8.4.3(a)

G a tree with n vertices $\Rightarrow \pi_k(G) = k(k-1)^{n-1}$

Inductive premise: with n = 1

G is the empty graph with 1 vertex

$\pi_k(G) = k = k(k-1)^{1-1}$

Inductive step

Inductive hypothesis: the formula is true for trees with a number of vertices < n

Inductive Thesis: the formula is true for a tree with n vertices

Let's prove the inductive step.
Let G be a tree with n vertices.
Let u be a vertex of G with $d(u)=1$,
Let e be the only edge incident on u.

$$\pi_k(G) = \pi_k(G-e) - \pi_k(G \cdot e)$$

$G \cdot e$ is what is left by eliminating e and u from G. So $G \cdot e$ is a tree with $n-1$ vertices.
$G-e$ has two components: one is u by itself, the other is equal to $G \cdot e$

So $\pi_k(G-e) = k \cdot \pi_k(G \cdot e)$.

By inductive hypothesis,

$$\pi_k(G \cdot e) = k(k-1)^{(n-1)-1}$$

number of vertices of the tree $G \cdot e$

Finally,

$$\pi_k(G) = \pi_k(G-e) - \pi_k(G \cdot e) =$$

$$= k \cdot k(k-1)^{n-2} - k(k-1)^{n-2} =$$

$$= k(k-1)^{n-2} \cdot (k-1) = k(k-1)^{n-1}$$

8.4.2 (a)

G simple with n vertices and m edges $\Rightarrow$ the coefficient of k^{n-1} is $-m$

By induction on the number of edges

Inductive premise:

if m is 0 then G is the empty graph with n vertices. So $\pi_k(G) = k^n - 0k^{n-1}$

Inductive step:

Inductive hypothesis: the formula holds for simple graphs with less than m edges

Inductive thesis: the formula holds for G with m edges.

Proof of the step:

$\pi_k(G) = \pi_k(G-e) - \pi_k(G \cdot e)$

$G-e$ has n vertices, $m-1$ edges

$G \cdot e$ has $n-1$ vertices, $m-1$ edges

$\pi_k(G-e) = k^n - (m-1)k^{n-1} + \dots$ (by inductive hypothesis)

$\pi_k(G \cdot e) = k^{n-1} - (m-1)k^{n-2} + \dots$ (" " ")

So,

$$\pi_k(G) = k^n - (m-1)k^{n-1} + \cdots$$

$$- k^{n-1} + \cdots$$

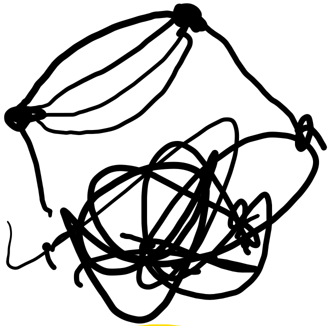
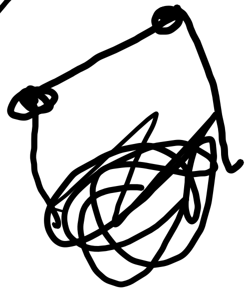
$$= k^n - mk^{n-1} + \cdots$$

8.4.2 (b) What could we say about a possibly existing graph G with $\pi_k(G) = k^4 - 3k^3 + 3k^2$?

$\nu(G) = 4 \quad \varepsilon(G) = 3$

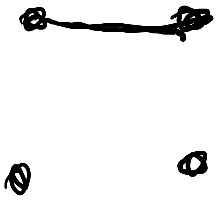
$\pi_k\Big($  $\Big) = \pi_k\Big($  $\Big) \cdot \pi_k\Big($  $\Big)$

