Uniform bound of the entanglement for the ground state of the quantum Ising model with large transverse magnetic field.

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Definition of the model

$L \geq 0 \qquad m \geq 0$

$$\Delta_m = \{-m, -m+1, \dots, m+L\} = [-m, m+L]$$

$$\mathcal{H} = \bigotimes_{x=-m}^{m+L} \mathbb{C}^2 \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sigma_x^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_x^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_x^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H_m = -J \sum_{\langle x,y \rangle} \sigma_x^{(3)} \sigma_y^{(3)} - h \sum_x \sigma_x^{(1)}$$

$J, h \geq 0$

$$\rho_m(\beta) = \frac{e^{-\beta H_m}}{tr\left(e^{-\beta H_m}\right)}$$

$$\rho_m = \lim_{\beta \to \infty} \rho_m(\beta) = |\psi_m\rangle\langle\psi_m|$$

$$\rho_m^L = tr_{\Delta_m \setminus [0,L]} \left(|\psi_m\rangle\langle\psi_m| \right)$$

Similarly one defines

$$\rho_m^L(\beta)$$

The trace is performed over

$$\left(\bigotimes_{x=-m}^{-1} \mathbb{C}^2 \right) \otimes \left(\bigotimes_{x=L+1}^{m+L} \mathbb{C}^2 \right)$$

corresponding to the spins in $\Delta_m \setminus [0, L]$.

The entanglement of the interval $[0, L]$ relative to its complement $\Delta_m \setminus [0, L]$ is defined as

$$S(\rho_m^L) = -\operatorname{tr}\left(\rho_m^L \log_2 \rho_m^L \right) =$$

$$= - \sum_{j=1}^{2^{L+1}} \lambda_j(\rho_m^L) \log_2 \lambda_j(\rho_m^L).$$

Representation

I interval $I \subset \mathbb{R}$. X_I space of functions from I to $\{-1, 1\}$. μ_I is the probability measure on X_I obtained from a Poisson pair process with intensity h, where the points of the process represent where the function switches value and μ_I is assumed to be invariant under sign inversion. Given Λ inverval

Λ interval $\Lambda \subset \mathbb{Z}$, we define the Gibbs measure on $X^{\Lambda}_{[-\frac{\beta}{2}, \frac{\beta}{2}]}$ with density

$$Z^{-1} \exp\left(-J \sum_{\langle x,y \rangle} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \sigma_x(t)\sigma_y(t)\,dt\right)$$

with respect to $\bigotimes_{x \in \Lambda} \mu_{[-\frac{\beta}{2}, \frac{\beta}{2}]}$.

This measure allows to represent $e_m(\beta)$ and e_m with $\Lambda \equiv \Delta_m$.

We subdivide the interval $[-\frac{\beta}{2}, \frac{\beta}{2}]$ into intervals of length $\xi = \frac{1}{N}$. A lattice is associated to the subdivision. We consider "spins" with value in the piecewise constant functions from $[0, \xi]$ to $\{-1, 1\}$. The spins of neighbouring sites in the vertical direction must satisfy an obvious compatibility condition. Therefore we consider a spin model on subsets $\Lambda \subset \mathbb{Z} \times \xi\mathbb{Z}$ with an interaction defined as follows.

The free measure on each site is $\mu_{[0,\xi]}$. If $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $|x_1 - y_1| = 1$ the interation between the spins σ_x and σ_y is given by

$$W(\sigma_x, \sigma_y) = J \int_0^{\xi} \sigma_x(t)\, \sigma_y(t)\, dt.$$

If $x=(x_1,x_2)$, $y=(x_1,x_2+\xi)$ then

$$W(\sigma_x,\sigma_y)=-\log\delta_{\sigma_x(\xi),\sigma_y(0)}.$$

The Gibbs measure on $\Lambda\subset\subset\mathbb{Z}\times\xi\mathbb{Z}$ can then be represented as a measure with density

$$Z^{-1}\exp\Big(-\sum_{\langle x,y\rangle}W(\sigma_x,\sigma_y)\Big)$$

with respect to the product measure

$$\prod_{x\in\Lambda}\mu_{[0,\xi]}.$$

First step

For $x=(x_1,x_2)$ and $y=(y_1,x_2)$ with $|x_1-y_1|=1$ we write

$$e^{-W(\sigma_x,\sigma_y)}=1+\big(e^{-W(\sigma_x,\sigma_y)}-1\big).$$

By expanding the product over all horizontal nearest neighbour bonds H_Λ we obtain

$$Z=\sum_{A\subset H_\Lambda}\int\prod_{\ell=(x,y)\in A}\big(e^{-W(\sigma_x,\sigma_y)}-1\big)e^{-\sum_{(x,y)\in V_\Lambda}W(\sigma_x,\sigma_y)}\,d\bigotimes_{x\in\Lambda}\mu_x(\underline{\sigma})$$

where V_Λ is the set of vertical nearest neighbour bonds in Λ.

