ESAME 08/02

1 A

$$\boxed{w = \frac{-3\sqrt{2}\, i}{1-i}}$$

Scrivere w in forma Trigonometrica:

$$w = \frac{-3\sqrt{2}\, i}{1-i} \cdot \frac{1+i}{1+i} = \frac{+3\sqrt{2} - 3\sqrt{2}\, i}{2} =$$

$$= 3\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right) = 3\left(\cos\left(\frac{7\pi}{4}\right) + \operatorname{sen}\left(\frac{7\pi}{4}\right)\right)$$

$$= 3\, e^{i\frac{7}{4}\pi}$$

Risolvere: $\boxed{z^3 = 3e^{i\frac{7}{4}\pi}}$

Sia $z = \rho e^{i\vartheta}$ allora

$$\begin{cases} \rho = \sqrt[3]{3} \\ 3\vartheta = \frac{7}{4}\pi + 2k\pi \quad k=0,\dots,2 \end{cases} \qquad \begin{cases} \rho = \sqrt[3]{3} \\ \vartheta = \frac{7}{12}\pi + \frac{2}{3}k\pi \quad k=0,..,2 \end{cases}$$

$$z_1 = \sqrt[3]{3} \cdot e^{i\frac{7}{12}\pi} \qquad z_2 = \sqrt[3]{3} \cdot e^{i\frac{15}{12}\pi} \qquad z_3 = \sqrt[3]{3} \cdot e^{i\frac{23}{12}\pi}$$

1 B

2

$$\boxed{w = \frac{5\sqrt{2}\, i}{i+1}}$$

Scrivo w in forma Trigonometrica:

$$w = \frac{5\sqrt{2}\, i}{i+1} \cdot \frac{i-1}{i-1} = \frac{-5\sqrt{2} - 5\sqrt{2}\, i}{-2} =$$

$$= 5\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= 5\left(\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right)$$

$$= 5\, e^{i\frac{\pi}{4}}$$

$$\boxed{\text{Risolvere} \quad z^3 = w}$$

Pongo $z = \rho\, e^{i\vartheta}$

$$\begin{cases} \rho = \sqrt[3]{5} \\ 3\vartheta = \frac{\pi}{4} + 2k\pi \end{cases} \quad k = 0, \dots, 2 \qquad \begin{cases} \rho = \sqrt[3]{5} \\ \vartheta = \frac{\pi}{12} + \frac{2k\pi}{3} \end{cases}$$

$$z_1 = \sqrt[3]{5} \cdot e^{i\frac{\pi}{12}}, \quad z_2 = \sqrt[3]{5}\, e^{i\frac{9}{12}\pi} \;\left(\tfrac{3}{4}\right), \quad z_3 = \sqrt[3]{5}\, e^{i\frac{17}{12}\pi},$$

2) A : $\lim_{x \to 0} \dfrac{\sin\left(x + \frac{x^2}{2}\right) - \log\left(1 + x - \frac{x^2}{2}\right) + 1 - \cos(x^2)}{e^{1+x^2} - e^{\cos x}}$ (3)

$\boxed{\text{Num}}$: $\cdot\ \sin\left(x + \frac{x^2}{2}\right) = \cancel{x} + \frac{x^2}{2} + o(x^2)$

$\cdot\ -\log\left(1 + x - \frac{x^2}{2}\right) = -\cancel{x} + \frac{x^2}{2} + \frac{x^2}{2} + o(x^2)$

$\cdot\ 1 - \cos(x^2) = \cancel{1} - \cancel{1} + o(x^3)$

$\rightarrow \boxed{\text{Num} = \frac{3}{2} x^2 + o(x^2)}$

$\boxed{\text{Den}}$: $\cdot\ e^{1+x^2} = e \cdot e^{x^2} = e\left(1 + x^2 + o(x^2)\right)$

$\cdot\ -e^{\cos x} = -e^{1 - \frac{1}{2}x^2 + o(x^2)} =$

$= -e \cdot \left(e^{-\frac{1}{2}x^2 + o(x^2)}\right) =$

$= -e\left(1 - \frac{1}{2}x^2 + o(x^2)\right)$

$\rightarrow \text{Den} = e\left[\cancel{1} + x^2 - \cancel{1} + \frac{1}{2}x^2 + o(x^2)\right]$

$= \frac{3}{2} e x^2 + o(x^2)$

Conclusione: $\lim_{x \to 0} \frac{N}{D} = \lim_{x \to 0} \dfrac{\frac{3}{2}x^2}{\frac{3}{2}e x^2} = \boxed{\frac{1}{e}}$

B) $$\lim_{x \to 0} \frac{e^{e^{x^2}} - e^{1+2x^2}}{\log(1+x^2) + e^{x+\frac{x^2}{2}} - 1 - \operatorname{sen}(x+x^2)}$$

