ESAME 22-01-2019

1 A $\boxed{\lim_{x \to 0} \frac{e^{x+x^3} - 1 + \log(1+\operatorname{sen} x) - 2\tan x}{\cos(x+\pi)\,(1-\sqrt{1+x^3})}}$

$\boxed{\text{Num}}$:

- $e^{x+x^3} - 1 = \cancel{x} + x^3 + \cancel{\frac{1}{2}x^2} + \frac{1}{6}x^3 + o(x^3)$
- $\log(1+\operatorname{sen} x) = \log\left(1 + x - \frac{1}{6}x^3 + o(x^3)\right)$

  $= \cancel{x} - \frac{1}{6}x^3 - \cancel{\frac{1}{2}x^2} + \frac{1}{3}x^3 + o(x^3)$
- $-2\tan x = \cancel{-2x} - \frac{2}{3}x^3 + o(x^3)$

$\boxed{\text{Num}} = x^3 + \cancel{\frac{1}{6}x^3} - \cancel{\frac{1}{6}x^3} + \frac{1}{3}x^3 - \frac{2}{3}x^3 + o(x^3)$

$= \boxed{\frac{2}{3}x^3 + o(x^3)}$

$\boxed{\text{Denom}}$:

- $\cos(x+\pi) \to -1$ per $x \to 0$
- $1 - \sqrt{1+x^3} = 1 - \left(1 + \frac{1}{2}x^3 + o(x^3)\right)$

  $= -\frac{1}{2}x^3 + o(x^3)$

Quindi $\lim_{x\to 0} \frac{\text{Num}}{\text{Den}} = \lim_{x \to 0} \frac{\frac{2}{3}x^3 + o(x^3)}{+\frac{1}{2}x^3 + o(x^3)} = \boxed{+\frac{4}{3}}$

(B) $$\lim_{x\to 0} \frac{e^{\tan x} + \log(1+x+x^3) - 1 - 2\,\text{sen}\,x}{e^{x+2}\left(\sqrt[5]{2x^3+1} - 1\right)}$$

Num

- $$e^{\tan x} - 1 = e^{x+\frac{1}{3}x^3+o(x^3)} - 1$$
$$= x + \frac{1}{3}x^3 + \cancel{\frac{1}{2}x^2} + \frac{1}{6}x^3 + o(x^3)$$

- $$\log(1+x+x^3) = x + x^3 - \cancel{\frac{1}{2}x^2} + \frac{1}{3}x^3 + o(x^3)$$

- $$-2\,\text{sen}\,x = -2x + \frac{1}{3}x^3 + o(x^3)$$

$$\text{Num} = \frac{1}{3}x^3 + \frac{1}{6}x^3 + \frac{1}{3}x^3 + \frac{1}{3}x^3 + x^3 + o(x^3)$$

$$= \frac{13}{6}x^3 + o(x^3)$$

Denom.:

- $$e^{x+2} \longrightarrow e^2 \quad \text{per } x \to 0$$

- $$\sqrt[5]{2x^3+1} - 1 = \left(\cancel{1} + \frac{2}{5}x^3 + o(x^3)\right) \cancel{-1}$$
$$= \frac{2}{5}x^3 + o(x^3)$$

$$\lim_{x\to 0} \frac{\text{Num}}{\text{Den}} = \lim_{x\to 0} \frac{\frac{13}{6}x^3}{\frac{2}{5}x^3} = \boxed{\frac{65}{12e^2}}$$

2 A $g(x) = f(e^x + f^2(x))$

$$g'(x) = f'(e^x + f^2(x)) \cdot [e^x + 2f(x) f'(x)]$$

$$\Rightarrow \boxed{g'(0)} = f'(1 + \underbrace{f^2(0)}_{=1}) \cdot [1 + 2f(0) f'(0)]$$

$$= f'(2) \cdot [1 + 2e] =$$

$$= 4(1 + 2e) = \boxed{4 + 8e}$$

B $g(x) = f(\operatorname{sen} x + f^3(x))$

$$g'(x) = f'(\operatorname{sen} x + f^3(x)) \cdot [\cos x + 3f^2(x) f'(x)]$$

$$g'(\pi) = f'(f^3(\pi)) \cdot [-1 + 3f^2(\pi) \cdot f'(\pi)]$$

$$= \underbrace{f'(1)}_{e} \cdot [-1 + 0] = \boxed{-e}$$

3 A $$\boxed{\int_0^{\frac{1}{3}} \frac{e^{\frac{9}{2}x}}{e^{3x}+1}\, dx}$$

Pongo $e^{3x} = t$

$\Rightarrow 3x = \log t$

$x = \frac{1}{3} \log t$

$dx = \frac{1}{3t}$

$$\Rightarrow \frac{1}{3}\int_1^e \frac{t^{\frac{3}{2}}}{t(1+t)}\, dt =$$

$$= \frac{1}{3} \int_1^e \frac{\sqrt{t}}{(1+t)}\, dt = 2\int_1^{\sqrt{e}} \frac{s^2}{1+s^2}\, ds = 2\int_1^{\sqrt{e}} \left(\underbrace{\frac{s^2+1}{1+s^2}}_{1} - \frac{1}{1+s^2} \right) ds$$