Given a set A of horizontal bonds $A \subset H_\Lambda$, we consider the set of sites $S(A)$ belonging to some bond of A. If two sites in $S(A)$ are separated by a vertical segment in $S(A)^c$, then we integrate over the corresponding spins.

If the sites are $x = (x_1, y_1)$ and $y = (x_1, y_2)$ with $y_2 \geq y_1 + 2\xi$, then the integral over intermediate spins gives

$$\frac{1 + e^{-2h(y_2 - y_1)}}{2} \quad \text{if } \sigma_x(\xi) = \sigma_y(0)$$

$$\frac{1 - e^{-2h(y_2 - y_1)}}{2} \quad \text{if } \sigma_x(\xi) \neq \sigma_y(0).$$

We perform then a second expansion adding and subtracting $\frac{1}{2}$.

In this way the partition function can be written as a sum over polymer configurations.

We take $\xi = \frac{1}{\sqrt{h}}$. The activity $\zeta(R)$ of a polymer R can then be estimated by

$$\left(e^{\frac{J}{\sqrt{h}}} - 1\right)^{\#(\text{hor. bonds})} e^{-2\sqrt{h}\,(\text{vert. bonds})}$$

$$\leq c(h)^{N(R)}$$ where

$$c(h) \doteq \max\left(e^{\frac{1}{\sqrt{h}}} - 1,\ e^{-2\sqrt{h}}\right)$$ and

$$N(R) = \#\binom{\text{hor bonds}}{\text{in } R} + \#\binom{\text{vert. bonds}}{\text{in } R}.$$

Kotecky and Preiss conditions are satisfied for the convergence of cluster expansion when h is sufficiently large.

Entanglement

In order to study the entanglement of the reduced state, one introduces a modified system (see [Grimmett, Osborne and Scudo]). The sites of $S_L = [0, L] \times \{0\}$ are doubled into two copies denoted respectively by S_L^+ and S_L^-. For $x \in S_L$ the corresponding sites in S_L^+, S_L^- are denoted respectively by x^+ and x^- that are connected respectively with the upper and lower part. The spin configuration of S_L^+, S_L^- are denoted respectively by $\sigma_L^+ = (\sigma_x^+, x \in S_L)$ $\sigma_L^- = (\sigma_x^-, x \in L)$ and take value in $\Sigma_L = \{-1, +1\}^{L+1}$.

the interaction is like that of the original system with the natural changes due to the definition of connection. $\Phi_{m,\beta}$ denotes the corresponding Gibbs measure on $\Lambda_{m,\beta}$ (with S_L split into S_L^+ and S_L^-. One can perform on this system the construction described above and the cluster expansion that is convergent for h sufficiently large.

Estimate of mixing

There is a constant $\eta > 0$ such that if $\frac{J}{h} < \eta$ there is a contstant C (uniformly in m and L)

$$C^{-1} \leq \frac{\Phi_{m,\beta}(\sigma_L^+ = \varepsilon^+, \sigma_L^- = \varepsilon^-)}{\Phi_{m,\beta}(\sigma_L^+ = \varepsilon^+)\,\Phi_{m,\beta}(\sigma_L^- = \varepsilon^-)} \leq C$$

for $\varepsilon_+, \varepsilon_- \in \Sigma_L$. $C \to 1$ as $\frac{J}{h} \to 0$.

Idea of the proof

The cluster expansion allows to write the ratio as

$$\exp\Big(\sum_{C} \Phi^T(C)\Big),$$

where the sum ranges over all clusters C of polymers that intersect both S_L^+ and S_L^- and the term $\Phi^T(C)$, the coefficients provided by the cluster expansion. The estimates of Kotetzky and

Preiss allows to obtain the bound.

The bound on the entanglement can then be obtained by following the same steps as in [GOS].

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