Does intelligence imply contradiction?

Action editor: Vasant Honavar

P. Frosini *

Department of Mathematics and ARCES, University of Bologna, I-40126 Bologna, Italy

Abstract

Contradiction is often seen as a defect of intelligent systems and a dangerous limitation on efficiency. In this paper we raise the question of whether, on the contrary, it could be considered a key tool in increasing intelligence in biological structures. A possible way of answering this question in a mathematical context is shown, formulating a proposition that suggests a link between intelligence and contradiction.

A concrete approach is presented in the well-defined setting of cellular automata. Here we define the models of "observer", "entity", "environment", "intelligence," and "contradiction". These definitions, which roughly correspond to the common meaning of these words, allow us to deduce a simple but strong result about these concepts in an unbiased, mathematical manner.

Evidence for a real-world counterpart to the demonstrated formal link between intelligence and contradiction is provided by three computational experiments.
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Keywords: Intelligence; Contradiction; Cellular automaton

1. Introduction

In this paper we are going to examine the relationship between intelligence and contradiction, hopefully clarifying the presence and importance of inconsistency in thought and in the processes trying to emulate it. To arrive at our objective, we shall need to put the concepts of "observer", "entity," and "environment" on a mathematical footing, so that formal definitions of intelligence and contradiction can be proposed.

This model will allow us to treat our controversial subject precisely, illustrating the possibility of a quantitative mathematical approach to the problem, and its intrinsic advantages.

2. Background: contradiction in science, mathematics, philosophy

Contradiction is undoubtedly one of the most interesting concepts in science. It has been studied since ancient times and it would be impossible to take into account all the literature on this subject, whether from a logical, philosophical, or psychological viewpoint (Piaget, 1974). Many scholars, from the Greek philosophers onwards, have studied contradiction often regarding it as a key presence in human thought processes. On the other hand, mathematical research views contradiction as incompatible with any workable theory and has studied inconsistency almost exclusively in terms of the danger it represents to formal structures. Even after Gödel published his famous "Second Theorem", mathematicians continued to consider contradiction simply as a nuisance to be eliminated. The situation did not change after the work of mathematicians such as Church, Kleene, Rosser, and Turing on the limitations of logical systems and computational machines, demonstrating

* Tel.: +39 051 2094478; fax: +39 051 2094490.
  E-mail address: frosini@dm.unibo.it

the weakness of a naïve approach to the concept of "mathematical truth". On this subject we also refer to De Long (1970), Lucas (1964), and Webb (1980). Paraconsistent logics (i.e., logics where not every statement follows from a contradiction) were created in order to constrain the presence of inconsistencies (cf. Anderson et al., 1975 for relevant logics, Jaskowski, 1948 for non-adjunctive systems, da Costa, 1974 for non-truth-functional logics, and Dunn, 1976 for many-valued systems).

Paradoxically, mathematics has almost always considered the problem of inconsistency of thought as either taboo or an irrelevant subject. In this sense there is an enormous difference between the research of mathematicians and that of philosophers. An interesting attempt to close the gap between the studies carried out in these two fields was made at the beginning of the previous century by the Russian mathematician, philosopher and theologian Florenskij (1914). We mention this work also because it contains a fascinating survey of the concept of contradiction. From an epistemological point of view, an interesting debate about this and other problems concerning mathematics has recently been raised by the mathematician and philosopher G.C. Rota (cf., e.g., Rota, 1997). Another key reference is the work done by G. Priest, concerning the relationship between contradiction and mathematical logic (cf., e.g., Priest, 2006).

Psychology and economics are also involved in research on contradiction. The concepts of *inconsistency between attitudes or behaviors (cognitive dissonance)* (cf. Festinger, 1957) and *time-inconsistent agent* (cf., e.g., Brocas & Carrillo, 2000; Strotz, 1956) are generally studied in these fields. However, it should be noted that the term "inconsistent" is often used in a precise or technical sense, depending on the particular scientific context.

We shall not make any attempt to review the extensive bibliography of the psychological, economical, philosophical and epistemological approaches to contradiction, since this would extend this paper far beyond our limited purposes.

Informally, we could define contradiction as the phenomenon in which a given entity evolves in two different ways (at different times) from the same initial state. In Section 4.4 we shall justify this definition by comparing it with alternative definitions. A typical example of this phenomenon could be that of a person who answers differently to the same question at different times. We could object that, in a deterministic and mechanical paradigm, those different answers simply reveal either different states of mind or a difference between the questions, but this objection is misleading. In fact, if we look at phenomena as events perceived by an observer, it does not make sense to consider differences that are not perceived by the observer. To us, the expression "same initial state" simply means that the observer does not perceive changes in the pair (entity, environment) that he/she is observing. Hence, we see contradiction as a concept that intrinsically depends on an observer. In fact, even the classical approach given by Turing (1950)

to the problem of testing intelligence suggests the key role of the observer as judge. In any case, a mathematical attempt to formalize the notions of intelligence and contradiction probably cannot avoid reference to the concept of observer, since such formalization cannot avoid involving a testing procedure, which requires the presence of an observer (possibly neither human nor intelligent). This does *not* imply that an entity cannot observe itself; indeed, an entity can perfectly well study the "intelligence" of another entity, and an agency inside a given entity can study the "intelligence" of other agencies inside the same entity (or even of itself!) (we refer to Minsky, 1986; Rychlak, 1991; Wooldridge & Jennings, 1995) for the (concept of agency). The role of the observer in judging intelligence has been studied by many researchers (cf., e.g., Goodnow, 1969; Jones & Nisbett, 1971). An important reference to the central role of the observer is contained in the fundamental work of Maturana and Varela (1992).

Experience shows us that contradictions are very common in the behavior of living beings and other complex systems. Thus, when a complex system is constructed, much effort usually goes into guaranteeing consistency in the defined structure. Mathematicians seem to be particularly disturbed by contradiction, although it is a vital part of reality. In the past, the existence of contradiction was studied as a formal and philosophical problem, but was ignored by mathematics and computer science. Nowadays the situation is quite different. Interest in artificial intelligence compels us to look at the occurrence of contradiction as a practical problem. As Minsky and others have pointed out, reasonable models of intelligence suppose the presence of internal conflicts that must be solved in order to make unambiguous decisions (cf., e.g., Minsky, 1986; Rich, 1983; Winston, 1984). Moreover, it is clear that an intelligent entity must be able to manage contradiction (as happens, for instance, in artificial vision when two different interpretations of an image conflict with each other), and people working in artificial intelligence are well aware that conflicts cannot be separated from decisions. In other words, an intelligent entity must be able to solve internal conflicts and change its vision of the world (see, e.g., Dennett, 1978). Furthermore, a significant proportion of software development and research is spent in detecting, analyzing and handling inconsistency in development processes and products (cf. Ghezzi & Nuseibeh, 1998) and there is a considerable amount of literature on this subject. We also refer to Wooldridge and Jennings (1995) for a discussion of the problem of inconsistency in agent theory.

In any cases the concept of contradiction is much more than just an inevitable practical problem, and even in software engineering many researchers have begun to accept inconsistencies not only as problems to solve but also as a reality to live with (cf., e.g., Balzer, 1991), and some have developed a body of research that seeks to "make inconsistency respectable" (cf. Gabbay & Hunter, 1990). It is also interesting to point out the presence of contradictions in

the behavior of Search Engines for the World Wide Web (cf. Bar-Ilan, 1998/99).

Besides this, a contradictory action frequently reveals itself to be a valuable quality allowing entities to survive changes in their world. The unusual behavior of a cell caused by genetic mutation can be seen as a sort of contradiction in the way we have previously described it, as long as the mutation (i.e., the cause of a change in behavior) is not perceptible. Hence contradiction can be seen as a virtue rather than as a defect. Furthermore, the constant presence of inconsistencies in our thoughts leads us to the following natural question: is contradiction accidental or is it the necessary companion of intelligence? As we pointed out previously, this question is no longer only important from a philosophical point of view, since any attempt to construct artificial entities capable of intelligent behavior demands an answer to this question.

The sole aim of this paper is to place this question in a mathematical framework and to propose a formal line of attack. In order to do this we have chosen to use the concept of cellular automaton (a structure invented by von Neumann (1966) to study the phenomenon of self-replication), since it combines simplicity of definition with the capability of simulating complex systems.

In our model, we obtain a result suggesting a strong link between contradiction and intelligence. Roughly speaking, our finding can be expressed in this way:

Any sufficiently intelligent entity must be contradictory.

Obviously, this result depends on some hypotheses that some readers may not agree with, and so our answer is far from being absolute: it is given only to point out a possible approach to the study of inconsistency in complex systems.

However, this result is not counter-intuitive, even in a deterministic world: in plain words it can be explained in the following way. Intelligence can be seen as the capability of an entity to survive changes in the environment by adapting to new conditions. If both the changes in the environment and the adaptation of the entity are sufficiently clear to an observer who is examining what is going on, then their presence can be perceived and there is no contradiction (since the different behavior is justified by the changes in the entity and the environment). On the other hand, if the changes in the environment and the adaptation of the entity become too complex and subtle for the observer to see the differences in all these data, then the behavior of the entity may begin to be seen as contradictory, since the observer cannot perceive the differences causing this change in behavior. Therefore, the entity may become contradictory for the observer when the intelligence of the entity produces behavior that is too complex for that particular observer. As an example, when someone changes his/her mind about something, we usually consider him/her to be contradictory if we cannot understand the details of the mental process prompting him/her to make this change in opinion. On the other hand, if we are able to understand the changes producing this different behavior ("the reasons of the change in opinion"), then no contradiction is perceived.

