Partial Fraction Decomposition

Algebraic techniques for determining the constants in the numerators of partial fractions are demonstrated in the examples that follow. Note that the techniques vary slightly, depending on the type of factors of the denominator: linear or quadratic, distinct or repeated.

**Example 1** Distinct Linear Factors

Write the partial fraction decomposition of \( \frac{x + 7}{x^2 - x - 6} \).

**Solution**

The expression is proper, so be sure to factor the denominator. Because \( x^2 - x - 6 = (x - 3)(x + 2) \), you should include one partial fraction with a constant numerator for each linear factor of the denominator. Write the form of the decomposition as follows.

\[
\frac{x + 7}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2} \quad \text{Write form of decomposition.}
\]

Multiplying each side of this equation by the least common denominator, \((x - 3)(x + 2)\), leads to the basic equation

\[
x + 7 = A(x + 2) + B(x - 3). \quad \text{Basic equation}
\]

Because this equation is true for all \(x\), you can substitute any convenient values of \(x\) that will help determine the constants \(A\) and \(B\). Values of \(x\) that are especially convenient are ones that make the factors \((x + 2)\) and \((x - 3)\) equal to zero. For instance, let \(x = -2\). Then

\[
-2 + 7 = A(-2 + 2) + B(-2 - 3) \quad \text{Substitute } -2 \text{ for } x.
\]

\[
5 = A(0) + B(-5) \quad \text{Substitute } 0 \text{ for } x.
\]

\[
5 = -5B 
\]

\[
-1 = B. \quad \text{Solve for } B.
\]

To solve for \(A\), let \(x = 3\) and obtain

\[
3 + 7 = A(3 + 2) + B(3 - 3) \quad \text{Substitute } 3 \text{ for } x.
\]

\[
10 = A(5) + B(0) \quad \text{Substitute } 3 \text{ for } x.
\]

\[
10 = 5A, \quad 2 = A. \quad \text{Solve for } A.
\]

So, the partial fraction decomposition is

\[
\frac{x + 7}{x^2 - x - 6} = \frac{2}{x - 3} + \frac{-1}{x + 2}.
\]

Check this result by combining the two partial fractions on the right side of the equation, or by using your graphing utility.

**CHECK POINT** Now try Exercise 23.
The next example shows how to find the partial fraction decomposition of a rational expression whose denominator has a repeated linear factor.

**Example 2  Repeated Linear Factors**

Write the partial fraction decomposition of \(\frac{x^4 + 2x^3 + 6x^2 + 20x + 6}{x^3 + 2x^2 + x}\).

**Solution**

This rational expression is improper, so you should begin by dividing the numerator by the denominator to obtain

\[
x + \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}.
\]

Because the denominator of the remainder factors as

\[x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x + 1)^2\]

you should include one partial fraction with a constant numerator for each power of \(x\) and \(x + 1\) and write the form of the decomposition as follows.

\[
\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.
\]

Write form of decomposition.

Multiplying by the LCD, \(x(x + 1)^2\), leads to the basic equation

\[
5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx.
\]

Basic equation

Letting \(x = -1\) eliminates the \(A\) - and \(B\)-terms and yields

\[
5(-1)^2 + 20(-1) + 6 = A(-1 + 1)^2 + B(-1)(-1 + 1) + C(-1)
\]

\[
5 - 20 + 6 = 0 + 0 - C
\]

\[
C = 9.
\]

Letting \(x = 0\) eliminates the \(B\)- and \(C\)-terms and yields

\[
5(0)^2 + 20(0) + 6 = A(0 + 1)^2 + B(0)(0 + 1) + C(0)
\]

\[
6 = A(1) + 0 + 0
\]

\[
6 = A.
\]

At this point, you have exhausted the most convenient choices for \(x\), so to find the value of \(B\), use any other value for \(x\) along with the known values of \(A\) and \(C\). So, using \(x = 1\), \(A = 6\), and \(C = 9\),

\[
5(1)^2 + 20(1) + 6 = A(1 + 1)^2 + B(1)(1 + 1) + 9(1)
\]

\[
31 = 6(4) + 2B + 9
\]

\[
-2 = 2B
\]

\[
-1 = B.
\]

So, the partial fraction decomposition is

\[
\frac{x^4 + 2x^3 + 6x^2 + 20x + 6}{x^3 + 2x^2 + x} = x + \frac{6}{x} + \frac{-1}{x + 1} + \frac{9}{(x + 1)^2}.
\]

**CHECKPOINT**

Now try Exercise 49.
The procedure used to solve for the constants in Examples 1 and 2 works well when the factors of the denominator are linear. However, when the denominator contains irreducible quadratic factors, you should use a different procedure, which involves writing the right side of the basic equation in polynomial form and equating the coefficients of like terms. Then you can use a system of equations to solve for the coefficients.