$G_1 \quad G_2 \qquad\qquad G_1 \qquad G_2$

$\pi_k\Big($  $\Big) = \pi_k\Big($  $\Big)$

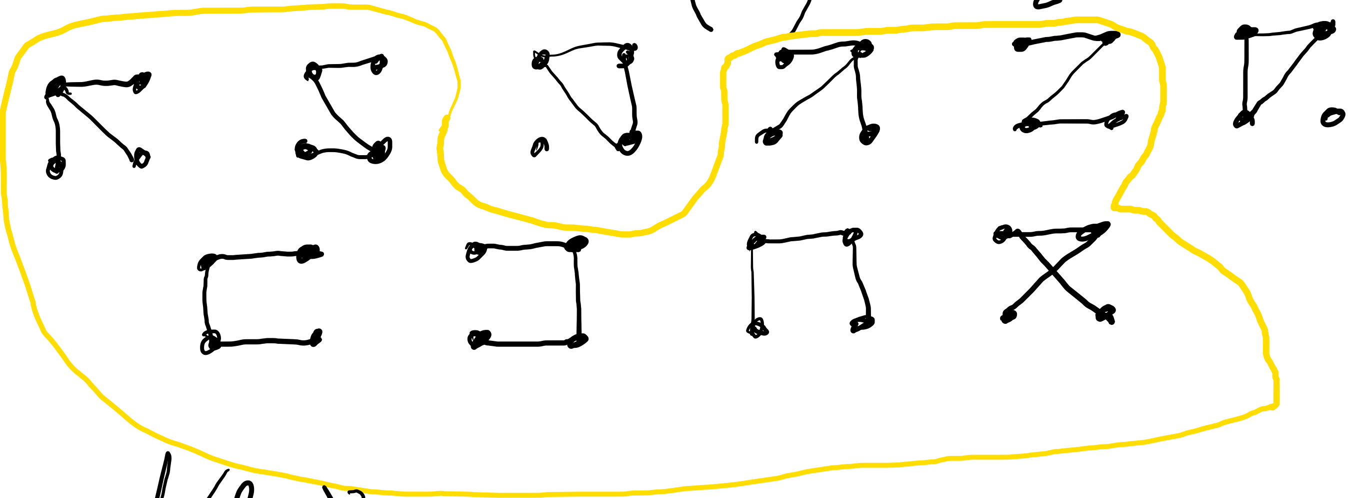
$\pi_k\Big($ 

$\Big) = \dfrac{\pi_k(G_1) \cdot \pi_k(G_2)}{k}$

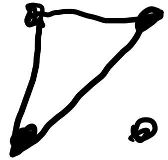


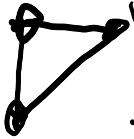
$$\binom{6}{3} = \frac{\cancel{6}\cdot 5\cdot 4}{\cancel{3}\cdot \cancel{2}} = 20$$

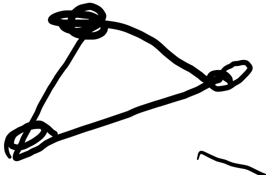
$$\binom{5}{2} = \frac{5\cdot 4}{2} = 10$$

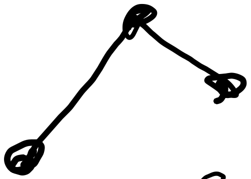
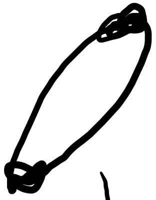
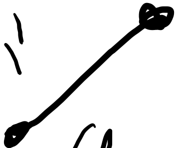


$$k(k-1)^3 =$$
$$= k(k^3-3k^2+3k-1) =$$
$$= k^4-3k^3+3k^2-k$$

$\pi_k\Big($  $\Big)$?