We shall devote this paper to formalizing this idea in a mathematical context.

3. Some notes about our epistemological approach

The subject we are going to treat is quite controversial. Terms such as "intelligence" and "contradiction" allow for so many different interpretations that our first aim is to clarify the epistemological setting we wish to use. The reader will find that all the ideas in this paper will be discussed from a point of view that stresses the belief that intelligence cannot be studied independently from the context in which it develops and is observed. This belief is based on the consideration that an analysis of phenomena where feedback between the observed object and the observer is not negligible requires this retroaction to be carefully examined. Intelligence is a typical case of this feedback, since the attempt to study the intelligence of an entity cannot ignore the possibility (the fact) that the entity tries to influence (influences) the observer. Studying intelligence independently from the context is effectively a contradiction in terms. While we are aware that our choice biases this whole paper, we wish to stress the framework we are using and to bring out some explicit links with some well-known lines of thought.

As previously explained, we believe that intelligence and contradiction are phenomena concerning the relation between an entity and an observer within an environment. As a consequence, we think it is nonsense to speak about intelligence and contradiction as concepts independent from the context, i.e., the experimental setting where intelligence is studied. The definitions of "entity", "environment", "intelligence," and "contradiction" considered in the following sections must always be taken with reference to a given observer and not as an absolute. This means that we should really say *entity detection*, *environment detection*, *intelligence detection*, and *contradiction detection made by a given observer* (in the same way as, in quantum mechanics, the concept of "particle" is replaced with that of "observation of a particle"). From this point of view we must not assume that our opinion about a phenomenon (e.g., the presence of an entity at a given place) is the one accepted by the considered observer: once it has been chosen, we cannot superimpose our own personal judgment on the one it expresses. Nevertheless we shall maintain the use of the words "entity", "intelligence," and "contradiction" for the sake of concision. In any cases the real and relative meanings of these terms will always have to be carefully recalled.

Some aspects of our approach are surely not new. The reader will find many links to ideas previously expressed by other researchers. The treatment of concepts such as entity and dependence on the observer is certainly related to the work by Maturana and Varela (1992). The hypothesis that intelligence is situated in the world, not

in disembodied systems such as theorem provers or expert systems, can be found in behaviorism and, in particular, in Brooks' research (cf., e.g., Brooks, 1991). The same can be said about the idea that intelligent behavior arises as a result of an agent's interaction with its environment, and that intelligence is ''in the eye of the beholder'' and is not an innate, isolated property. The interdisciplinary methodology, together with the use of mathematical concepts and the assertion that any experience is subjective, may remind us of the approach by Bateson (1972, 1979). The importance given to a global point of view and the impossibility of splitting up the problem of perception into independent components derives from *Gestaltpsychologie* (cf., e.g., Katz, 1944).

A common base to these approaches might be found in the use of a phenomenological and global framework. In other words, our attitude of mind is to think of intelligence as an emergent property that cannot be studied at a unique level. In particular, we look at perception and comprehension as intrinsically related processes (cf. Hofstadter, 1995), and assume that intelligence cannot be examined without reference to the act of perceiving. All of our research is centered on the hypothesis that any attempt to study intelligence and contradiction cannot ignore this phenomenological and global point of view, which is strongly dependent on the choice of an observer. This implies that our attention is not given to isolated entities, but to relations between observers and entities acting within an environment.

Like any other epistemological framework, the framework we are going to use in this paper can be criticized or even rejected. While we shall motivate any choices we make, the reader should consider our research within the setting we have described.

Note 1. In Section 4 we shall give formal definitions of the concepts we have mentioned in this section. We shall proceed by setting out some hypotheses in our model, in order to emulate some properties of the real world: for the sake of clarity we shall first informally describe each property we wish to emulate, and then we shall give its counterpart in the formal mathematical language of cellular automata. In Section 5 we shall obtain the above mentioned result concerning the connection between contradiction and intelligence. In Section 6 we shall present the results of three computational experiments supporting the line of thought expressed in this paper. In Section 7 some controversial points and our corresponding answers will be presented.

4. A way of formalizing the problem

4.1. A cellular automaton as a ''world'' in which we can study entities

The first thing we need is a mathematical structure through which we can try to give an acceptable formaliza-

tion of such concepts as *entity*, *environment*, *intelligence*, and *contradiction*. Obviously, we are not interested in all the phenomena involving such complex concepts, but only in constructing a simple model to preserve some key facts of a real case. Cellular automata are good candidates for this, since many authors have shown their usefulness in representing many complex phenomena. In particular, they have proved capable of emulating many physical and biological systems. The literature available on this subject is considerable and we refer to the bibliography in Wolfram (1994) for many references. Furthermore, it is well known that many cellular automata have the property of universal computation – that is, they can emulate every Turing machine. Therefore, any computation that can be achieved by a Turing machine can be performed by many cellular automata, too. For example, it is well known that Conway's famous cellular automaton *Life* (cf., e.g., Wolfram, 1994) has this property. So, in principle, all algorithms we can implement on a computer can also be implemented in *Life*. Obviously, this implementation would not be practical and would take a great deal of space and time for execution, but this is a common problem for Turing machines and here we are interested only in a theoretical approach. Moreover, in spite of their huge theoretical capabilities, cellular automata have the advantage of being very easily defined.

Some people may think that such a simple structure cannot emulate or reproduce intelligence. In particular, some may simply maintain that a Turing machine *cannot* have intelligence, for various reasons (cf. Searle, 1984). We do not want to enter into this debate, but we stress that most of the tools available for developing artificial intelligence (including discrete neural networks) can be emulated by a Turing machine, so that everything we use at the moment to study intelligence from a discrete-mathematical point of view can be reduced *in principle* to the functioning of a cellular automaton. Therefore, it is reasonable to choose a cellular automaton as a model for our proposals.

In this paper we shall often refer to an instance $\mathcal{C}^*$ of the well-known cellular automaton *Life* (see Fig. 1), showing a moving structure commonly known as a *glider*. This simple structure allows for the construction of the logical gates AND, OR, NOT, and on the basis of this, it has been proven that *Life* can emulate every Turing machine (cf., e.g., Wolfram, 1994). We have chosen this example both because of its simplicity and because of its relevance to the theory of cellular automata. Obviously, it is hard to view this as a model of a world populated by structures endowed with intelligence, but probably such an interesting model would require a cellular automaton with a huge number of non-zero cells, so our toy example is a more economical way of making our definitions clear.

In any case we shall justify our choice of these definitions by showing their appropriateness to the real world. In order to do so, we shall use a more complex (but still simple) example that is not explicitly implemented in a cellular automaton, since it would be too large. However, this
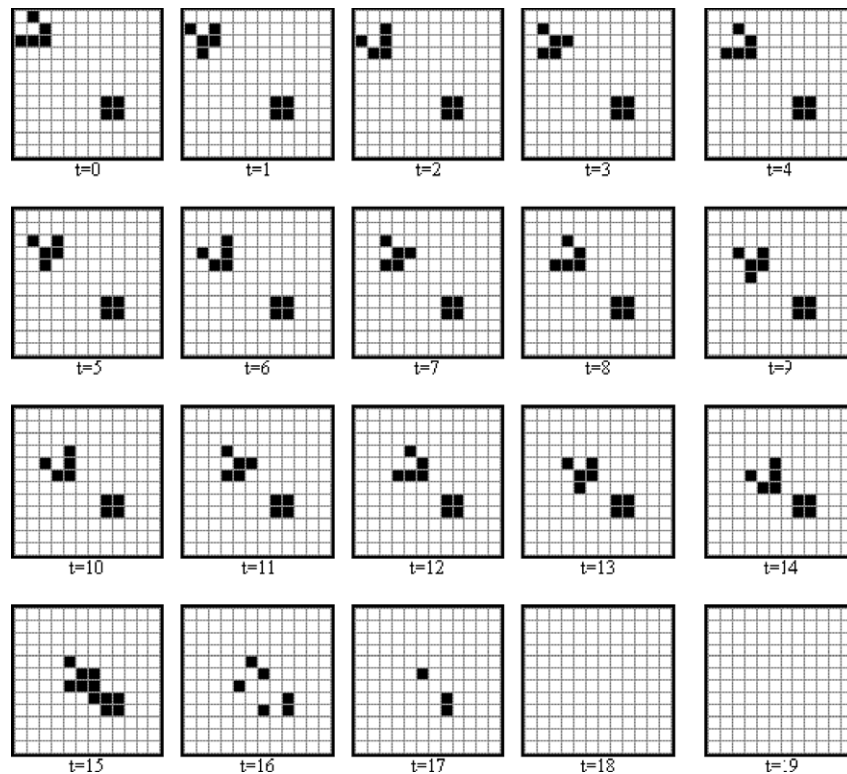
Fig. 1. The first 20 consecutive states $s_0, s_1, \ldots, s_{19}$ in the evolution of the cellular automaton $\mathcal{C}^*$, showing a "glider" crashing against a "block".

implementation is possible in principle, because of the properties previously mentioned. We proceed analogously when we informally speak about an algorithmic procedure without explicitly and formally giving a complete definition of the Turing machine simulating the procedure. Since we shall refer to this latter example throughout this paper, we begin from its description:

Example 1. ("*FIGHT*"). At the time of this writing, contests between virtual robots are increasingly common all over the World Wide Web. In general, this type of game is given by an implementation on a server of a fight between programs emulating virtual robots. The task of each virtual robot is to destroy its opponents using a set of permitted actions inside a given virtual arena. Each program usually runs on the same server, and a specific routine (the referee) examines the state of the fight in real time. To summarize, we have a program containing various subroutines representing the various virtual robots, and another subroutine implementing the referee. In this kind of game it is easy to identify the concepts of "world", "environment" and "entity": the world is the program implementing the arena with the fighting robots and the judging referee, the part of the arena external to the considered entity may be interpreted as the environment, while each entity is represented by a virtual robot. We point out that no human observer usually watches the game, and that all "perceptions" and "judgments" belongs to the referee. For each time step, the referee identifies the virtual robots, their positions in the space, and their states

(dead or alive), so that it can decide the result of the competition. Observe that, in this particular game, the referee is not affected by the destructive actions of the virtual robots, but we can easily imagine more complex games, where the robots can influence the decisions of the referee, as happens in the real world. Obviously, we can think of a concrete implementation of the previous game in a large cellular automaton, even if we do not explicitly describe it. In the following we shall often refer to this particular cellular automaton in order to clarify and justify some concepts, and we shall call it FIGHT. Before proceeding, we point out that in FIGHT it makes intuitive sense to speak about the intelligence of a virtual robot (or, if we prefer, of its human programmer) by considering its ability to survive in the contest. We shall return to this idea in the next sections.