Num:

$$\bullet \quad e^{e^{x^2}} = e^{(1+x^2+o(x^2))}$$

$$= e \cdot e^{x^2+o(x^2)} = e\left[1+x^2+o(x^2)\right]$$

$$\bullet \quad -e^{1+2x^2} = -e \cdot \left[e^{2x^2}\right] = -e\left[1+2x^2+o(x^2)\right]$$

$$\to \boxed{\text{Num}} = e\left[\cancel{1}+x^2-\cancel{1}-2x^2+o(x^2)\right]$$

$$= \boxed{-ex^2+o(x^2)}$$

Den:

$$\bullet \quad \log(1+x^2) = \cancel{x^2}+o(x^2)$$

$$\bullet \quad e^{x+\frac{x^2}{2}} - 1 = \cancel{1}+\cancel{x}+\frac{x^2}{2}+\frac{x^2}{2}+o(x^2) \cancel{-1}$$

$$\bullet \quad -\operatorname{sen}(x+x^2) = -\cancel{x}-\cancel{x^2}+o(x^2)$$

$$\to \boxed{\text{Den} = x^2+o(x^2)}$$

Conclusione:

$$\lim_{x \to 0} \frac{N}{D} = \lim_{x \to 0} \frac{-ex^2}{x^2} = \boxed{-e}$$

3 A

$$\boxed{\int_0^1 \frac{2x\log(x^2+3)}{(x^2+3)^3}\,dx} \overset{\text{per parti}}{=} \left[-\frac{1}{2}\cdot\frac{1}{(x^2+3)^2}\cdot\log(x^2+3)\right]_0^1 ⑤$$

$$+\frac{1}{2}\int_0^1 \frac{1}{(x^2+3)^2}\cdot\frac{2x}{(x^2+3)}$$

$$=\left[-\frac{1}{2}\frac{1}{(x^2+3)^2}\log(x^2+3)\right]_0^1 + \frac{1}{2}\underbrace{\int_0^1 \frac{2x}{(x^2+3)^4}}$$

$$\parallel \qquad + \frac{1}{2}\cdot\left[-\frac{1}{3}\frac{1}{(x^2+3)^3}\right]_0^1$$

B

$$\boxed{\int_0^1 \frac{4x\log(2+2x^2)}{(2+2x^2)^2}} \rightsquigarrow \text{è analogo}$$

(4) $\boxed{A}$ $\displaystyle\int_0 \frac{\arctan x + x^{\alpha}}{x^2 + x^{2\alpha}}\,dx$

in $\boxed{0}$: $f(x) \sim \dfrac{x + x^{\alpha}}{x^2 + x^{2\alpha}}$

- se $\alpha \geq 1$ $f(x) \sim \dfrac{x}{x^2} = \dfrac{1}{x}$ NON conv
- se $\alpha < 1$ $f(x) \sim \dfrac{1}{x^{\alpha}}$ $\boxed{\text{CONV } \forall \alpha < 1}$

$\boxed{\text{in } +\infty}$: $f(x) \sim \dfrac{x^{\alpha}}{x^2 + x^{2\alpha}}$

($\arctan x \underset{x \to +\infty}{\longrightarrow} \frac{\pi}{2}$)

- se $\alpha < 1$:

$f(x) \sim \dfrac{x^{\alpha}}{x^2} = \dfrac{1}{x^{2-\alpha}}$

conv $\Leftrightarrow$ $2-\alpha > 1$

$\Leftrightarrow$ $\boxed{\alpha < 1}$

Converge $\Leftrightarrow$ $\boxed{0 < \alpha < 1}$

$\boxed{B}$ $\displaystyle\int_0^{+\infty} \frac{\arctan(x^2) + x^{2\beta}}{x^3 + x^{3\beta}}\,dx$

in $\boxed{0}$: $f(x) \sim \dfrac{x^2 + x^{2\beta}}{x^3 + x^{3\beta}}$

- se $\beta \geq 1$ $f(x) \sim \dfrac{1}{x}$ NON conv
- se $\beta < 1$ $f(x) \sim \dfrac{1}{x^{\beta}}$ CONV

in $(+\infty)$ $f(x) \sim \dfrac{x^{2\beta}}{x^3 + x^{3\beta}}$

se $\beta < 1$ $\to$ $f(x) \sim \dfrac{x^{2\beta}}{x^3}$

conv $\Leftrightarrow$ $3 - 2\beta > 1$ $\Leftrightarrow$ $\beta < \frac{2}{2}$

Conclusione: converge $\Leftrightarrow$ $\boxed{0 < \beta < 1}$

ok

⑤ $f(x) = \log\left(2x^2 + x + |x-1|\right)$

$\boxed{\text{Dom}}$: $2x^2 + x + |x-1| > 0 \iff$

$$\begin{cases} 2x^2 + x + x - 1 > 0 \\ x \geq 1 \end{cases} \cup \begin{cases} 2x^2 + \cancel{x} + 1 - \cancel{x} > 0 \\ x < 1 \end{cases}$$

$$\begin{cases} 2x^2 + 2x - 1 > 0 \\ x \geq 1 \end{cases} \cup \begin{cases} 2x^2 + 1 > 0 \\ x < 1 \end{cases}$$