$\pi_k\Big($  $\Big) =$



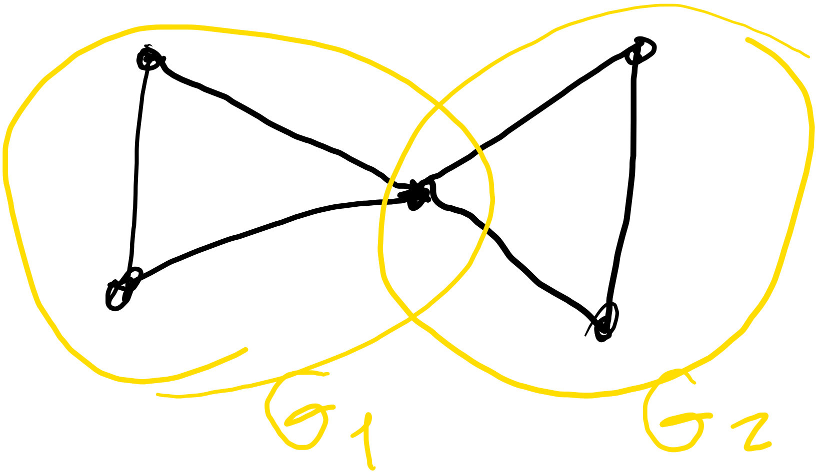
 $-$  , 

$k(k-1)^2$ $\qquad$ $k(k-1)^1$

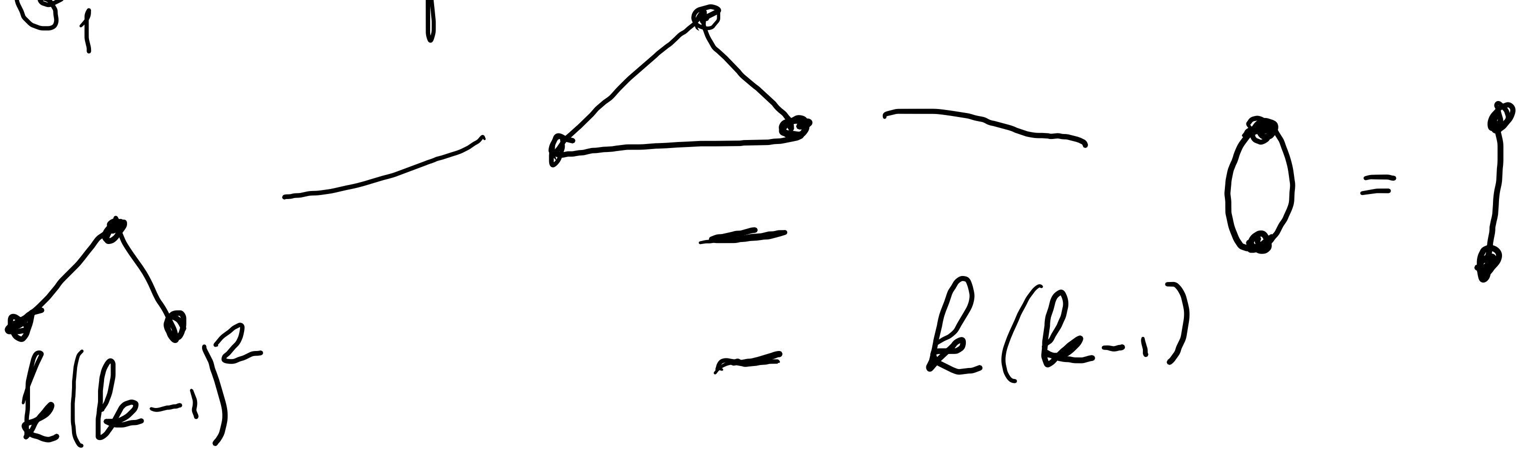
$$k^3-2k^2+k$$

$$-\ k^2+k$$

$$k^3-3k^2+2k$$



G_1 isomorphic to G_2, isomorphic to



$$
\begin{array}{r}
k^3 - 2k^2 + k \\
-\ k^2 + k \\
\hline
k^3 - 3k^2 + 2k
\end{array}
$$

$\pi_k(G_1) = \pi_k(G_2) =$

$$\pi_k(G) = \frac{\pi_k(G_1) \cdot \pi_k(G_2)}{k} = \frac{(k^3 - 3k^2 + 2k)^2}{k} =$$

$$= \frac{k^6 + 9k^4 + 4k^2 - 6k^5 + 4k^4 - 12k^3}{k} = \frac{k^6 - 6k^5 + 13k^4 - 12k^3 + 4k^2}{k} =$$

$$= k^5 - 6k^4 + 13k^3 - 12k^2 + 4k$$

A	B	$A \wedge B$	$(A \wedge B) \wedge B$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$A \wedge B \wedge B = A \wedge B$

A	B	C	$A \wedge B$	$A \wedge B \wedge C$	$(A \wedge B) \vee (A \wedge B \wedge C)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F