For specific and concrete reference to the theme of competition between robots we can also refer to the project RoboCup (see, e.g., Kitano, 1998), involving teams of robot soccer players.

Now that we have justified our choice of a cellular automaton as a model, let us return to our formal approach. The environment in which we shall formalize the concepts we are interested in is a two-dimensional cellular automaton $\mathcal{C}$. By that we mean a regular lattice of sites (called *cells*), in which each cell contains a value chosen from the set $\{0, 1\}$. This lattice is subject to evolution from an arbitrary initial state I. At each time-step, this evolution changes the value contained in each cell c by

following a (usually local) rule that does not depend on the absolute position of *c*: this rule determines the new value of the cell *c* and depends only on the values contained in the cells belonging to a specified local neighborhood of *c*.[1]

Remark 2. Some authors confine the definition of cellular automaton to the case of the local evolution rule (e.g., involving only the 3×3 neighborhood of each cell) and prefer to call *lattice dynamical systems* the structures we are using. By contrast, we are following the approach given in Wolfram (1994), which allows the use of larger (bounded) neighborhoods. However, from a theoretical point of view, any evolution rule depending on an $n \times n$ neighborhood can be emulated by a 3×3-neighborhood rule applied to another cellular automaton with *m* possible values for each cell (instead of the two values $0, 1$), and so the condition of locality is important only from a practical point of view. However, we wish to stress once again that the aim of this paper is not to consider *efficient* cellular automata, but only to point out some general phenomena arising in all those cellular automata that have certain properties.

Now, we give a simple cellular automaton $\mathcal{C}^*$ in order to make our definitions clear. We do not imply that $\mathcal{C}^*$ is interesting as a model, but only that it is suitable as an example. We shall refer back to $\mathcal{C}^*$ in the remainder of this paper as well.

Example 3. (The glider in "Life") In Fig. 1 we show some 12×12 matrices representing 20 consecutive states during the evolution of a cellular automaton $\mathcal{C}^*$ following Conway's rule.[2] By calling *x* the number of the eight neighbors of a cell that are non-zero, we can state the evolution rule as follows: if $x = 2$ then the cell takes the same value as in the previous time step (i.e., we maintain its color); if $x = 3$, then the cell takes on value 1 (i.e., we set it black); in all other cases the cell takes on value 0 (i.e., we set it white). Observe that updating happens for all cells at the same time. In this way, for every state *s* in the set Σ of all the possible states of $\mathcal{C}$ we define the consecutive state $f(s)$ by following the evolution rule *f*.

We recall that cellular automata can be regarded as discrete dynamical systems and that they are theoretically capable of simulating every Turing machine. Moreover, they seem to be a suitable structure in which to study self-reproducing entities (cf., e.g., Arbib, 1966; Langton, 1984; von Neumann, 1966). Considerable literature about cellular automata exists and we shall point to it for more details about the theory (cf., e.g., Burks, 1974; Codd, 1968; Gutowitz, 1991; Packard & Wolfram, 1985; Toffoli & Margolus, 1987).

Our model $\mathcal{C}$ is the evolving "universe" in which we shall study the phenomena of intelligence and contradiction. However, we are *not* assuming that $\mathcal{C}$ can emulate *all* the physical properties and laws of the real world. We simply mean that cellular automata are models capable of emulating a set of properties of complex entities that is sufficient to explain the presence of the contradictions we see in the real world. Obviously, we cannot be sure that this correspondence is not accidental, but as is well known, no model can be mathematically proved adequate to describe a real phenomenon: this can only be verified experimentally. We shall come back to the concept of contradiction in Sections 4.4, 5 and 6.

4.2. An observer judges the presence of entities

Before speaking about contradiction, we must define the concept of "existence" for an entity. Given this concept, we shall be able to discuss whether an entity is contradictory or not. We point out that in the real world the presence of an entity is strictly connected to the presence of an observer perceiving this entity, and hence existence is subjective, at least from an operative point of view (cf., e.g., McGinn, 1982). In fact, it is common for different people to see different entities in the same environment (see Fig. 3). This is usual in visual perception (cf., e.g., Marr, 1982; Winston, 1984) and, for a physicist, this position would be quite natural. An important reference to this issue can be found in the work of Maturana and Varela (1992).

It may not seem so obvious from a practical point of view, and one might imagine that complex entities exist independently of any observer. For example, someone might argue that in real life the existence of a living being at a certain position and time is absolute, since we are looking at macroscopic phenomena where the indeterminacy of quantum mechanics plays no role. An answer to this objection is easily formulated: if the concept of entity were not dependent on the observer, then no animal could hunt using camouflage, physicians' diagnoses would always be identical and no man could "mistake his wife for a hat" (Sacks, 1985). Scientists would always see the same causes for each phenomenon, and all people would agree in judging who the heroes and villains in a movie or a political event are. In reality, the problem is not whether the concept of entity is subjective or not, but whether we can avoid taking this subjectivity into account or not. Our opinion is that

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[1] More formally, $\mathcal{C}$ can be defined in the following way. Denote by Σ the set of functions from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}_2$, where $\mathbb{Z}$ is the set of the integers and $\mathbb{Z}_2$ is the cyclic group of order 2. Moreover, let *I* be an element in Σ. Each pair in $\mathbb{Z} \times \mathbb{Z}$ is called a *cell*. We define the two-dimensional cellular automaton $\mathcal{C}$ as a pair (I, f) where *f* is a function from Σ to Σ. We shall call *states of* $\mathcal{C}$ the functions in Σ and *initial state of* $\mathcal{C}$ the state *I*. In plain words, each state of $\mathcal{C}$ is a choice of the contents in the cells of the lattice representing $\mathcal{C}$. The function *f* will be called the *evolution rule of the cellular automaton*. A state β of $\mathcal{C}$ will be said to be *consecutive* to a state α of $\mathcal{C}$ if $\beta = f(\alpha)$. After *t* steps in the evolution of the cellular automaton we shall call *present state of* $\mathcal{C}$ (or *state at time t of* $\mathcal{C}$) the state $s_t = f^t(I)$, obtained by applying *f* *t* times to *I* (we recall that each state $f^t(I)$ is a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}_2$).

[2] Formally speaking, the matrices represent the finite sublattice of $\mathbb{Z} \times \mathbb{Z}$ given by the set $\{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, n-1\}$ (the cell $(0,0)$ is the top left one and all the cells outside this sublattice contain the value 0). Each matrix gives a function in Σ. The first matrix represents the initial state *I*. Black cells and white cells denote cells containing 1 and 0, respectively.

the study of artificial intelligence cannot neglect this subjectivity. By referring to our cellular automaton FIGHT, we emphasize that only the referee can judge the state of the virtual robots. We must not forget that there is usually no human observer to confirm or contest the referee's decisions during the game. The possibility of verifying such decisions is only hypothetical since no human operator could examine all of them directly.

Therefore, we need a formal concept of "observer" in order to proceed. In an experiment, an observer is an individual who examines both the state of a studied entity (a reagent, a cell, an animal species…) and the condition of the environment where the experiment happens (a laboratory, a tissue culture, an ecosystem…). The researcher acting as observer perceives the events and reports them according to his/her own opinion and subjective rules of judgment. Informally, we might say that an observer is a "black box" $\square$ capable of identifying the states of a particular entity and of the environment judged relevant to its future behavior. Thus, from a mathematical viewpoint, an observer might be defined as an ordered pair of functions (ps_{ent}, ps_{ENV}), describing the ways the observer judges the perceived states (ps) of the studied entity and of the environment during time, respectively.

In FIGHT, the observer (i.e., the referee) searching for a virtual robot R can be seen as a pair of functions $\square = (ps_{ent}, ps_{ENV})$, too. The functions ps_{ent} and ps_{ENV} take each state s_t of the game to the states that the software judging the competition associates with the robot and environment in question. For example, the perceived states $ps_{ent}(s_t)$ and $ps_{ENV}(s_t)$ could describe the health of the robot and how crowded the environment is. The absence (death) of the virtual robot at time t would be revealed by the equality $ps_{ent}(s_t) = 0$.

Formally, we give the following

Definition 4. (Observer) Let us choose two finite nonempty sets $\mathcal{P}_{ent}$ and $\mathcal{P}_{ENV}$, which will be called *sets of perceptible states for the entity and its environment*, respectively. We shall assume that $\mathcal{P}_{ent}$ contains a privileged element 0. We shall call an *observer* any function $\square = (ps_{ent}, ps_{ENV}) : \Sigma \to \mathcal{P}_{ent} \times \mathcal{P}_{ENV}$.

