ESAME ANALISI I - 08/01/19 (1)

(1) A $\boxed{W = \frac{1+\sqrt{3}i}{1-\sqrt{3}i}}$

Scrivere $w$ in forma trigonometrica:

$$W = \frac{1+\sqrt{3}i}{1-\sqrt{3}i} \cdot \frac{1+\sqrt{3}i}{1+\sqrt{3}i} = \frac{-2+2\sqrt{3}}{4} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$\Rightarrow |W| = \left(\frac{1}{4} + \frac{3}{4}\right)^{1/2} = 1 \qquad \arg(w) = \frac{2}{3}\pi$$

$$\Rightarrow W = e^{i\frac{2}{3}\pi} = \cos\left(\frac{2}{3}\pi\right) + i \operatorname{sen}\left(\frac{2}{3}\pi\right)$$

Risolvo: $z^6 = \frac{1+\sqrt{3}i}{1-\sqrt{3}i} = e^{i\frac{2}{3}\pi}$

Sia $z = \rho \cos\vartheta$

$$\rightarrow \begin{cases} \rho = 1 \\ 6\vartheta = \frac{2}{3}\pi + 2k\pi \end{cases} \qquad k = 0, \dots, 5$$

$$\boxed{z_k = e^{i\left(\frac{\pi}{9} + \frac{k}{3}\pi\right)} \qquad k = 0, \dots, 5}$$

B L'altro analogo:

$$W = \frac{1-\sqrt{3}i}{1+\sqrt{3}i} \cdot \frac{1-\sqrt{3}i}{1+\sqrt{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$\rightarrow |W| = 1 \qquad \arg(w) = \frac{4}{3}\pi \rightarrow w = \cos\left(\frac{4\pi}{3}\right) + i \operatorname{sen}\left(\frac{4\pi}{3}\right)$$

Soluzioni di $z^4 = W \longrightarrow \boxed{z_k = e^{i\left(\frac{\pi}{3} + \frac{k}{2}\pi\right)} \qquad k = 0, \dots, 3}$

2) A) $\lim_{x \to 0} \dfrac{e^{x-x^2} - \arctan(x-x^3) - \cos(2x+x^2)}{\sin(x+\pi)\left(2-\sqrt{4-x^2}\right)}$ (2)

$\boxed{\text{Num}}$ :

- $e^{x-x^2} = 1 + x - x^2 - \frac{1}{2}x^2 + o(x^2)$
- $\arctan(x-x^3) = x + o(x^2)$
- $-\cos(2x+x^2) = -1 + \frac{1}{2}(2x+x^2)^2 + o(x^2)$

$\rightarrow \text{Num} = \cancel{1} + \cancel{x} - \frac{3}{2}x^2 - \cancel{x} - \cancel{1} + 2x^2 + o(x^2)$

$= \frac{1}{2}x^2 + o(x^2)$

$\boxed{\text{Denom}}$ : $\sin(x+\pi) = -\sin x = -x + o(x)$

$2 - \sqrt{4-x^2} = 2\left(1 - \left(1 - \frac{x^2}{4}\right)^{\frac{1}{2}}\right)$

$= 2\left(\cancel{1} - \cancel{1} + \frac{1}{8}x^2 + o(x^2)\right) = \frac{1}{4}x^2 + o(x^2)$

$\lim_{x \to 0} \frac{\text{Num}}{\text{Dem}} = \lim_{x \to 0} \dfrac{\frac{1}{2}\cancel{x^2}}{-x \cdot \frac{1}{4}\cancel{x^2}}$ $\left\langle \begin{array}{l} \lim_{x \to 0^+} f(x) = -\infty \\ \lim_{x \to 0^-} f(x) = +\infty \end{array} \right.$

(B) $\lim_{x \to 0} \dfrac{\sqrt{4+x} - 2 + \sin\left(\frac{x}{4}\right) - \log\left(1 + \frac{x}{2} - x^2\right)}{e^{x+1}(\cos x - 1)}$ (3)