### Example 3  Distinct Linear and Quadratic Factors

Write the partial fraction decomposition of

\[ \frac{3x^2 + 4x + 4}{x^3 + 4x} \].

#### Solution

This expression is proper, so factor the denominator. Because the denominator factors as

\[ x^3 + 4x = x(x^2 + 4) \]

you should include one partial fraction with a constant numerator and one partial fraction with a linear numerator and write the form of the decomposition as follows.

\[ \frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \]

Write form of decomposition.

Multiplying by the LCD, \(x(x^2 + 4)\), yields the basic equation

\[ 3x^2 + 4x + 4 = Ax(x^2 + 4) + (Bx + C)x \].

Basic equation

Expanding this basic equation and collecting like terms produces

\[ 3x^2 + 4x + 4 = Ax^3 + 4Ax + Bx^2 + Cx \]

\[ = (A + B)x^2 + Cx + 4A. \]

Polynomial form

Finally, because two polynomials are equal if and only if the coefficients of like terms are equal, you can equate the coefficients of like terms on opposite sides of the equation.

\[ 3x^2 + 4x + 4 = (A + B)x^2 + Cx + 4A \]

Equate coefficients of like terms.

You can now write the following system of linear equations.

\[
\begin{align*}
A + B &= 3 & \text{Equation 1} \\
C &= 4 & \text{Equation 2} \\
4A &= 4 & \text{Equation 3}
\end{align*}
\]

From this system you can see that \(A = 1\) and \(C = 4\). Moreover, substituting \(A = 1\) into Equation 1 yields

\[ 1 + B = 3 \implies B = 2. \]

So, the partial fraction decomposition is

\[ \frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{2x + 4}{x^2 + 4}. \]

**CHECK POINT** Now try Exercise 33.
The next example shows how to find the partial fraction decomposition of a rational expression whose denominator has a *repeated* quadratic factor.

### Example 4  Repeated Quadratic Factors

Write the partial fraction decomposition of \( \frac{8x^3 + 13x}{(x^2 + 2)^2} \).

#### Solution

Include one partial fraction with a linear numerator for each power of \((x^2 + 2)\).

\[
\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2} \quad \text{Write form of decomposition.}
\]

Multiplying by the LCD, \((x^2 + 2)^2\), yields the basic equation

\[
8x^3 + 13x = (Ax + B)(x^2 + 2) + Cx + D
\]

Basic equation

\[
= Ax^3 + 2Ax + Bx^2 + 2B + Cx + D
\]

Polynomial form

\[
= Ax^3 + Bx^2 + (2A + C)x + (2B + D).
\]

Equating coefficients of like terms on opposite sides of the equation

\[
8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A + C)x + (2B + D)
\]

produces the following system of linear equations.

\[
\begin{align*}
A &= 8 & \text{Equation 1} \\
B &= 0 & \text{Equation 2} \\
2A + C &= 13 & \text{Equation 3} \\
2B + D &= 0 & \text{Equation 4}
\end{align*}
\]

Finally, use the values \(A = 8\) and \(B = 0\) to obtain the following.

\[
2(8) + C = 13 \quad \text{Substitute 8 for } A \text{ in Equation 3.}
\]

\[
C = -3
\]

\[
2(0) + D = 0 \quad \text{Substitute 0 for } B \text{ in Equation 4.}
\]

\[
D = 0
\]

So, using \(A = 8, B = 0, C = -3,\) and \(D = 0\), the partial fraction decomposition is

\[
\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2}.
\]

Check this result by combining the two partial fractions on the right side of the equation, or by using your graphing utility.

**CHECK Point** Now try Exercise 55.
Guidelines for Solving the Basic Equation

**Linear Factors**

1. Substitute the *zeros* of the distinct linear factors into the basic equation.
2. For repeated linear factors, use the coefficients determined in Step 1 to rewrite the basic equation. Then substitute *other* convenient values of $x$ and solve for the remaining coefficients.

**Quadratic Factors**

1. Expand the basic equation.
2. Collect terms according to powers of $x$.
3. Equate the coefficients of like terms to obtain equations involving $A, B, C, \ldots$ and so on.
4. Use a system of linear equations to solve for $A, B, C, \ldots$.