Note 2. $\mathcal{P}_{ent}$ and $\mathcal{P}_{ENV}$ can be interpreted as personal descriptions of the states that the observer perceives for the entity and its environment. The value $0 \in \mathcal{P}_{ent}$ can be seen as a judgment of absence for the entity in question with respect to the examined state of the cellular automaton. It is important to point out that in the real world these perceptions do not retain all the information about each event. On the contrary, they usually replace the real world with a more compact representation. For example, a physical or biological experiment is not described by giving all possible information about the laboratory where the experiment is done, but a set of quantitative and qualitative data that are judged influential or important for the results of the experiments. So $\mathcal{P}_{ent}$ and $\mathcal{P}_{ENV}$ may consist of formulas, verbal statements, or any other kind of data considered useful for the description of what happens in the experiment.

The hypothesis that $\mathcal{P}_{ent}$ and $\mathcal{P}_{ENV}$ are finite sets is important. It means that our observers are assumed to have limited capabilities, and it will play a key role in our proof of the proposition stated in Section 5. We emphasize that this hypothesis corresponds to the fact that in reality the observers can have neither infinite memory nor unbounded computational capabilities. We consider this as self-evident, but for skeptics, many references are available in the literature. As an example, Wooldridge and Jennings (1995) take for granted that all real agents are resource-bounded. They also confront the famous Logical Omniscience Problem, which arises from the assumption of unbounded inference capabilities (cf. Stalnaker, 1991). Therefore, our hypothesis seems to be quite natural.

Once again, we point out that nothing is taken for granted about the working of the observer $\square$. It is like a "black box" that decides – in an unknown way – whether at a specific time a certain type of entity is present or not, and what the states of this entity are, as well as the states of the environment influencing its future behavior according to the judgment of the observer. So any kind of decisional mechanism is acceptable inside the black box. The symbol "$\square$" has been chosen to suggest this fact.

As an example of an observer in $\mathcal{C}^*$, we may consider a process that displays the location of the glider and the state taken on by the environment where it is moving.[3]

An observer in FIGHT would be a more interesting example, but its precise definition in some programming language could take many pages of this paper. However, it is not difficult to imagine how it would work. For every set X of cells, the observer could compare the contents of X with some set of stored patterns. In this way it would determine whether or not X contains a given virtual robot R and whether it is "alive" or not. Similarly, it could determine the state of the neighborhood judged relevant to the future evolution of the robot.

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[3] From a formal viewpoint, we are considering the function $\square^* = (ps_{ent}^*, ps_{ENV}^*)$, where $ps_{ent}^*(s_t)$ is an element in the set $\mathcal{P}_{ent}^*$ containing all possible non-empty subsets of $\mathbb{Z} \times \mathbb{Z}$ and the symbol 0, and $\mathcal{P}_{ENV}^*$ is the set of all possible states for the "matrix" displayed in Fig. 1 (or, alternatively, a set of qualitative descriptions of these states). In other words, in this case $\mathcal{P}_{ent}^*$ represents the set of all possible locations for the glider, while $\mathcal{P}_{ENV}^*$ can be seen as the set of all states that the environment can take on. Obviously, this is only one among many possible choices for the sets $\mathcal{P}_{ent}^*$ and $\mathcal{P}_{ENV}^*$. An analogous observer who recognizes the block that is going to be destroyed by the glider could be considered. In our example, $ps_{ent}^*(s_t)$ is the set representing the "body of the glider" for $0 \leqslant t \leqslant 14$ and $ps_{ent}^*(s_t) = 0$ for $t \geqslant 15$. It is worth noting that our formalization would allow to represent a fuzzy disappearing of the considered entity. It would be sufficient to take the time t to a fuzzy set instead of a set. This can be easily obtained by changing the sets $\mathcal{P}_{ent}^*$ and $\mathcal{P}_{ENV}^*$.

It is obvious but important to stress that an observer does not usually perceive the environment as coinciding with the whole set $\mathbb{Z} \times \mathbb{Z}$, representing our "universe". It is quite clear that a psychologist observing a patient cannot consider all possible data in the universe in order to examine the reaction of the individual. The psychologist must select a small set of data belonging to a small environment (the patient's answers, drawings, expressions...). Thinking of an observer knowing and processing all the data in the universe is similar to imagining a psychologist capable of using all the data in the patient's life. This is not only practically impossible: it could also be completely misleading from a theoretical point of view, since omniscient observers are totally different from real-world observers.

Now we turn to the task of formally defining the concept of "entity". In real life, an entity usually appears to be stable in our perception of it. Obviously, this trivial remark hides one of the greatest philosophical debates in history, and a discussion of the subject would require a much longer paper. Here we confine ourselves to referring to the interesting essay "The primacy of identity" in Rota (1997). In fact, we are only interested in giving an acceptable definition for practical purposes and we merely point out that stability and coherence in perception constitute the key factor in determining "existence" from a subjective point of view (cf., e.g., Marr, 1982). Therefore, it seems natural to define the existence of an entity as persistence in perception with respect to a given observer. In plain words, we shall call an entity each maximal sequence of consecutive non-trivial (i.e., different from 0) images of the function ps_{ent}. From the semantic viewpoint, such a sequence shows that the observer perceives the existence of the considered structure (e.g., a glider) during the corresponding sequence of time steps. Maximality expresses the request that our sequence is as long as possible. We will formalize this concept in the next definition.

Definition 5. (Entity and lifetime) Each maximal sequence of "consecutive" perceived states in $\mathcal{P}_{ent} - \{0\}$ will be called an *entity* with respect to the observer $\square$. In other words, an entity with respect to $\square$ is defined as a sequence $(ps_{ent}(s_t), ps_{ent}(s_{t+1}), \ldots, ps_{ent}(s_{t+q}))$ with $ps_{ent}(s_t)$, $ps_{ent}(s_{t+1}), \ldots, ps_{ent}(s_{t+q}) \neq 0$, $ps_{ent}(s_{t+q+1}) = 0$ and $ps_{ent}(s_{t-1}) = 0$ (if $t > 0$). We shall call the set $\{t, t+1, \ldots, t+q\}$ the *lifetime* of the entity. The value $ps_{ent}(s_{t+h})$ ($0 \leqslant h \leqslant q$) will be called the *state of the entity perceived by* $\square$ *at time* $t + h$.

With reference to Figs. 1 and 2, the sequence $(ps^*_{ent}(s_0), ps^*_{ent}(s_1), \ldots, ps^*_{ent}(s_{14}))$ gives an example of an entity (the "glider") "perceived" by the observer $\square^*$ in $\mathcal{C}^*$.

Similarly, it makes sense to consider the environment of an entity.

Definition 6. (Environment) If $\mathcal{E} = (ps_{ent}(s_t), ps_{ent}(s_{t+1}), \ldots, ps_{ent}(s_{t+q}))$ is an entity, then the sequence $(ps_{ENV}(s_t), ps_{ENV}(s_{t+1}), \ldots, ps_{ENV}(s_{t+q}))$ will be called the *environment*
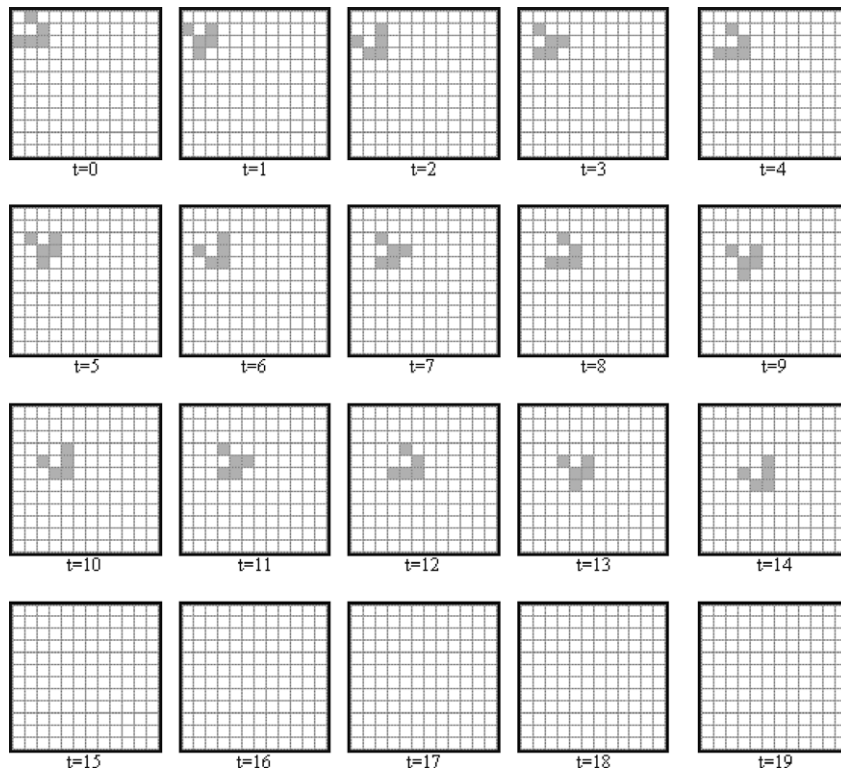

Fig. 2. The "body" $ps^*_{ent}(s_t)$ of the glider (displayed in grey at the first 15 times in the evolution of $\mathcal{C}^*$). The block does not appear in this figure. For $t \geqslant 15$ we have $ps^*_{ent}(s_t) = 0$, meaning that the glider is not found on the scene by the observer, since it has been destroyed in the collision against the block visible in Fig. 1.
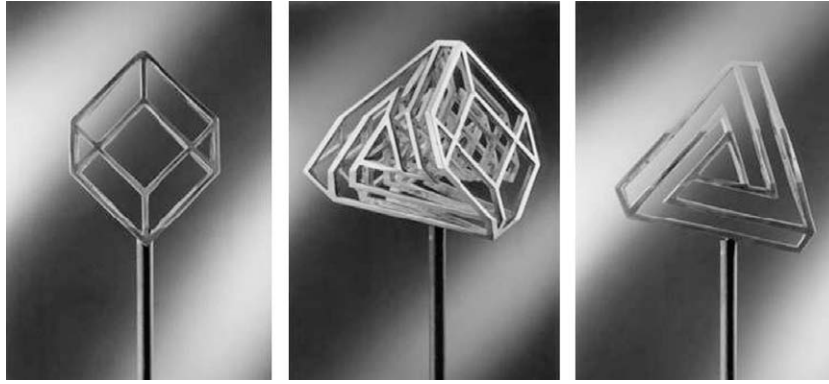
Fig. 3. Different observers can see different entities in the same environment. This photograph depicts three views of the same bronze sculpture by Guido Moretti, showing a Necker cube transforming into an impossible triangle.

