcal{F} \otimes_A M) = \Gamma\left(Y, \mathrm{R}^i f_* \left(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\tilde{M}\right)\right)$$

and the natural maps

$$\varphi^i(y)\colon T^i(A) \otimes_A k(y) = \mathrm{R}^i f_*(\mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to \mathrm{H}^i(X_y, \mathcal{F}_y),$$

for $i \geq 0$ and $y \in Y$, as in [**Har77**, III.12]. For every point $y \in Y$, $\mathrm{H}^1(X_y, \mathcal{L}_y) = 0$ implies that $\varphi^1(y)$ is surjective, then it is bijective ([**Har77**, Theorem III.12.11(a)]); hence $\mathrm{R}^1 f_*(\mathcal{F}) = 0$. According to [**Har77**, Theorem III.12.11(b)], the map $\varphi^0(y)$ is bijective, for every y. Since $\varphi^{-1}(y)$ is vacuously surjective, [**Har77**, Theorem III.12.11(b)] implies that $\mathrm{R}^0 f_*\mathcal{F} = f_*\mathcal{F}$ is locally free and its rank is determined by $\varphi^0(y)$.

Since $\varphi^0(y)$ is bijective for every point $y \in Y$, [**EGA**, Proposition III.12.10] implies that the functor T^0 is right exact. The bijectivity of (B.1) follows from a) $\Rightarrow$ d) of [**EGA**, Théorème III.7.7.5], because the homomorphism (B.1) is exactly the homomorphism (7.7.5.3). □

LEMMA B.2. *Let $f\colon X \to Y$ be a proper morphism of locally noetherian schemes, let $\mathcal{F}$ be a coherent sheaf on X such that $\mathcal{F}$ is flat over Y and, for every point $y \in Y$, $\mathrm{H}^0(X_y, \mathcal{F}_y) = 0$, where $\mathcal{F}_y$ is the restriction of $\mathcal{F}$ to the fibre X_y.*

Then $f_*\mathcal{F} = 0$ and for every morphism $g\colon Y' \to Y$ and every cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

we have that $(f')_*(g')^*\mathcal{F} = 0$.

LEMMA B.3. *Let $f\colon X \to Y$ be a proper flat morphism of locally noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X such that $\mathcal{F}$ is flat over Y and, for every point $y \in Y$, $\mathrm{H}^2(X_y, \mathcal{F}_y) = 0$, where $\mathcal{F}_y$ is the restriction of $\mathcal{F}$ to the fibre X_y.*

Then $\mathrm{R}^1 f_*(\mathcal{F})$ is a locally free $\mathcal{O}_Y$-module of rank $\dim_{k(y)} \mathrm{H}^1(X_y, \mathcal{F}_y)$ at $y \in Y$ and for every morphism $g\colon Y' \to Y$ and every cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

the natural homomorphism

$$g^*(\mathrm{R}^1 f_*\mathcal{F}) \longrightarrow \mathrm{R}^1 f'_*(g'^*\mathcal{F})$$

is an isomorphism.

The proofs of Lemma B.2 and Lemma B.3 are analogous to the one of Lemma B.1 and are omitted.

LEMMA B.4. *Let $f\colon X \to Y$ be proper flat morphism of locally noetherian schemes, let $\mathcal{L} \in \mathrm{Pic}(X)$ be an invertible sheaf on X such that, for every point $y \in Y$, the restriction $\mathcal{L}_y$ to the fibre X_y is very ample relative to $\operatorname{Spec} k(y)$ and $\mathrm{H}^1(X_y, \mathcal{L}_y) = 0$. Then $\mathcal{L}$ is very ample relative to f.*

PROOF. We can suppose that Y is affine and noetherian. Denote by $\mathcal{E}$ the coherent sheaf $f_*\mathcal{L}$. We use notations from the proof of Lemma B.1.

Since the invertible sheaf $\mathcal{L}_y$ is generated by its global sections on X_y, the surjectivity of $\varphi^0(y)$ implies that the homomorphism $f^*\mathcal{E} \to \mathcal{L}$ is surjective at points of X_y. Therefore $f^*\mathcal{E} \to \mathcal{L}$ is surjective. According to [**EGA**, Proposition II.4.2.3], this defines a Y-morphism $i\colon X \to \mathbb{P}(\mathcal{E})$. It suffices to show that i is a closed immersion in a neighbourhood of every point y. This is implied by the fact that the induced morphism $i_y : X_y \to \mathbb{P}(\mathcal{E})_y$ is a closed immersion (because $\mathcal{L}_y$ is very ample) and by [**EGA**, Proposition III.4.6.7(i)]. □

A morphism of schemes is called *of Cohen-Macaulay with pure relative dimension n* if it is flat, locally of finite presentation, and its fibres are Cohen-Macaulay schemes with pure dimension n.

In [**Har66**] and [**Con00**, Chapters 3 and 4], for every proper Cohen-Macaulay morphism $f : X \to Y$ of relative dimension n, one defines the *dualizing sheaf* $\omega_f \in \mathrm{QCoh}(X)$ and the *trace map* $\gamma_f : \mathrm{R}^n f_*(\omega_f) \to \mathcal{O}_Y$. The construction is Zariski local on the source and compatible with base change. Another construction of the dualizing sheaf is developed in [**Kle80**]. In the following proposition we recall some properties of the dualizing sheaf and the trace map.

Proposition B.5. *Let $f : X \to Y$ be a proper Cohen-Macaulay morphism with pure relative dimension n. Then:*

(1) if f is smooth, then $\omega_f \simeq \det \Omega^1_{X/Y}$;
(2) ω_f is flat, of finite presentation;
(3) ω_f is invertible if and only if all fibres of f are Gorenstein;
(4) if there exists a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & P \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

where i is a closed immersion and π is proper smooth with pure relative dimension N, then

$$\omega_f \simeq \mathcal{E}xt_P^{N-n}(i_*\mathcal{O}_X, \omega_\pi);$$

(5) γ_f is surjective;
(6) if f has geometrically reduced and geometrically connected fibres, then γ_f is an isomorphism.

Proof. See Chapters 3 and 4 of [**Con00**]. □

Theorem B.6 (Grothendieck duality). *Let $f : X \to Y$ be a proper Cohen-Macaulay morphism with pure relative dimension n, let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank, and let $m \in \mathbb{Z}$ be an integer. Suppose that $\mathrm{R}^i f_* \mathcal{F}$ is locally free of finite rank on Y for all $i > m$.*

Then, for every $\mathcal{G} \in \mathrm{QCoh}(Y)$ and every $i \geq m$, the natural map

$$\mathrm{R}^{n-i} f_*(\mathcal{F}^\vee \otimes \omega_f \otimes f^*\mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R}^i f_*(\mathcal{F}), \mathcal{G})$$

induced by

$$\mathrm{R}^i f_*(\mathcal{F}) \otimes \mathrm{R}^{n-i} f_*(\mathcal{F}^\vee \otimes \omega_f \otimes f^*\mathcal{G}) \longrightarrow \mathrm{R}^n f_*(\omega_f \otimes f^*\mathcal{G}) \simeq$$

$$\simeq \mathrm{R}^n f_*(\omega_f) \otimes \mathcal{G} \xrightarrow{\gamma_f \otimes 1} \mathcal{G}$$

is an isomorphism.

PROOF. See [**Con00**, Theorem 5.1.2]. □

Notice that, if Y is the spectrum of a field and $\mathcal{G} = \mathcal{O}_Y$, the theorem above implies the Serre duality ([**Har77**, Corollary III.7.7]).

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