Keep in mind that for *improper* rational expressions such as

\[
\frac{N(x)}{D(x)} = \frac{2x^3 + x^2 - 7x + 7}{x^3 + x - 2}
\]

you must first divide before applying partial fraction decomposition.

**CLASSROOM DISCUSSION**

**Error Analysis** You are tutoring a student in algebra. In trying to find a partial fraction decomposition, the student writes the following.

\[
\frac{x^2 + 1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1}
\]

\[
\frac{x^2 + 1}{x(x - 1)} = \frac{A(x - 1) + Bx}{x(x - 1)}
\]

Basic equation

By substituting $x = 0$ and $x = 1$ into the basic equation, the student concludes that $A = -1$ and $B = 2$. However, in checking this solution, the student obtains the following.

\[
\frac{-1}{x} + \frac{2}{x - 1} = \frac{(-1)(x - 1) + 2(x)}{x(x - 1)}
\]

\[
= \frac{x + 1}{x(x - 1)}
\]

\[
\neq \frac{x^2 + 1}{x(x - 1)}
\]

What is wrong?
7.4 EXERCISES

VOCABULARY: Fill in the blanks.
1. The process of writing a rational expression as the sum or difference of two or more simpler rational expressions is called ________ ________ ________.
2. If the degree of the numerator of a rational expression is greater than or equal to the degree of the denominator, then the fraction is called ________.
3. Each fraction on the right side of the equation \( \frac{x - 1}{x^2 - 8x + 15} = \frac{-1}{x - 3} + \frac{2}{x - 5} \) is a ________ ________.
4. The ________ ________ is obtained after multiplying each side of the partial fraction decomposition form by the least common denominator.

SKILLS AND APPLICATIONS

In Exercises 5–8, match the rational expression with the form of its decomposition. [The decompositions are labeled (a), (b), (c), and (d).]

(a) \( \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{x - 2} \)       (b) \( \frac{A}{x} + \frac{B}{x - 4} \)
(c) \( \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{x - 4} \)       (d) \( \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \)

5. \( \frac{3x - 1}{x(x - 4)} \)                        6. \( \frac{3x - 1}{x^2 - 4} \)
7. \( \frac{3x - 1}{x^2 + 4} \)                        8. \( \frac{3x - 1}{x^2 - 4} \)

In Exercises 9–18, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

9. \( \frac{3}{x^2 - 2x} \)                                10. \( \frac{x - 2}{x^2 + 4x + 3} \)
11. \( \frac{9}{x^2 - 7x^2} \)                              12. \( \frac{x^2 - 3x + 2}{4x^3 + 11x^2} \)
13. \( \frac{4x^2 + 3}{x^2 - 1} \)                        14. \( \frac{6x + 5}{(x + 2)^2} \)
15. \( \frac{2x - 3}{x^2 + 10x} \)                       16. \( \frac{x - 6}{2x^2 + 8x} \)
17. \( \frac{1}{x(x^2 + 1)^2} \)                          18. \( \frac{x + 4}{x^2(3x - 1)^2} \)

In Exercises 19–42, write the partial fraction decomposition of the rational expression. Check your result algebraically.

19. \( \frac{1}{x^2 + x} \)                                20. \( \frac{3}{x^2 - 3x} \)
21. \( \frac{1}{2x^2 + x} \)                              22. \( \frac{5}{x^2 + x - 6} \)
23. \( \frac{3}{x^2 + x - 2} \)                          24. \( \frac{x + 1}{x^2 - x - 6} \)
25. \( \frac{1}{x^2 - 1} \)                                26. \( \frac{1}{4x^2 - 9} \)
27. \( \frac{x^2 + 12x + 12}{x^3 - 4x} \)                28. \( \frac{x + 2}{x(x^2 - 9)} \)
29. \( \frac{3x}{(x - 3)^2} \)                            30. \( \frac{2x - 3}{(x - 1)^2} \)
31. \( \frac{4x^2 + 2x - 1}{x^2(x + 1)} \)                32. \( \frac{6x^2 + 1}{x^2(x - 1)^2} \)
33. \( \frac{x^2 + 2x + 3}{x^3 + x} \)                    34. \( \frac{2x}{x^3 - 1} \)
35. \( \frac{x}{x^3 - 3x^2 - 4x + 12} \)                  36. \( \frac{x + 6}{x^3 - 3x^2 - 4x} \)
37. \( \frac{2x^2 + x + 8}{(x^2 + 4)^2} \)                38. \( \frac{x^2}{x^4 - 2x^2 - 8} \)
39. \( \frac{x}{16x^4 - 1} \)                            40. \( \frac{3}{x^4 + x} \)
41. \( \frac{x^2 + 5}{(x + 1)(x^2 - 2x + 3)} \)          42. \( \frac{x^2 - 4x + 7}{(x + 1)(x^2 - 2x + 3)} \)

In Exercises 43–50, write the partial fraction decomposition of the improper rational expression.