Fig. 4. An observer judges the intelligence of an entity by measuring her survival capability in the considered environment.

of $\mathcal{E}$. The value $ps_{ENV}(s_{t+h})$ $(0 \leqslant h \leqslant q)$ will be called the *state of the environment perceived by $\square$ at time $t + h$*.

Note that we do not set any particular constraint on the observer's judgment about the concept of environment influencing the entity's behavior.

Obviously, human observers are much more complex than the ones we have defined. Proximity in position during time, for instance, is important for recognizing the presence of an entity in our world, in most cases. However, this and other properties are not necessary in order to derive the proposition about intelligence and contradiction that we wish to obtain in Section 5. For this reason we did not require these hypotheses in our definitions.

In our model, at each time τ, each observer $\square$ tries to find the entity it is capable of recognizing. The result of that search (that is, the pair $\square(s_\tau)$) represents both the state it perceives for the entity and the state of the environment judged relevant to its future behavior. If $ps_{ent}(s_\tau) = 0$, the entity is not found in the "universe" at time τ by the given observer. Each "maximal chain of consecutive non-zero perceived states" is an entity. Obviously, other kinds of choices would be possible, but we are not interested in enumerating all of them: we only wish to point out the consequences of a reasonable definition.
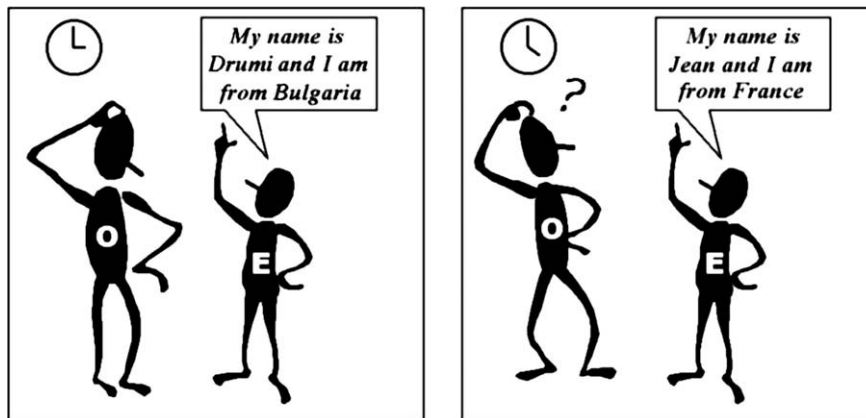

Fig. 5. A simple example of contradiction: the observer *O* perceives two contrasting behaviors of *E* without seeing any relevant difference in *E* and his environment, causing the behavioral change.

Fig. 6. *The subjective nature of contradiction.* Ludwig Wittgenstein is generally considered to have changed his thinking considerably over his philosophical career, since he denied his own *Tractatus Logico–Philosophicus*. While an expert in Wittgenstein's thought might be able to explain his change in opinion on the basis of the knowledge of his philosophical research and experience, a common observer might judge his behavior to be an example of contradiction.

Remark 7. In this paper we are not interested in discussing the complexity of the search performed by the observer. On this subject we refer to Tsotsos (1989) as an example of an approach to the problem.

4.3. A definition of the intelligence of an entity

This is obviously a key point. It is clear that at this time it is not possible to say exactly what intelligence is but, on the other hand, we certainly do not need or wish to enter into the debate concerning this problem. However, it is equally clear that we are not looking for the *answer* to this huge problem but for a reasonable formal idealization allowing us to proceed in a mathematical context.

Many definitions of intelligence have been proposed in the past, among others, biological, computational, epistemological, anthropological and sociological (cf., e.g., Khalfa, 1994). Themes such as multiple intelligences (cf., e.g., Gardner, 1985; Horn, 1986), cultural relativism (cf. Mugny & Carugati, 1989) or the behaviorist interpretation of intelligence (cf. Brooks, 1991) have been studied by many researchers. We simply refer to the excellent survey and the rich bibliography contained in Sternberg (1990). However, in order to follow a mathematical approach to our problem, we *need* a formal definition allowing for a quantitative comparison of intelligence across different entities. This eliminates all high-level or self-referential definitions: references to concepts as complex as intelligence are not useful to our goal. Therefore, expressions like "the attitude to solving problems" (cf. Minsky, 1986) cannot be a good definition, since they require clarification of very difficult concepts (in this case the concept of "problem"). The classical Turing test (cf. Turing, 1950) is perhaps the most famous attempt to define intelligence through an experimental framework.

Unfortunately, this test and its various reformulations are not suitable for a mathematical approach, since they occur more as tools in a philosophical debate than as practical procedures. In particular, they give neither a formal definition nor a quantification of intelligence and do not make clear any criterion of judgment for the observer, so that it is difficult to imagine of translating this approach into a mathematical structure. In any case the Turing test suggests that measuring intelligence strongly depends on an experiment made by an observer acting as a judge. After all, this is the way intelligence is commonly measured, and it is not surprising that I.Q. tests reflect the thoughts and opinions of the psychologists who prepare them. So it seems natural to look at intelligence as something we can measure by a test made by an observer. Since a test is a practical process, we cannot think of the observer as an omniscient individual, capable of perceiving and examining all data about the entity it is studying. More realistically, all it can do is subject the entity to some tests and formulate its own opinion about the results. As an example, our opinion about the intelligence of someone is not based on a complete knowledge of his/her life but on some particular experiences concerning his/her behavior. On the other hand, a classical way of approaching the goal of formally defining intelligence is that of looking at it as the capability that an entity has to adapt to changes in the environment (cf., e.g., Sternberg, 1990). From this point of view, intelligence can be measured by quantifying success in adaptation. Such success can simply be expressed by the length of life of the entity considered: this is the approach we have chosen in this paper.

It may be opportune to observe that the structure of a classical intelligence test can easily fit into this framework. The role of observer is taken by the psychologist administrating the test, which usually consists of some trials and problems that must be overcome by the person examined. Overcoming a difficulty (such as solving a problem) can be seen as a form of survival inside a particular game. Obviously, when we use the word "survival" we do not necessarily mean survival in a biological sense. In our setting, surviving simply means remaining a player in the game (see Fig. 4).

When we say that intelligence may be expressed by the length of life of the considered entity, we do not at all mean that it is *explicitly* represented in this way (see Remark 8 below), but that it *can* be represented in this way if suitable language is used.

For example, the efforts to solve puzzles in a mathematical competition are not usually described as an attempt to survive. Nevertheless, the number of puzzles that each participant has solved during the contest could be formally seen as a length of life.

Our final comment is that we are *not* necessarily thinking of the observer as the creator of the difficulties that the examined person must face. This may happen in the case of the intelligence test, but we are mostly interested in cases when the observer does not completely understand the problems presented and the corresponding solutions. These are cases in which the most interesting phenomena

of contradiction may happen, as we shall see in the following.

For the reasons we have explained, the intelligence of an entity in a cellular automaton $\mathcal{C}$ can be seen as the number of consecutive states of $\mathcal{C}$ during which the studied entity exists with respect to a given observer.

Remark 8. ("The man and the sequoia") An easy but misleading criticism of our approach could be the assertion that there is very little relation between length of life and intelligence. For example, we could observe that if we consider a human being (a man, say) and a sequoia in a forest, it is likely that the man will "survive" for a far shorter time than the sequoia, but this is not a good reason for thinking that the former is less intelligent than the latter.

To have this opinion means to forget that our definition of intelligence strongly depends on the choice of a suitable model where intelligence can be measured as ability to survive.[4]

As usual, choosing a model is a matter that depends on the aspects of reality that we are interested in. For example, there would be no point in making a biological simulation of a chess-player in order to measure her skill in chess, since no one would judge her intelligence by examining her metabolism and immunological efficiency. We should rather test her in the "virtual world" of chess games, where threats consist not of disease but of the opponents' moves.

Similarly, a comparison of intelligence between the man and the sequoia (with respect to the "human" concept of intelligence) should be done in a model in which dangers and difficulties consist of what a human being considers to be problems to be solved.

For example, we might choose a model representing a physical world in which the man and the sequoia are in direct competition for survival. In such a model, we can imagine that the former could easily destroy the latter, revealing the latter's relative lack of intelligence.

This kind of test is similar to what we do when we think about the intellectual deficiency of a living being. We do not look for a real proof of incapacity to react to "dangers". We simply simulate in our brain what would happen if such dangers occurred to the considered living being, by referring to a model represented in our imagination. In a "virtual world" of this kind, the lack of intelligence of the sequoia could easily be expressed in terms of a short duration of life.