$\boxed{\text{Num}}$ : • $\sqrt{4+x} - 2 = 2\left(\left(1 + \frac{x}{4}\right)^{\frac{1}{2}} - 1\right)$

$$= 2\left(\cancel{1} + \frac{1}{8}x - \frac{1}{8}\cdot\frac{x^2}{16} \cancel{-1} + o(x^2)\right)$$

$$= \frac{1}{4}x - \frac{1}{64}x^2 + o(x^2)$$

• $\sin\left(\frac{x}{4}\right) = \frac{x}{4} + o(x^2)$

• $-\log\left(1 + \frac{x}{2} - x^2\right) = -\frac{x}{2} + x^2 + \frac{1}{8}x^2 + o(x^2)$

$$\text{Num} = \cancel{\frac{1}{4}x} - \frac{1}{64}x^2 + \cancel{\frac{1}{4}x} - \cancel{\frac{x}{2}} + x^2 + \frac{1}{8}x^2 + o(x^2)$$

$$= \frac{71}{64}x^2 + o(x^2)$$

$\boxed{\text{Denom}}$ : $e^{x+1}(\cos x - 1) = (e + o(1))\left(-\frac{1}{2}x^2 + o(x^2)\right)$

$$\lim_{x \to 0} \frac{\text{Num}}{\text{Denom}} = \lim_{x \to 0} \frac{\frac{71}{64}x^2}{-\frac{e}{2}x^2} = \boxed{-\frac{71}{32e}}$$

③ A $\displaystyle\int_0^{\pi/2} e^{\operatorname{sen}^2(x)} \underbrace{(\operatorname{sen}x\cos x)^3}_{\operatorname{sen}x\cos x\,(\operatorname{sen}^2x\cos^2x)} dx$ ④

Pongo $t = \operatorname{sen}^2 x \to dt = 2\operatorname{sen}x\cos x\, dx$

$\Rightarrow \frac{1}{2}\displaystyle\int_0^1 e^t\, t\cdot(1-t)\, dt$

$= \frac{1}{2}\Big[\underbrace{e^t\cdot t(1-t)}_{0}\Big]_0^1 - \frac{1}{2}\displaystyle\int_0^1 e^t(1-2t)\,dt$

$= -\frac{1}{2}\big[e^t(1-2t)\big]_0^1 - \cancel{\frac{2}{2}}\displaystyle\int_0^1 e^t\, dt$

$= +\frac{e}{2} + \frac{1}{2} - \big[e^t\big]_0^1 = -\frac{e}{2} - \frac{1}{2}$

B Analogo ( pongo $t = \operatorname{sen}^2(2x)$ )

④ A $\displaystyle\int_0^{+\infty} \overbrace{\frac{1+\sqrt{x}}{x^{2\alpha}+x^{4\alpha}}}^{f(x)}\, dx \qquad \alpha\in\mathbb{R}^+$

• $\displaystyle\int_0^1 f(x)\,dx$ : $f(x) \sim \frac{1}{x^{2\alpha}}$ per $x\to 0$

$\Rightarrow$ Int. converge $\Leftrightarrow 2\alpha<1 \Leftrightarrow \alpha<\frac{1}{2}$

• $\displaystyle\int_1^{+\infty} f(x)\,dx$ : $f(x) \sim \frac{\sqrt{x}}{x^{4\alpha}} = \frac{1}{x^{4\alpha-1/2}}$ per $x\to+\infty$

Int conv $\Leftrightarrow 4\alpha - \frac{1}{2} > 1 \Leftrightarrow \alpha > \frac{3}{8}$

SOL²

$\boxed{\frac{3}{8} < \alpha < \frac{1}{2}}$

(B) Analogamente: $\int_0^{+\infty} \frac{2+\sqrt{x}}{x^{3\alpha}+x^{6\alpha}}\,dx$