$s = \sqrt{t}$

$t = s^2$

$dt = 2s\, ds$

$$2\left\{ [s]_1^{\sqrt{e}} - [\operatorname{arctg} s]_1^{\sqrt{e}} \right\}$$

B $\int_0^{\frac{1}{5}} \frac{e^{\frac{15}{2}x}}{2+e^{5x}}\,dx$ Amalogamente pag

$e^{5x} = t \to dx = \frac{1}{5t}\,dt$

$\Rightarrow \frac{1}{5}\int_1^{e} \frac{t^{3/2}}{t(1+t)}\,dt \longrightarrow$ come il precedente

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4 A $\sum_{n=1}^{+\infty} \underbrace{(n^{2\alpha}+n)\cdot \sin\left(\frac{1}{n^3+n^2}\right)}_{a_n}$

per $n \to +\infty$ $a_n \sim \frac{n^{2\alpha}+n}{n^3+n^2}$

se $\boxed{\alpha \leq \frac{1}{2}}$ $a_n \sim \frac{n}{n^3} = \frac{1}{n^2} \longrightarrow$ converge

se $\boxed{\frac{1}{2} < \alpha \leq 3}$ $a_n \sim \frac{n^{2\alpha}}{n^3} = \frac{1}{n^{3-2\alpha}}$ conv $\Leftrightarrow 3-2\alpha > 1$

$\Leftrightarrow \alpha < 1$

$\longrightarrow \boxed{\frac{1}{2} < \alpha < 1}$

se $\alpha > 3$: $a_n \sim \frac{n^{2\alpha}}{n^{\alpha}} = n^{\alpha}$ NON CONV

Soluzione : $\boxed{0 \leq \alpha < 1}$

B) $$\sum_{n=1}^{+\infty} \underbrace{\left(1 - e^{\frac{1}{n^5+n^\alpha}}\right)\left(n^{3\alpha} + n^2\right)}_{a_n}$$

$$a_n \sim \frac{n^{3\alpha} + n^2}{n^5 + n^\alpha}$$

- se $\underline{\alpha \le 2/3}$ $\quad a_n \sim \dfrac{n^2}{n^5} = \dfrac{1}{n^3}$ $\quad$ conv

- se $\dfrac{2}{3} < \alpha \le 5$ $\quad a_n \sim \dfrac{n^{3\alpha}}{n^5} = \dfrac{1}{n^{5-3\alpha}}$ $\quad$ conv $\Leftrightarrow 5 - 3\alpha > 1$

$\Leftrightarrow \alpha < \dfrac{4}{3}$

conv $\Leftrightarrow$ $\boxed{2/3 < \alpha < 4/3}$

- se $\alpha > 5$ $\quad a_n \sim \dfrac{n^{3\alpha}}{n^\alpha} = n^{2\alpha}$ $\quad$ NON CONV MAI

Quindi concludo: conv $\Leftrightarrow$ $\boxed{\cancel{2/3} < \alpha < 4/3}$

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(5) STUDIO FUNZIONE: (6)

$$f(x) = e^{\frac{|x-1|}{x+2}}$$

- DOMINIO: $x \neq -2$

- LIMITI $\lim_{x \to 2^-} f(x) = \emptyset$ $\lim_{x \to 2^+} f(x) = +\infty$

$\lim_{x \to -\infty} f(x) = \frac{1}{e}$ $\lim_{x \to +\infty} f(x) = e$

- MONOTONIA:

$$f'(x) = e^{\frac{|x-1|}{x+2}} \cdot \frac{\operatorname{sgn}(x-1)(x+2) - |x-1|}{(x+2)^2}$$

$$= \frac{e^{\frac{|x-1|}{x+2}}}{(x+2)^2} \cdot \operatorname{sgn}(x-1)\left[\cancel{x}+2-\cancel{x}+1\right] \geq 0$$