Remark 9. ("The oscillating pendulum") It is important to underline once again that our definition of intelligence strongly depends on the choice of the observer. Obviously, if the observer is quite different from a human observer and

has very limited capabilities, the correspondent definition of intelligence will be very unusual. We make clear our position by giving another example. Let us consider an oscillating pendulum and an observer looking at it. On the basis of our approach, one might criticize our definition by claiming that the observer perceives an indefinitely long "life" of the pendulum, since it never stops. It is worthy to remark that in this way he would assume to consider an observer that is completely different from a human one. Indeed, a human observer interested in examining the pendulum would have a lot of information available about it, in her memory, and some brain activity concerning her perceptions. The "right" model should not describe the physical world where the pendulum is oscillating, but the computational structure (her brain) where the pendulum is tested and its behavior checked. In her brain, the observer could easily imagine to stop or even destroy the pendulum. In this model, that is the most natural for a human observer, the lack of intelligence of the pendulum could be easily revealed. Considering a different model (e.g. just representing the physical evolution of the pendulum) would mean to choose some kind of mechanical observer that simply registers a list of actions much like a camera can do, without any usual mental activity. Judging the intelligence of the pendulum by examining the regularity of its oscillations would be much like judging the intelligence of a chess player by examining the regularity of his heartbeats. It should not be surprising if the choice of an unusual observer produces a concept of intelligence that is not the most natural one.

Formally we give the following definition.

Definition 10. (Intelligence of an entity) Let us assume that an entity $\mathcal{E} = (ps_{ent}(s_t), ps_{ent}(s_{t+1}), \ldots, ps_{ent}(s_{t+q}))$ with respect to an observer $\square$ is given. Then we say that q, i.e., $|lifetime| - 1$, is its intelligence.

Hence, e.g., the intelligence of the entity represented in Fig. 2 is 14. The simplicity of this example should not deceive the reader. More complex cases could be easily shown, which are not so trivial and might be interesting for applications. As an example among many, we could consider the problem of quantifying the efficiency of a given commercial software agent A. A natural way to do this could be simulating a standard test market M and testing A inside M. In this case the intelligence of A (i.e., the lifetime during which the agent can survive in the standard market) might be taken as a useful reference for comparison between similar agents.

We underline that the concept of intelligence, like the concept of entity, is strictly dependent on the chosen observer. While we have already justified this position, we refer the interested reader to Brooks, 1991 for further discussion of the idea that intelligence is "in the eye of the observer".

Note 3. It is important to point out that measuring intelligence is becoming a key problem in computer science. As an example, the use of collaborative agent systems

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[4] When we say "choice of a model" we mean both choosing the observer and the cellular automaton $\mathcal{C}$ representing the phenomenon we are studying (we must remember that the observer operates on the states of $\mathcal{C}$).

requires the ability to measure the extent to which a set of collaborative agents is able to accomplish the goals it was built for (cf., e.g., Nwana, 1996). In other words, we want to know if it is reliable or not, and to compare its "intelligence" to that of other collaborative agent systems pursuing the same aim (e.g., think of controlling a nuclear installation or a chemical plant). This necessity makes the measurement of intelligence a more and more important task in software engineering, and gives another practical motivation to our research.

4.4. A definition of the contradictory nature of an entity

Following the dictionary ("contradiction." Merriam-Webster OnLine: Collegiate Dictionary. 2000. http://www.merriam-webster.com/dictionary.htm (6 August 2001)), the word *contradiction* has the following meanings in the ordinary language:

(1) act or an instance of contradicting;
(2) *a*: a proposition, statement, or phrase that asserts or implies both the truth and falsity of something;*b*: a statement or phrase whose parts contradict each other ("a round square is a contradiction in terms");
(3) *a*: logical incongruity;*b:* a situation in which inherent factors, actions, or propositions are inconsistent or contrary to one another.

What is common to these definitions is a conflict of behavior, as happens when a statement is both asserted and negated, either by different subjects or by a single individual. For example, we call a human being contradictory if he/she supports both a statement and its negation. If we accept the point of view that the concepts of intelligence and contradiction depend on the judgment of an observer, we can reformulate the previous definitions by saying that contradiction is a phenomenon in which an observer perceives that an individual or a group of individuals produces behaviors which are, in some sense, incompatible (see Fig. 5). These types of behavior include opinions and assertions but are not necessarily limited to these. As we know (see previous definition 3b) actions can also be contradictory, and the word "contradictory" is often used to denote a change in behavioral rules ("He is contradictory: in the past he defended this cause, while now he attacks it").

Therefore, a common property can be found in our definitions: an entity can be said to be contradictory if faced with the same circumstances, it does not exhibit the same behavior (see Fig. 6). In other words, the ordinary use of the term *contradictory* refers to a change in behavior of the *same* entity.

So it is reasonable to call an entity contradictory, if it happens that, at different times, it reacts differently to the same state of its own body and of the environment where it lives – that is, the same action is considered to produce different results (cf. Piaget, 1974). At the end of this section

we shall propose a mathematical formalization of this definition.

Some possible objections to our approach to contradiction should be considered.

The first objection concerns the classical use of the term "contradiction" in mathematical logic. We know that (roughly speaking) a theory is contradictory if in such a theory it is possible to prove both a statement α and its negation $\neg\alpha$. At first glance our approach to contradiction seems to ignore this classical use. It could seem that there is no relation between the meaning we are speaking about and the one studied by logicians and mathematicians. This is not the case in our context, since the concept of contradictory theory we use in logic can be seen as a particular case with respect to our definition. This point can be clarified by an example. Let us assume that a theory S endowed with a finite set A of axioms is contradictory, in the sense we have previously described, i.e., in S it is possible to prove both a statement α and its negation $\neg\alpha$. Now we can imagine a Turing machine T accepting the set A and the formula α as input data and producing all possible valid proofs of length l, with l progressively increasing. In other words, T will produce all possible valid proofs of length 1, then all possible valid proofs of length 2, and so on. If T finds a proof of α, it writes down TRUE in a precise location of its infinite tape. Similarly, if T finds a proof of $\neg\alpha$, it writes down FALSE at the same location. The statement written at that location represents the answer given by T to the question "Is α true or false in S?". Obviously, in both the cases examined the previous contents of the cell are erased, while the absence of any symbol at the considered location must be interpreted as the fact that T cannot prove either α or $\neg\alpha$ (and hence answer the question) until the present computational step.

Since S is assumed to be contradictory, there will be two times (or steps) in the functioning of T at which the answers will be different, corresponding to the times at which T will discover a proof of α and $\neg\alpha$, respectively. Hence the reaction of the Turing machine will appear to be contradictory (in the sense we specified) to an observer, under the hypothesis we implicitly made, that time is not considered input data for T. Here we are only assuming that the observer asked to judge contradiction recognizes T as a valid prover for S and maintains this opinion about the identity of T during all its functioning. This example shows that our approach to the concept of contradiction includes the classical notion of logical contradiction as a particular case.

We are aware that our viewpoint can be criticized by asserting that contradiction is an absolute concept in mathematical logic, independently of the opinion of the particular observer. Even if we respect this position, we cannot avoid doubting that it is completely acceptable from a scientific point of view. Excluding on principle any reference to an observer when affirming (in a formal sense) that "Theory X is contradictory" means to maintain an idealistic approach that might be harmful for further progress in artificial intel-

ligence. Although many mathematicians support an idealistic vision of mathematics and mathematical logic, none of them would probably accept a statement without checking the corresponding proof, so implicitly requiring that an observer expresses his/her opinion about the statement considered. In fact, we can probably assert that mathematicians are not only interested in the existence of proofs, but, above all, in the discovery of proofs. Moreover, the history of mathematics is full of wrong statements that have been corrected when some expert (observer) has changed his/her point of view and found some mistake. Expunging the role of the observer and his/her judgment means separating mathematics (and hence mathematical logic) from the research by which it is produced, and putting knowledge into a limbo where truth is both untouchable (since we want it to be stable in time) and potentially transient (since progress and research can change it). In some sense, we could say that the price of certainty, from an idealistic point of view, is to give up the study of reality.

While we do not insist on this subject, we refer to the discussion in Davis and Hersh (1981) about the difficulties inherent in an idealistic approach to mathematics.

Another possible objection concerns the meaning of the expression "equivalent conditions". If the conditions are really equivalent one might think that two different behaviors are not possible in a deterministic setting, and hence that no contradiction could appear. Once again, we stress that in our model the only judge of equivalence can be the chosen observer. As happens in reality, there is no point in asserting that two conditions are different if we cannot perceive any difference between them. The contrary position may be interesting in philosophy but (perhaps) much less in computer science. The assertion that there is no room for contradiction in the presence of complete and universal knowledge is perhaps valid, but not very useful in practice, and it may imply the non-existence of equivalent conditions, thus destroying the concept of science as we usually interpret it. As an example of what we are saying, let us imagine that a proof of a contradiction (in the mathematical sense) is discovered for a given, relevant and useful theory. Assume that the proof is checked and verified by all the qualified experts in the world, and imagine that we can make the same verification of correctness. How plausible would it be to argue that the given theory is nonetheless free from contradictions and that the so-called "proof" of a contradiction in it must contain one or more invisible flaws? It is highly implausible, and we would probably rely on the opinion of experts and on our own verification, without considering the existence of invisible data and errors that could potentially modify our position.

Analogously, when we speak about "equivalent conditions" for an observer, we should not think of an incompetent judgment due to lack of information or the presence of errors, since, in doing so, we would simply superimpose our own personal judgment on the opinion of the chosen observer. This act would be equivalent to a change of observer.

We can thus introduce and propose the following definition.