43. \( \frac{x^2 - x}{x^2 + x} \)                          44. \( \frac{x^2 - 4x}{x^2 + x + 6} \)
45. \( \frac{2x^3 - x^2 + 5}{x^4 + 3x + 2} \)             46. \( \frac{x^3 + 2x^2 - x + 1}{x^2 + 3x - 4} \)
47. \( \frac{x^4}{(x - 1)^3} \)                            48. \( \frac{16x^4}{(2x - 1)^3} \)
49. \( \frac{x^4 + 2x^3 + 4x^2 + 8x + 2}{x^3 + 2x^2 + x} \) 50. \( \frac{2x^4 + 8x^3 + 7x^2 - 7x - 12}{x^3 + 4x^2 + 4x} \)
In Exercises 51–58, write the partial fraction decomposition of the rational expression. Use a graphing utility to check your result.

51. \( \frac{5 - x}{2x^2 + x - 1} \)  
52. \( \frac{3x^2 - 7x - 2}{x^3 - x} \)  
53. \( \frac{4x^2 - 1}{2x(x + 1)^2} \)  
54. \( \frac{3x + 1}{2x^3 + 3x^2} \)  
55. \( \frac{x^2 + x + 2}{(x^2 + 2)^2} \)  
56. \( \frac{x^3}{(x + 2)^2(x - 2)^2} \)  
57. \( \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} \)  
58. \( \frac{3x^3 - x + 3}{x^2 + x - 2} \)

**GRAPHICAL ANALYSIS** In Exercises 59 and 60, (a) write the partial fraction decomposition of the rational function, (b) identify the graph of the rational function and the graph of each term of its decomposition, and (c) state any relationship between the vertical asymptotes of the graph of the rational function and the vertical asymptotes of the graphs of the terms of the decomposition. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

59. \( y = \frac{x - 12}{x(x - 4)} \)  
60. \( y = \frac{2(4x - 3)}{x^2 - 9} \)

61. **ENVIRONMENT** The predicted cost \( C \) (in thousands of dollars) for a company to remove \( p\% \) of a chemical from its waste water is given by the model

\[
C = \frac{120p}{10,000 - p^2}, \quad 0 \leq p < 100.
\]

Write the partial fraction decomposition for the rational function. Verify your result by using the table feature of a graphing utility to create a table comparing the original function with the partial fractions.

62. **THERMODYNAMICS** The magnitude of the range \( R \) of exhaust temperatures (in degrees Fahrenheit) in an experimental diesel engine is approximated by the model

\[
R = \frac{5000(4 - 3x)}{(11 - 7x)(7 - 4x)}, \quad 0 < x \leq 1
\]

where \( x \) is the relative load (in foot-pounds).

(a) Write the partial fraction decomposition of the equation.

(b) The decomposition in part (a) is the difference of two fractions. The absolute values of the terms give the expected maximum and minimum temperatures of the exhaust gases for different loads.

\[
Y_{\text{max}} = |1\text{st term}| \quad Y_{\text{min}} = |2\text{nd term}|
\]

Write the equations for \( Y_{\text{max}} \) and \( Y_{\text{min}} \).

(c) Use a graphing utility to graph each equation from part (b) in the same viewing window.

(d) Determine the expected maximum and minimum temperatures for a relative load of 0.5.

**EXPLORATION**

**TRUE OR FALSE?** In Exercises 63–65, determine whether the statement is true or false. Justify your answer.

63. For the rational expression \( \frac{x}{(x + 10)(x - 10)^2} \), the partial fraction decomposition is of the form \( \frac{A}{x + 10} + \frac{B}{(x - 10)^2} \).

64. For the rational expression \( \frac{2x + 3}{x^2(x + 2)^2} \), the partial fraction decomposition is of the form \( \frac{Ax + B}{x^2} + \frac{Cx + D}{(x + 2)^2} \).

65. When writing the partial fraction decomposition of the expression \( \frac{x^3 + x - 2}{x^2 - 5x + 14} \), the first step is to divide the numerator by the denominator.