$\Leftrightarrow$ $\operatorname{sgn}(x-1) \geq 0$

se $x > 1 \Rightarrow f \nearrow$

$x < 1 \Rightarrow f \searrow$

- $x = -1$ pto di MIN LOCALE
- $\inf f = \emptyset$
- $\sup f = +\infty$

PTI di NON DERIVABILITÀ

$x = +1$ è pto di non derivabilità

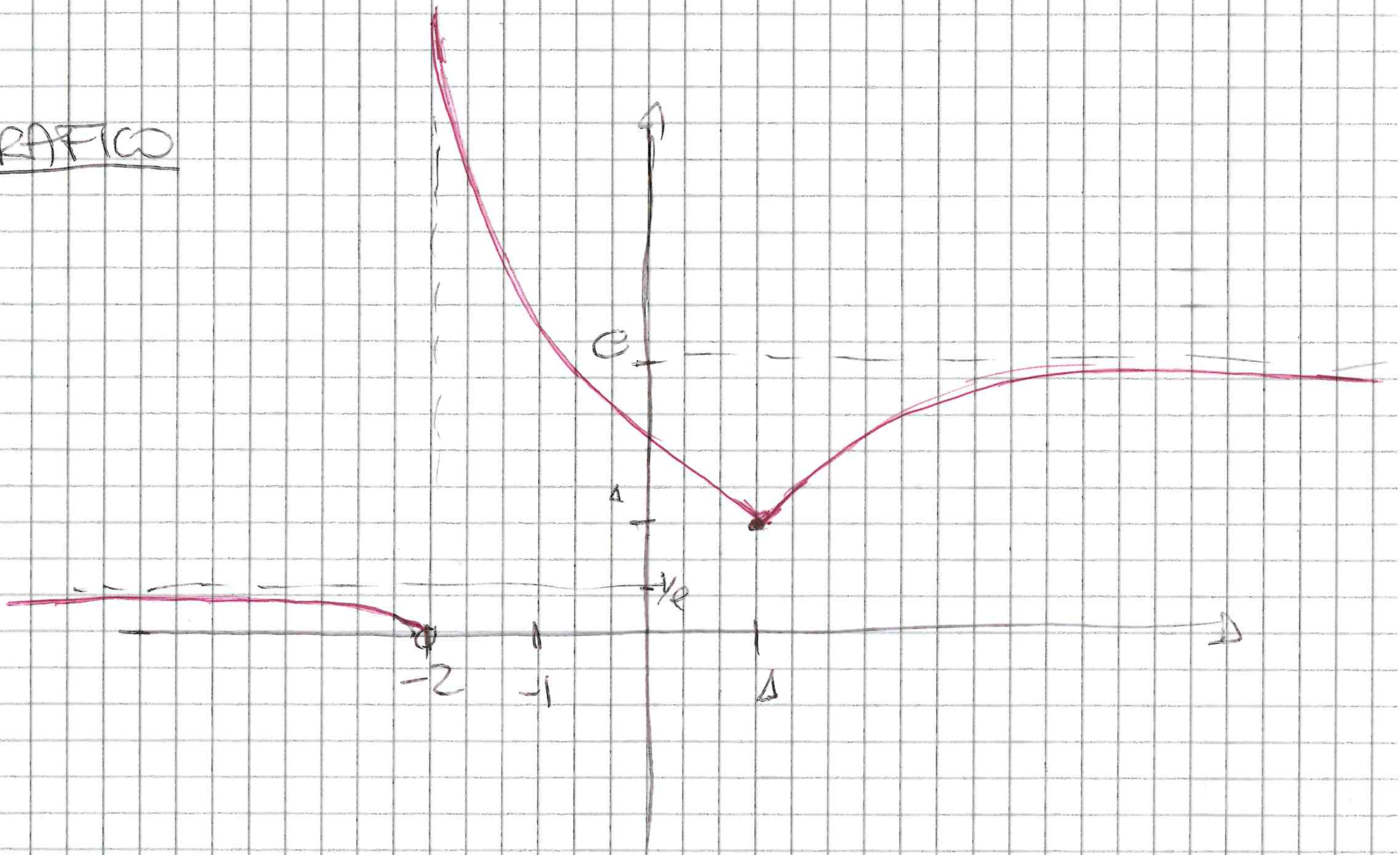
infatti

$$\lim_{x \to 1^+} f'(x) = \frac{1}{9} \cdot 1 \cdot 3 = \frac{3}{9} = \frac{1}{3}$$

$$\lim_{x \to 1^-} f'(x) = \frac{1}{9} \cdot (-1) \cdot 3 = -\frac{1}{3}$$

$\neq$

GRAFICO



L'altro è analogo

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6 A $\begin{cases} y' = t \cos^2 y \\ y(0) = \pi/4 \end{cases}$ $\displaystyle\int_0^t \frac{y'}{\cos^2 y} dt = \int_0^t s\, ds$

$$\Rightarrow \operatorname{tg}(y(t)) - \underbrace{\operatorname{tg}(\pi/4)}_{=1} = \frac{1}{2} t^2$$

$$\Rightarrow \boxed{y(t) = \arctan\left(\frac{1}{2}t^2 + 1\right)}$$

(B) $\begin{cases} y' = \frac{y}{1+t^2} \\ y(0)=2 \end{cases}$ $\int_0^t \frac{y'}{y}\,dy = \int_0^t \frac{1}{1+s^2}\,ds$ (8)

$\Rightarrow \log(y(t)) - \log 2 = \operatorname{arctg} t$

$\Rightarrow \boxed{y(t)} = e^{\log 2} \cdot e^{\operatorname{arctg} t} = \boxed{2e^{\operatorname{arctg} t}}$