Definition 11. (Contradictory entity) Let us assume that an entity $\mathcal{E} = (ps_{ent}(s_t), ps_{ent}(s_{t+1}), \ldots, ps_{ent}(s_{t+q}))$ with respect to the observer $\square$ is given. If natural numbers a, b $(a, b \leqslant q)$ exist such that $ps_{ent}(s_{t+a}) = ps_{ent}(s_{t+b})$ and $ps_{ENV}(s_{t+a}) = ps_{ENV}(s_{t+b})$ (i.e. $\square(s_{t+a}) = \square(s_{t+b})$), but $ps_{ent}(s_{t+a+1}) \neq ps_{ent}(s_{t+b+1})$, then we shall say that such an entity is *contradictory*.

In other words, our definition means that, while the observer perceives equivalent states for the entity and the environment at times $t + a$ and $t + b$, it is assumed that the entity reacts differently to these states.

Remark 12. ("Is a flipping coin a contradictory entity?") A simple question may arise immediately after giving our definition of contradiction. Should we consider a flipping coin, giving many different results, an example of contradictory entity? This kind of question is important to make our position clear. Once again, the point is the choice of the model and the observer. If we decide to choose a "human observer" we cannot rule out his/her usual distinguishing features. A human observer knows that the coin is a disk made of inert metal and that it can be easily stopped and destroyed. Obviously, the model we are interested in must be large and complex enough to represent the evolution of the observer's brain, where the information is stored and the "physical" coin is substituted with its mental representation (cf. Remark 9). This representation can be checked by the observer's mind. The fact that in this model the coin does not oppose its destruction reveals that the coin has no intelligence and hence, by definition, no contradictory behavior (no contradiction is possible if $q = 0$, according to Definition 11). Obviously, since any kind of observer can be chosen in the model, we could also choose a different and non-human observer, having no memory and no preexistent opinion about the coin, and unable to check the reaction of the coin to hypothetical situations. However, it should not be surprising that the choice of this unusual observer would lead to an unusual evaluation of intelligence and contradiction.

Another useful concept is that of the deterministic environment, formalized by the following definition:

Definition 13. (Deterministic environment) Let us assume that an environment $(ps_{ENV}(s_t), ps_{ENV}(s_{t+1}), \ldots, ps_{ENV}(s_{t+q}))$ with respect to the observer $\square$ is given. If for any pair of natural numbers (a, b) verifying $a, b \leqslant q$, $ps_{ent}(s_{t+a}) = ps_{ent}(s_{t+b})$ and $ps_{ENV}(s_{t+a}) = ps_{ENV}(s_{t+b})$ (i.e. $\square(s_{t+a}) = \square(s_{t+b})$) the equality $ps_{ENV}(s_{t+a+1}) = ps_{ENV}(s_{t+b+1})$ holds, then we say that the considered environment is *deterministic*.

According to the previous definition, if the environment is deterministic its future state depends on the present state of the entity and the environment (i.e., all that the observer knows about the examined "world"). In any case, this

dependence is *not* required to be explicit and computable, and the observer may not be able to anticipate the future environmental state.

Some environments appear to be deterministic, while others do not. Even far away from quantum mechanics, it may happen that the environment evolves in an unpredictable way, according to the observer's judgment. For example, the weather evolution may be predictable or unpredictable, depending on the computational capabilities of the observer looking at it and on the information that is available to him, expressed by the states he can perceive.

From a formal point of view it may be interesting to observe that, following our definitions, an environment is deterministic if and only if it is non-contradictory as an entity, with respect to the dual observer that exchanges the roles of ps_{ent} and ps_{ENV} (provided we add the required special symbol 0 to $\mathcal{P}_{ENV}$).

5. The key result in our model

In the model we have established the following result can be proved, as a trivial consequence of the pigeonhole principle. This result shows that determinacy is forced to break down when the observer examines an intelligent enough entity.

Proposition 14. *Assume $\mathcal{E}$ is an entity having a finite lifetime and a deterministic environment with respect to an observer $\square$ for the cellular automaton $\mathcal{C}$. Let k be the product of the cardinalities of the sets $\mathcal{P}_{ent}$ and $\mathcal{P}_{ENV}$. Then, if the intelligence of $\mathcal{E}$ is strictly greater than k, the entity $\mathcal{E}$ must be contradictory.*

Proof. Let $L = \{t,\ t+1,\ldots,t+q\}$ be the lifetime of $\mathcal{E}$. From $q > k = |\mathcal{P}_{ent}| \cdot |\mathcal{P}_{ENV}|$ it follows that in L two time steps $t+a$ and $t+b$ $(a < b)$ must exist such that $ps_{ent}(s_{t+a}) = ps_{ent}(s_{t+b})$ and $ps_{ENV}(s_{t+a}) = ps_{ENV}(s_{t+b})$. Suppose $\mathcal{E}$ is not contradictory. Then for each time step $\tau \in L$, the values $ps_{ent}(s_\tau)$ and $ps_{ENV}(s_\tau)$ would determine both $ps_{ent}(s_{\tau+1})$ (since by assumption $\mathcal{E}$ is non-contradictory) and $ps_{ENV}(s_{\tau+1})$ (since by assumption the environment of $\mathcal{E}$ is deterministic). Therefore, the equalities $ps_{ent}(s_\tau) = ps_{ent}(s_{\tau+b-a}) \neq 0$, $\quad ps_{ENV}(s_\tau) = ps_{ENV}(s_{\tau+b-a})$ would hold for every $\tau \geqslant t + a$. Thus, $ps_{ent}(s_\tau)$ and $ps_{ENV}(s_\tau)$ would be periodic functions in τ for $\tau \geqslant t + a$ and the lifetime of $\mathcal{E}$ would be infinite, contradicting our hypothesis. Hence our thesis is proved. $\square$

The previous result can be reformulated in the following way: if an entity is intelligent enough with respect to a given observer, then either the entity appears to be contradictory (and hence its behavior is unpredictable) or the environment is not deterministic (and hence no prediction can be made). This statement requires that the entity has a finite lifetime and the observer has bounded capabilities, and suggests that in the real world the previously described limitation about determinacy should be expected in intelligent systems.

Remark 15. Some comments should be made about the stipulation that the lifetime of entity $\mathcal{E}$ is finite. From a technical point of view, this stipulation is made in order to exclude the possibility of an observer judging a structure that endlessly repeats the same configurations to be alive. In the real world and in realistic models this type of endless repetition cannot occur, since mechanisms break down and living beings die sooner or later (some remains are usually left but the observer does not recognize them as being alive, as in the case of biological death). In this fashion, our stipulation characterizes the structures that are most interesting for our proposals.

Suitable limitations to our choice of model would allow us to exclude entities with an infinite lifetime, but we preferred to accept all models and simply point out the ones we think are most significant.

Remark 16. From the observer's viewpoint, the contradictory behavior of the studied entity implies that its actions are unpredictable. In fact, the observer cannot foresee the next state of a contradictory entity as a consequence of its present state and the state of the environment. Thus, the statement we have proved implies the following assertion, valid for a deterministic environment:

Any sufficiently intelligent entity is unpredictable.

This point of view is supported by various research. In particular, the project *Copycat* (Hofstadter, 1984a, 1984b) suggests that non-determinism is very important for intelligence. The detailed description of some cognitive processes points out the necessity of non-deterministic behavior in order to allow the discovery and efficient manipulation of analogies. For an introduction to this project and its implications we also refer to Hofstadter (1995).

Many examples stressing the importance of the link between intelligence and unpredictable behavior might be done, showing how unforeseeable actions can be useful for survival. As an example of this kind, we could refer to the techniques that many animals adopt for escaping predators (think of a rabbit avoiding a pursuing fox by making unpredictable zigzag bounds across a field).

6. Computational experiments

In our model a proposition about the link between intelligence and contradiction has been proved. The next question is to what extent our approach can be connected to the real world. Some research proving the existence of this link is available in literature. For example, the result of the experiments described in Mattei (2000) might be interpreted as evidence of the relationship between intelligence and contradiction. However, in order to get further data that maintain the statement expressed in our framework, we have carried out some tests. In this section we shall give the results of three computational experiments. Each of

these is an idealization of a real phenomenon involving intelligence and contradiction. In each case the results support the thesis that there is an upper bound beyond which only contradictory entities "survive" (we wish to stress that, in our context, "to survive" simply means to get the best performance in the game undertaken). The experiments to be described are very austere, and deliberately so, in order to make them simple and comprehensible, but more complex and realistic examples could easily be obtained by introducing more parameters.

Experiment 1. (*Up and Down.*) We begin with an experiment showing the computation of a threshold analogous to the one we spoke about in Section 5, in a concrete case. We consider a simple solitaire game, called *Up and Down*. We have a deck of n cards having different values. Before playing, each player chooses a strategy – that is, a sequence of $n - 1$ words in the set {*up,down*}: $w_1, \ldots, w_{n-1}$. Then the cards in the deck are placed on the table one after the other, and we get a sequence of cards $c_1, \ldots, c_n$. The player wins if and only if $w_i = up$ when $c_i < c_{i+1}$ and $w_i = down$ when $c_i > c_{i+1}$ for every i. In other words, a player "overcomes the difficulty of the game" when he/she always guesses correctly the rises and falls in the sequence of cards on the table. The rise-and-fall structure of the deck can be thought of as a simple environment E with respect to which the player tries to survive by guessing the behavior suitable for E. In this simulation, the player's states that are supposed to be perceived by the observer are the player's wait for a new game and his/her choice of a strategy (if he/she is still "in the game"). Note that a single strategy may work for many different orderings of the deck of cards.