66. **CAPSTONE** Explain the similarities and differences in finding the partial fraction decompositions of proper rational expressions whose denominators factor into (a) distinct linear factors, (b) distinct quadratic factors, (c) repeated factors, and (d) linear and quadratic factors.

In Exercises 67–70, write the partial fraction decomposition of the rational expression. Check your result algebraically. Then assign a value to the constant \( a \) to check the result graphically.

67. \( \frac{1}{a^2 - x^2} \)  
68. \( \frac{1}{x(x + a)} \)  
69. \( \frac{1}{y(a - y)} \)  
70. \( \frac{1}{(x + 1)(a - x)} \)

71. **WRITING** Describe two ways of solving for the constants in a partial fraction decomposition.
What you should learn

• Sketch the graphs of inequalities in two variables.
• Solve systems of inequalities.
• Use systems of inequalities in two variables to model and solve real-life problems.

Why you should learn it

You can use systems of inequalities in two variables to model and solve real-life problems. For instance, in Exercise 83 on page 547, you will use a system of inequalities to analyze the retail sales of prescription drugs.

The Graph of an Inequality

The statements $3x - 2y < 6$ and $2x^2 + 3y^2 \geq 6$ are inequalities in two variables. An ordered pair $(a, b)$ is a solution of an inequality in $x$ and $y$ if the inequality is true when $a$ and $b$ are substituted for $x$ and $y$, respectively. The graph of an inequality is the collection of all solutions of the inequality. To sketch the graph of an inequality, begin by sketching the graph of the corresponding equation. The graph of the equation will normally separate the plane into two or more regions. In each such region, one of the following must be true.

1. All points in the region are solutions of the inequality.
2. No point in the region is a solution of the inequality.

So, you can determine whether the points in an entire region satisfy the inequality by simply testing one point in the region.

Example 1 Sketching the Graph of an Inequality

Sketch the graph of $y \geq x^2 - 1$.

Solution

Begin by graphing the corresponding equation $y = x^2 - 1$, which is a parabola, as shown in Figure 7.19. By testing a point above the parabola $(0, 0)$ and a point below the parabola $(0, -2)$, you can see that the points that satisfy the inequality are those lying above (or on) the parabola.

Be careful when you are sketching the graph of an inequality in two variables. A dashed line means that all points on the line or curve are not solutions of the inequality. A solid line means that all points on the line or curve are solutions of the inequality.

Now try Exercise 7.
The inequality in Example 1 is a nonlinear inequality in two variables. Most of the following examples involve linear inequalities such as \( ax + by < c \) (\( a \) and \( b \) are not both zero). The graph of a linear inequality is a half-plane lying on one side of the line \( ax + by = c \).

## Example 2 Sketching the Graph of a Linear Inequality

Sketch the graph of each linear inequality.

a. \( x > -2 \)  

b. \( y \leq 3 \)

### Solution

a. The graph of the corresponding equation \( x = -2 \) is a vertical line. The points that satisfy the inequality \( x > -2 \) are those lying to the right of this line, as shown in Figure 7.20.

b. The graph of the corresponding equation \( y = 3 \) is a horizontal line. The points that satisfy the inequality \( y \leq 3 \) are those lying below (or on) this line, as shown in Figure 7.21.

### TECHNOLOGY

A graphing utility can be used to graph an inequality or a system of inequalities. For instance, to graph \( y = x - 2 \), enter \( y = x - 2 \) and use the shade feature of the graphing utility to shade the correct part of the graph. You should obtain the graph below. Consult the user’s guide for your graphing utility for specific keystrokes.

![Graph of \( y = x - 2 \)](image)

### CHECK Point

Now try Exercise 9.

## Example 3 Sketching the Graph of a Linear Inequality

Sketch the graph of \( x - y < 2 \).

### Solution

The graph of the corresponding equation \( x - y = 2 \) is a line, as shown in Figure 7.22. Because the origin \((0, 0)\) satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line. Regardless of which point you choose, you will see that it does not satisfy the inequality.)

### TECHNOLOGY

A graphing utility can be used to graph an inequality or a system of inequalities. For instance, to graph enter \( x - y < 2 \) and use the shade feature of the graphing utility to shade the correct part of the graph. You should obtain the graph below. Consult the user’s guide for your graphing utility for specific keystrokes.

![Graph of \( x - y < 2 \)](image)

### CHECK Point

Now try Exercise 15.