• $\int_0^1 f(x)\,dx$    $f(x) \sim \frac{2}{x^{3\alpha}}$ per $x \to$

int. conv $\iff 3\alpha < 1 \iff \alpha < \frac{1}{3}$

• $\int_1^{+\infty} f(x)\,dx$ :    $f(x) \sim \frac{\sqrt{x}}{x^{6\alpha}} = \frac{1}{x^{6\alpha - 1/2}}$ per $x \to +\infty$

int conv $\iff 6\alpha - 1/2 > 1 \iff 6\alpha > \frac{3}{2}$

$\iff \alpha > \frac{1}{4}$

Conclusione: converge per $\boxed{\frac{1}{4} < \alpha < \frac{1}{3}}$

5) $f(x) = e^{\sqrt{\frac{x^2-1}{x-2}}}$

DOMINIO: $\frac{x^2-1}{x-2} \geq 0$    $x^2-1 \geq 0 \iff x \leq -1 \vee x \geq 1$

$x-2 > 0 \iff x > 2$

| | $-1$ | $1$ | $2$ | |
|---|---|---|---|---|
| + | − | + | + | |
| − | − | − | + | |
| − | + | − | + | |

$\Rightarrow \boxed{D = [-1,1] \cup (2,+\infty)}$

LIMITI

- $\lim_{x \to 2^+} e^{\sqrt{\frac{x^2-1}{x-2}}} = +\infty$

- $\lim_{x \to +\infty} e^{\sqrt{\frac{x^2-1}{x-2}}} = +\infty$

MONOTONIA:

$$f'(x) = e^{\sqrt{\frac{x^2-1}{x-2}}} \cdot \frac{\sqrt{\frac{x-2}{x^2-1}}}{2} \cdot \frac{2x(x-2) - x^2 + 1}{(x-2)^2}$$

$$f'(x) \geq 0 \iff x^2 - 4x + 1 \geq 0$$

$$x_{1,2} = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3}$$

$$f'(x) \geq 0 \iff x \leq 2 - \sqrt{3} \cup x \geq 2 + \sqrt{3}$$

Quindi:

$f \nearrow$ in $[-1, 2-\sqrt{3}) \cup (2+\sqrt{3}, +\infty)$

$f \searrow$ in $(2-\sqrt{3}, 1] \cup (2, 2+\sqrt{3})$

$x = 2 - \sqrt{3}$ è pto di MAX locale

$x = 2 + \sqrt{3}$ è pto di MIN locale

$x = \pm 1$ pti di MIN. ASSOLUTO

$\sup f = +\infty$

PTI di NON DERIVABILITÀ

Studio $\lim_{x \to \pm 1} f'(x) = \infty \to \pm 1$ pti di non derivabilità

GRAFICO

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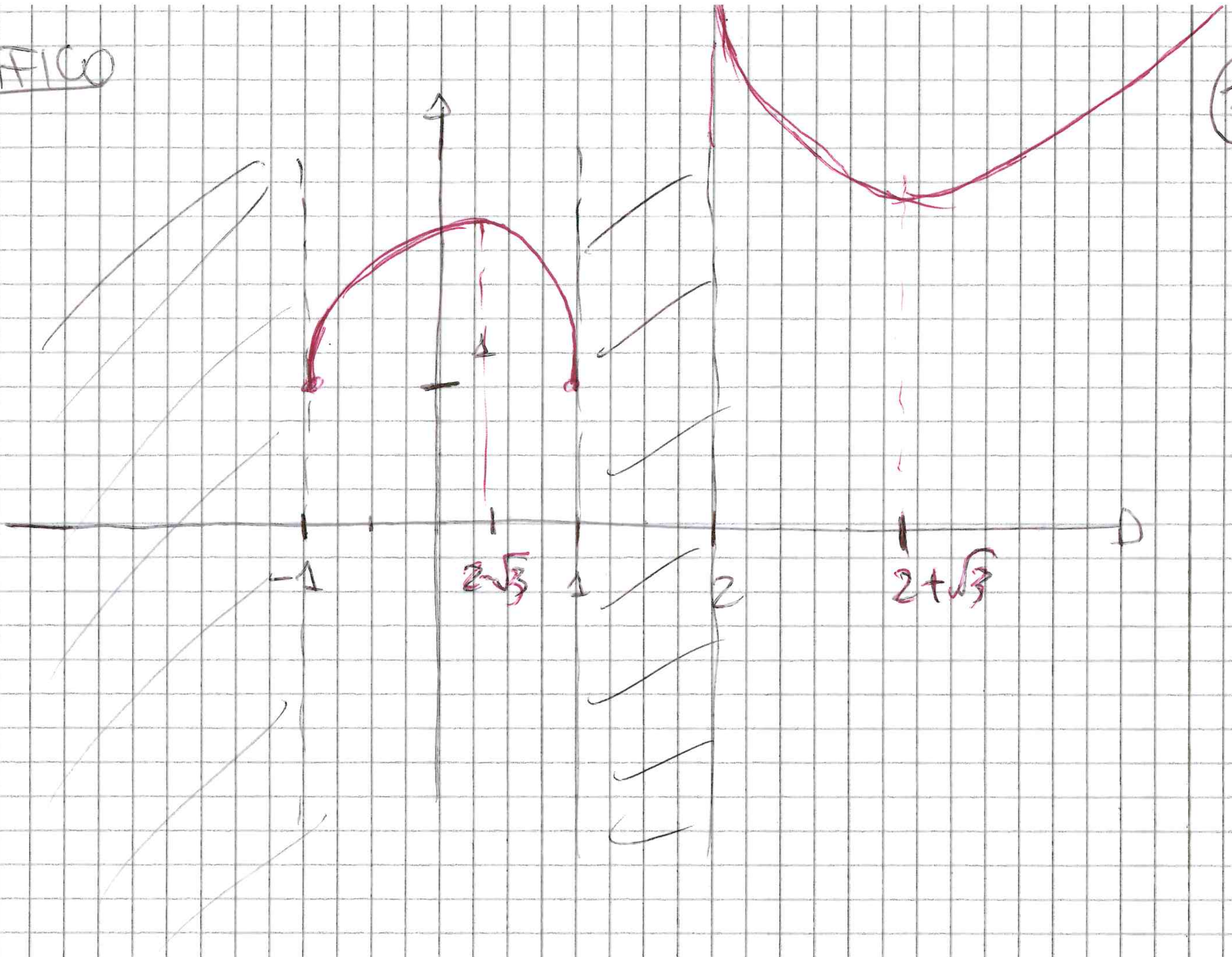


$\boxed{6}$
$$\begin{cases} y' = t\,e^{y-t} \\ y(0) = 1 \end{cases} \rightarrow \quad e^{-y}\,y' = t\,e^{-t}$$

$$\rightarrow \quad \int_0^t e^{-y}\,y'\,dt = \int_0^t s\,e^{-s}\,ds$$

$$\Rightarrow \quad -e^{-y(t)} + e^{-y(0)} = \left[-e^{-s}\cdot s\right]_0^t + \int_0^t e^{-s}\,ds$$

$$\Rightarrow \quad -e^{-y(t)} + e^{-1} = -e^{-t}\cdot t - e^{-t} + 1$$

$$\Rightarrow \quad e^{-y(t)} = e^{-t}\cdot t + e^{-t} + \frac{1}{e} - 1$$

$$\Rightarrow \quad -y(t) = \log\left(e^{-t}\cdot t + e^{-t} + \frac{1}{e} - 1\right)$$

$$\Rightarrow \quad \boxed{y(t) = -\log\left(e^{-t}\cdot t + e^{-t} + 1/e - 1\right)}$$