$\forall x \in [1, +\infty)$ $\qquad$ $\forall x \in (-\infty, 1)$

$$\Rightarrow \boxed{D = \mathbb{R}}$$

$\boxed{\text{SEGNO di } f}$: $f > 0 \iff 2x^2 + x + |x-1| > 1$

$$\begin{cases} 2x^2 + x + x - 1 > 1 \\ x \geq 1 \end{cases} \cup \begin{cases} 2x^2 + \cancel{x} + \cancel{1} - \cancel{x} > \cancel{1} \\ x < 1 \end{cases}$$

$$\begin{cases} 2x^2 + 2x > 2 \\ x \geq 1 \end{cases} \cup \begin{cases} 2x^2 > 0 \\ x < 1 \end{cases}$$

$\Updownarrow$

$\forall x \geq 1$ $\qquad$ $\forall x < 1, x \neq 0$

$\Rightarrow f(x) > 0 \ \forall x \neq 0, \quad f(x) = 0 \iff x = 0$

LIMITI :

8

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \log\left(x^2\left(2 + \underbrace{\frac{1}{x}}_{\to 0} + \underbrace{\frac{(x-1)}{x^2}}_{\to 0}\right)\right)$$

$$= \boxed{+\infty}$$

$$\Rightarrow \boxed{\sup_{\mathbb{R}} f = +\infty}$$

MONOTONIA:

$$f(x) = \begin{cases} \log(2x^2+2x-1) & \text{se } x \geq 1 \\ \log(2x^2+1) & \text{se } x < 1 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{4x+2}{2x^2+2x-1} & \text{se } x \geq 1 \\ \dfrac{4x}{2x^2+1} & \text{se } x < 1 \end{cases}$$

Studio il segno di f':

- se $x \geq 1$: $\dfrac{4x+2}{\underbrace{2x^2+2x-1}_{>0}} \geq 0 \iff 4x+2 \geq 0 \iff x \geq -1/2$

$\rightarrow f'(x) \geq 0 \quad \forall x \geq 1$

$\rightarrow$ $\boxed{f \nearrow \text{ in } (1, +\infty)}$

- se $x < 1$ $f'(x) = \dfrac{4x}{2x^2+1} > 0 \iff x > 0$

$\rightarrow$ $\boxed{f \searrow \text{ in } (-\infty, 0)}$

$\boxed{f \nearrow \text{ in } (0, 1)}$

0 è pto di MIN ASSOLUTO

PTI di NON DERIVABILITA':

studio $\lim_{x \to 1^{\pm}} f'(x)$:

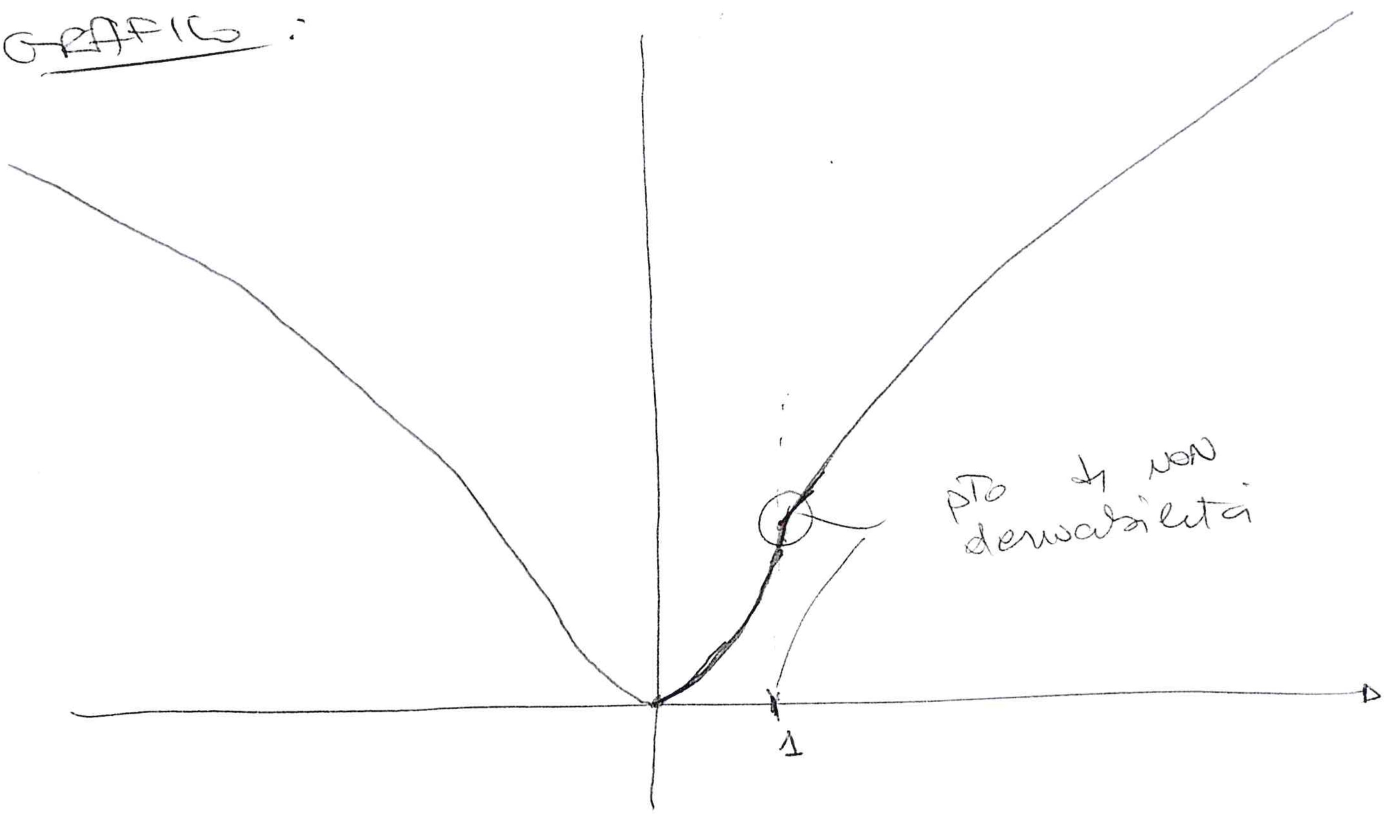
$$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} \frac{4x+2}{2x^2+2x-1} = \frac{6}{3} = 2$$

$$\neq$$

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} \frac{4x}{2x^2+1} = \frac{4}{3}$$

$\rightarrow$ $x = 1$ PTO di NON DERIVABILITA'

GRAFICO:



6 (A) $y'' + 4y' + 5y = e^{3t}$ (*)

Eq caratteristica:

$\lambda^2 + 4\lambda + 5 = 0 \rightarrow \lambda_{1,2} = -2 \pm \sqrt{4-5} = -2 \pm i$

$\rightarrow$ soluz generale eq. omogenea è: $y_{om}(t) = C_1 e^{-2t} \cdot \cos(t) + C_2 e^{-2t} \operatorname{sen}(t)$,

Cerco una soluz particolare dell'eq. non omogenea della forma: $\tilde{y}(t) = A e^{3t}$

$\rightarrow \tilde{y}'(t) = 3A e^{3t}$, $\tilde{y}''(t) = 9A e^{3t}$

Impongo che $\tilde{y}$ sia soluzione:

$9A + 12A + 5A = 1 \longrightarrow 26A = 1 \rightarrow A = \frac{1}{26}$

Conclusione: integrale generale di (*) è dato da:

$$\boxed{y(t) = C_1 e^{-2t} \cos t + C_2 e^{-2t} \operatorname{sen}(t) + \frac{1}{26} e^{3t}}$$

B) $\boxed{2y'' + 6y' + 5y = e^{2t}}$ (*)

Eq caratteristica: $2\lambda^2 + 6\lambda + 5 = 0 \to \lambda_{1,2} = \frac{-3 \pm \sqrt{9-10}}{2} = -\frac{3}{2} \pm \frac{i}{2}$

→ soluz generale eq. omogenea → $y_{om}(t) = C_1 e^{-\frac{3t}{2}} \cdot \cos\left(\frac{t}{2}\right) + C_2 e^{-\frac{3t}{2}} \operatorname{sen}\left(\frac{t}{2}\right)$

Cerco una soluz. particolare della non omogenea della forma $\tilde{y}(t) = Ae^{2t} \to \tilde{y}'(t) = 2Ae^{2t}$

$\tilde{y}''(t) = 4Ae^{2t}$

→

$8A + 12A + 5A = 1 \to A = \frac{1}{25}$

Conclusione: integrale generale di (*) è dato da

$$\boxed{y(t) = C_1 e^{-\frac{3t}{2}} \cos\left(\frac{t}{2}\right) + C_2 e^{-\frac{3t}{2}} \operatorname{sen}\left(\frac{t}{2}\right) + \frac{1}{25} e^{2t}}$$