To graph a linear inequality, it can help to write the inequality in slope-intercept form. For instance, by writing \( x - y < 2 \) in the form

\[ y > x - 2 \]

you can see that the solution points lie above the line \( x - y = 2 \) (or \( y = x - 2 \)), as shown in Figure 7.22.
Systems of Inequalities

Many practical problems in business, science, and engineering involve systems of linear inequalities. A solution of a system of inequalities in \( x \) and \( y \) is a point \((x, y)\) that satisfies each inequality in the system.

To sketch the graph of a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is common to every graph in the system. This region represents the solution set of the system. For systems of linear inequalities, it is helpful to find the vertices of the solution region.

Example 4  
Solving a System of Inequalities

Sketch the graph (and label the vertices) of the solution set of the system.

\[
\begin{align*}
\text{Inequality 1} & : & x - y & < 2 \\
\text{Inequality 2} & : & x & > -2 \\
\text{Inequality 3} & : & y & \leq 3
\end{align*}
\]

Solution

The graphs of these inequalities are shown in Figures 7.22, 7.20, and 7.21, respectively, on page 539. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate system, as shown in Figure 7.23. To find the vertices of the region, solve the three systems of corresponding equations obtained by taking pairs of equations representing the boundaries of the individual regions.

**Study Tip**

Using different colored pencils to shade the solution of each inequality in a system will make identifying the solution of the system of inequalities easier.

**Example 4**

Solving a System of Inequalities

Sketch the graph (and label the vertices) of the solution set of the system.

\[
\begin{align*}
\text{Inequality 1} & : & x - y & < 2 \\
\text{Inequality 2} & : & x & > -2 \\
\text{Inequality 3} & : & y & \leq 3
\end{align*}
\]

**Solution**

The graphs of these inequalities are shown in Figures 7.22, 7.20, and 7.21, respectively, on page 539. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate system, as shown in Figure 7.23. To find the vertices of the region, solve the three systems of corresponding equations obtained by taking pairs of equations representing the boundaries of the individual regions.

Vertex A: \((-2, -4)\)  
Vertex B: \((5, 3)\)  
Vertex C: \((-2, 3)\)

Note in Figure 7.23 that the vertices of the region are represented by open dots. This means that the vertices are not solutions of the system of inequalities.

**CHECK Point**  
Now try Exercise 41.
For the triangular region shown in Figure 7.23, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown in Figure 7.24. To keep track of which points of intersection are actually vertices of the region, you should sketch the region and refer to your sketch as you find each point of intersection.

**Example 5  Solving a System of Inequalities**

Sketch the region containing all points that satisfy the system of inequalities.

\[
\begin{align*}
  x^2 - y &\leq 1 & \text{Inequality 1} \\
  -x + y &\leq 1 & \text{Inequality 2}
\end{align*}
\]

**Solution**

As shown in Figure 7.25, the points that satisfy the inequality

\[ x^2 - y \leq 1 \]

are the points lying above (or on) the parabola given by

\[ y = x^2 - 1. \]  

Parabola

The points satisfying the inequality

\[ -x + y \leq 1 \]

are the points lying below (or on) the line given by

\[ y = x + 1. \]  

Line

To find the points of intersection of the parabola and the line, solve the system of corresponding equations.

\[
\begin{align*}
  x^2 - y &= 1 \\
  -x + y &= 1
\end{align*}
\]

Using the method of substitution, you can find the solutions to be \((-1, 0)\) and \((2, 3)\).

So, the region containing all points that satisfy the system is indicated by the shaded region in Figure 7.25.

**CHECK Point**  Now try Exercise 43.
When solving a system of inequalities, you should be aware that the system might have no solution or it might be represented by an unbounded region in the plane. These two possibilities are shown in Examples 6 and 7.

### Example 6  A System with No Solution

Sketch the solution set of the system of inequalities.

\[
\begin{align*}
  x + y &> 3 & \text{Inequality 1} \\
  x + y &< -1 & \text{Inequality 2}
\end{align*}
\]

**Solution**

From the way the system is written, it is clear that the system has no solution, because the quantity \(x + y\) cannot be both less than \(-1\) and greater than \(3\). Graphically, the inequality \(x + y > 3\) is represented by the half-plane lying above the line \(x + y = 3\), and the inequality \(x + y < -1\) is represented by the half-plane lying below the line \(x + y = -1\), as shown in Figure 7.26. These two half-planes have no points in common. So, the system of inequalities has no solution.

![Figure 7.26](image)

**Check Point** Now try Exercise 45.

### Example 7  An Unbounded Solution Set

Sketch the solution set of the system of inequalities.

\[
\begin{align*}
  x + y &< 3 & \text{Inequality 1} \\
  x + 2y &> 3 & \text{Inequality 2}
\end{align*}
\]

**Solution**

The graph of the inequality \(x + y < 3\) is the half-plane that lies below the line \(x + y = 3\), as shown in Figure 7.27. The graph of the inequality \(x + 2y > 3\) is the half-plane that lies above the line \(x + 2y = 3\). The intersection of these two half-planes is an **infinite wedge** that has a vertex at \((3, 0)\). So, the solution set of the system of inequalities is unbounded.

**Check Point** Now try Exercise 47.
Applications

Example 9 in Section 7.2 discussed the *equilibrium point* for a system of demand and supply equations. The next example discusses two related concepts that economists call *consumer surplus* and *producer surplus*. As shown in Figure 7.28, the consumer surplus is defined as the area of the region that lies **below** the demand curve, **above** the horizontal line passing through the equilibrium point, and to the right of the $p$-axis. Similarly, the producer surplus is defined as the area of the region that lies **above** the supply curve, **below** the horizontal line passing through the equilibrium point, and to the right of the $p$-axis. The consumer surplus is a measure of the amount that consumers would have been willing to pay **above** what they **actually paid**, whereas the producer surplus is a measure of the amount that producers would have been willing to receive **below** what they **actually received**.

**Example 8  Consumer Surplus and Producer Surplus**

The demand and supply equations for a new type of personal digital assistant are given by

\[
\begin{align*}
 p &= 150 - 0.00001x & \text{Demand equation} \\
 p &= 60 + 0.00002x & \text{Supply equation}
\end{align*}
\]

where $p$ is the price (in dollars) and $x$ represents the number of units. Find the consumer surplus and producer surplus for these two equations.

**Solution**

Begin by finding the equilibrium point (when supply and demand are equal) by solving the equation

\[60 + 0.00002x = 150 - 0.00001x,\]

In Example 9 in Section 7.2, you saw that the solution is $x = 3,000,000$ units, which corresponds to an equilibrium price of $p = 120$. So, the consumer surplus and producer surplus are the areas of the following triangular regions.

\[
\begin{align*}
 \text{Consumer Surplus} & \quad \text{Producer Surplus} \\
 p & \leq 150 - 0.00001x & p & \geq 60 + 0.00002x \\
 p & \geq 120 & p & \leq 120 \\
 x & \geq 0 & x & \geq 0
\end{align*}
\]

In Figure 7.29, you can see that the consumer and producer surpluses are defined as the areas of the shaded triangles.

\[
\begin{align*}
 \text{Consumer surplus} &= \frac{1}{2}(\text{base})(\text{height}) \\
 &= \frac{1}{2}(3,000,000)(30) = 45,000,000 \\
 \text{Producer surplus} &= \frac{1}{2}(\text{base})(\text{height}) \\
 &= \frac{1}{2}(3,000,000)(60) = 90,000,000
\end{align*}
\]

**CHECK**  Now try Exercise 71.
**Example 9  Nutrition**

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up a system of linear inequalities that describes how many cups of each drink should be consumed each day to meet or exceed the minimum daily requirements for calories and vitamins.

**Solution**

Begin by letting \( x \) and \( y \) represent the following.

\[
x = \text{number of cups of dietary drink X} \\
y = \text{number of cups of dietary drink Y}
\]

To meet or exceed the minimum daily requirements, the following inequalities must be satisfied.

\[
\begin{align*}
60x + 60y & \geq 300 & \text{Calories} \\
12x + 6y & \geq 36 & \text{Vitamin A} \\
10x + 30y & \geq 90 & \text{Vitamin C} \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

The last two inequalities are included because \( x \) and \( y \) cannot be negative. The graph of this system of inequalities is shown in Figure 7.30. (More is said about this application in Example 6 in Section 7.6.)

**Classroom Discussion**

Creating a System of Inequalities  Plot the points \((0, 0), (4, 0), (3, 2),\) and \((0, 2)\) in a coordinate plane. Draw the quadrilateral that has these four points as its vertices. Write a system of linear inequalities that has the quadrilateral as its solution. Explain how you found the system of inequalities.
VOCABULARY: Fill in the blanks.

1. An ordered pair \((a, b)\) is a ________ of an inequality in \(x\) and \(y\) if the inequality is true when \(a\) and \(b\) are substituted for \(x\) and \(y\), respectively.

2. The ________ of an inequality is the collection of all solutions of the inequality.

3. The graph of a ________ inequality is a half-plane lying on one side of the line \(ax + by = c\).

4. A ________ of a system of inequalities in \(x\) and \(y\) is a point \((x, y)\) that satisfies each inequality in the system.

5. A ________ ________ of a system of inequalities in two variables is the region common to the graphs of every inequality in the system.

6. The area of the region that lies below the demand curve, above the horizontal line passing through the equilibrium point, to the right of the \(p\)-axis is called the ________ ________.

SKILLS AND APPLICATIONS

In Exercises 7–20, sketch the graph of the inequality.

7. \(y < 5 - x^2\)
8. \(y^2 - x < 0\)
9. \(x \geq 6\)
10. \(x < -4\)
11. \(y > -7\)
12. \(10 \geq y\)
13. \(y < 2 - x\)
14. \(y > 4x - 3\)
15. \(2y - x \geq 4\)
16. \(5x + 3y \geq -15\)
17. \((x + 1)^2 + (y - 2)^2 < 9\)
18. \((x - 1)^2 + (y - 4)^2 > 9\)
19. \(y \leq \frac{1}{1 + x^2}\)
20. \(y > \frac{-15}{x^2 + x + 4}\)

In Exercises 21–32, use a graphing utility to graph the inequality.

21. \(y < \ln x\)
22. \(y \geq 2 - \ln(x + 3)\)
23. \(y < 4^{x-5}\)
24. \(y \leq 2^{2x-0.5} - 7\)
25. \(y \geq \frac{5}{6}x - 2\)
26. \(y \leq 6 - \frac{1}{3}x\)
27. \(y < -3.8x + 1.1\)
28. \(y \geq 20.74 + 2.66x\)
29. \(x^2 + 5y - 10 \leq 0\)
30. \(2x^2 - y - 3 > 0\)
31. \(\frac{5}{12}y - 3x^2 - 6 \geq 0\)
32. \(-\frac{1}{3}x^2 - \frac{3}{8}y < \frac{-1}{4}\)

In Exercises 33–36, write an inequality for the shaded region shown in the figure.

33. \(y\)
34. \(y\)
35. \(y\)
36. \(y\)

In Exercises 37–40, determine whether each ordered pair is a solution of the system of linear inequalities.

37. \[
\begin{align*}
   x &\geq -4 \\
   y &< -3 \\
   y &\leq -8x - 3 \\
   -4x + 2y &< 7
\end{align*}
\]
(a) (0, 0)  (b) (−1, −3)  (c) (−4, 0)  (d) (−3, 11)

38. \[
\begin{align*}
   -2x + 5y &\geq 3 \\
   y &< 4 \\
   -4x + 2y &< 7
\end{align*}
\]
(a) (0, 2)  (b) (−6, 4)  (c) (−8, −2)  (d) (−3, 2)

39. \[
\begin{align*}
   3x + y &> 1 \\
   y &\leq -\frac{3}{2}x^2 - 4 \\
   -15x + 4y &> 0
\end{align*}
\]
(a) (0, 10)  (b) (0, −1)  (c) (2, 9)  (d) (−1, 6)

40. \[
\begin{align*}
   x^2 + y^2 &\geq 36 \\
   -3x + y &\leq 10 \\
   \frac{2}{3}x - y &\geq 5
\end{align*}
\]
(a) (−1, 7)  (b) (−5, 1)  (c) (6, 0)  (d) (4, −8)

In Exercises 41–54, sketch the graph and label the vertices of the solution set of the system of inequalities.

41. \[
\begin{align*}
   x + y &\leq 1 \\
   -x + y &\leq 1 \\
   y &\geq 0
\end{align*}
\]
(a) (0, 0)  (b) (2, 0)  (c) (0, 1)  (d) (1, 0)

42. \[
\begin{align*}
   3x + 4y &< 12 \\
   x &> 0 \\
   y &> 0
\end{align*}
\]
(a) (3, 0)  (b) (0, 3)  (c) (0, 0)  (d) (4, −1)

43. \[
\begin{align*}
   x^2 + y &\leq 7 \\
   x &\geq -2 \\
   y &\geq 0
\end{align*}
\]
(a) (−2, 0)  (b) (2, 0)  (c) (2, 1)  (d) (−1, 1)

44. \[
\begin{align*}
   4x^2 + y &\geq 2 \\
   x &\leq 1 \\
   y &\leq 1
\end{align*}
\]
(a) (1, 2)  (b) (−1, 1)  (c) (1, 0)  (d) (0, 1)