Since a normal observer knows no relation connecting the present ordering of the pack to any future ordering of the pack, in this experiment all environmental states may be considered equivalent (as not influential on future events).

Therefore, according to our observer-oriented framework, it is clear that every change of strategy constitutes contradictory behavior, since the observer never perceives differences between the games.

In our experiment we considered all the possible strategies in the game by taking $n = 10, 11, 12$. Then we computed all possible shufflings of decks of sizes 10, 11, and 12. For each player we obtained the corresponding number of victories. We assumed that the players are not contradictory, i.e., they do not change their strategies during the set of games.

We found that the maximum number m of victories for a single player is 50,521 out of 3,628,800, 353,792 out of 39,916,800, 2,702,765 out of 479,001,600 for $n = 10, 11, 12$, respectively.

Therefore, any player winning a strictly greater number of games (with different orderings of the cards) is *forced* to be contradictory, i.e., to change his/her strategy during the set of games. In fact, it is easy to see that there are contradictory players who are able to win $m + 1$ times: it is sufficient to consider a player who has already won m games

using the same strategy, and who then changes strategy in order to win further games.

Obviously, the maximum m is the logical equivalent of the upper bound k we mentioned in the proposition we gave in Section 5.

It is relevant to point out that an analogous computation could easily be performed for strategies depending on the values of the cards already placed on the table. In this case we would also get bounds on the number of victories, beyond which contradictory behavior is unavoidable if we require different orderings for the cards.

Experiment 2. (*Co-operative/non-co-operative behavior.*) We simulated an interaction between individuals from the point of view of co-operative/non-co-operative behavior. We assumed that when two individuals meet, each of them can act either co-operatively or non-co-operatively. In the case of co-operation, each of them gets a positive pay off $g_{cc} = 2$, while when both of them act non-co-operatively the pay off is zero for both ($g_{nn} = 0$). If their behavior is different, the co-operative individual receives a negative pay off -1 ($g_{cn} = -1$) while the non-co-operative individual gets a positive pay off 1 ($g_{nc} = 1$). In other words, we have assumed that reciprocal co-operative behavior produces the maximum pay off, while every non-co-operative individual is supposedly trying to steal resources from co-operative individuals.

In our simulation we randomly assign a co-operative/non-co-operative stance to each of a set E of m people. Analogously, we randomly assign a co-operative/non-co-operative attitude to each one in a set I of n individuals. Then we assume that each individual x in I enters the environment represented by the set E and meets each person in this set, thus obtaining a total pay off dictated by the set of strategies.

We took $m = 20$ and $n = 1000$.

In this experiment we assume that the observer of the game can perceive the psychological status of player $x \in I$, but not that of the people in E whom player x is going to meet. Therefore the observer's knowledge of the environment and the entity are limited to the state of the game and the player's stance, in this case.

Before any of the m meetings any individual x can change his/her stance, and the probability of this change is set at p. Obviously, according to our framework, if x changes his/her behavior (by moving from a co-operative to a non-co-operative stance or vice versa) he/she becomes contradictory, since the observer never perceives differences between the meetings he/she observes, except for their results.

Finally, in the set I, the individual $\bar{x}$ who has achieved the maximum gain (i.e., the winner of the game) is determined. In every simulation two outcomes are possible: the winner $\bar{x}$ of the game is either contradictory or is not. By repeating our simulation *100* times, we calculated the percentage of winners that were contradictory.

We point out that we chose probability p so that non-contradictory individuals were as likely as contradictory ones ($p \approx 0.034$). Furthermore, we chose our pay off matrix

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in such a way that the expected value for the gain from each meeting was the same both for co-operative and non-co-operative individuals (i.e., 0.5).

In this experiment we found that the percentage of contradictory winners was 100%. In other words, all winners were contradictory, showing that contradictory individuals are much more likely to be winners in this type of situation.

Incidentally, we point out that a large bibliography exists for the co-operative/non-co-operative behavior tested in this experiment, examined from various points of view. A very interesting treatment from a biological point of view can be found in Dawkins, 1989.

Experiment 3. *(Stockholders and share prices.)* We simulated the behavior of a set of stockholders during a week. Each stockholder can buy or sell one kind of share and owns 10,000 units of cash assets and 10 shareholdings, at the beginning. The price of each share is an integer in the set {900, 1000, 1100}, varying daily. We assume that the price on day $t + 1$ is determined by the price on day t. This dependence is chosen randomly and is assumed to be unknown to the stockholders at the beginning of the week. The initial price of the shares is chosen randomly as well. On any day each stockholder can buy or sell an arbitrary number of shares at the price for that day, with the obvious constraint that he/she can neither spend an amount greater than his/her cash assets at that time, nor sell more shares than he/she owns.

Our experiment consists of 50 tests. In each test we have two groups of stockholders. Group *A* contains *100* non-contradictory stockholders. On each day of the week the number of shares to be sold or bought is chosen randomly, but we require that if, in the presence of a price *p*, the stockholder sells or buys a number *x* of shares, he/she makes the same choice every day the price takes the same value *p*. Group *B* contains *100* stockholders who are allowed to be contradictory. Therefore, in this case the number of shares to be sold or bought is chosen randomly on each day of the week, without any constraint on behavior in the presence of the same market price.

At the end of the week we compute the final capital of each stockholder in both groups, given by adding the stockholder's final cash assets to the value of the shares owned according to the final market price. The greatest final capitals $c(A)$ and $c(B)$ are found, by running through all the stockholders in each of the two groups. If $c(A) > c(B)$, then in *A* a non-contradictory stockholder exists whose final capital is greater than the final capitals of all the stockholders in *B*. If $c(A) < c(B)$, then in *B* a possibly contradictory stockholder exists, whose final capital is greater than all the final capitals of all the non-contradictory stockholders in *A*. We carried this experiment out *50* times, and we found that the former case *never* arose, while the latter arose *29* times (more than half the total number), thus demonstrating that contradictory behavior is often required to obtain the maximum total profit.

In our experiment it is quite natural to interpret the share price as the perceived environment, while the selling–buying action of the stockholder and his/her wait for a new price can be seen as the information available to the observer about the entity. The dependence of the share price on the price assigned on the previous day corresponds to the stipulation that the environment is deterministic.

Remark 17. As well as the experiments we have carried out, there is a wealth of further evidence supporting the relationship between intelligence and contradiction. The development of Genetic Programming and Genetic Algorithms, for example, is based on the concept that increasing the capability of problem-solving requires changes in behavioral rules without the observer realizing the exact procedure of these changes. However, we cannot enter into this wide area of study in the present paper.

7. Some controversial points: our answers

During the drawing-up of this paper, many useful comments were made by anonymous referees and by other readers. Since this constructive criticism has contributed considerably to this research we decided to collect together the main objections and questions about the concepts we are discussing, in order to highlight both these remarks and problems and the answers we gave in the paper. Obviously, this list must be perceived only as a concise résumé of the approach we developed in the previous sections.

- **Objection a:** *"The definition of intelligence seems to be arbitrary and not well justified. Many different definitions are possible, but intelligence is certainly not as simple as a number denoting a lifetime."*
  Answer: We see two possible mutually exclusive reasons on which to base this objection: (1) It is hopeless to try to make a mathematical definition of such a complex and elusive concept as intelligence; (2) The idea of a mathematical definition is acceptable, but the one proposed seems to be inadequate.
  Objection (1) is tantamount to rejecting the idea that intelligence can be scientifically studied. Science is widely understood as the proposal and working-out of precise models of limited aspects of reality, and the checking of how well these match reality itself. Our model, focusing on success in adaptation, allows a quantitative approach to the concept of intelligence and a predictive result about contradiction.
  As for (2), the intelligence we perceive in playing chess, proving theorems, deciding purchases and sales in a market, solving puzzles (and so on) can be seen as the ability to survive in an environment where the threats are represented, respectively, by chess opponents, logical errors, financial crises and the puzzles themselves. However, we are not suggesting that intelligence is accurately modelled by the length of an entity's survival in an arbitrarily chosen mathematical situation. Such a lifetime must be considered within a suitable model, which often involves the observer's "brain" and its predictive ability.

We discussed, in Remark 8, the example of the sequoia and the man, showing that if we take the proper model then the length of life is larger for the latter, contrary to naïve expectations. Therefore, in view of the motivations and the examples given in Section 4.3, a convincing criticism of our approach should be based on counterexamples showing some kind of intelligence that *cannot* be reduced (in the sense specified in the paper) to survival capability.

- **Objection b:** *"What is the practical usefulness of measuring intelligence by a single number?"*
  Answer: Obviously, saying that the intelligence of the glider in Fig. 2 is 14 is not very interesting. On the other hand, saying that the intelligence of a commercial software agent is x (since it can "survive" x time cycles in a given standard test market Y) could be much more interesting for a possible purchaser. In fact, the need for quantification of and comparison between various software agents' performances is without doubt going to be ever more relevant in software engineering, according to many experts.

- **Objection c:** *"The definition of contradiction involves criteria that are far less stringent than would be required to conform with common usage in logic."*
  Answer: We showed that the concept of contradiction is not only a matter of mathematical logic. Additionally, we pointed out that the meaning of "contradiction" used in mathematical logic is subsumed under our definition (see Section 4.4), if we accept the key role of the observer.

- **Objection d:** *"To describe an agent's behavior as inconsistent merely on the grounds that the agent adopts a different strategy when dealing with the same particular facet of its environment on different occasions seems to be an implausibly weak criterion of contradictory behavior. To modify one's strategies in the light of changing external conditions is not inconsistent. Intelligence is applying different strategies to different circumstances."*
  Answer: Again, if we accept the key role of the observer, this observation is misleading. The expressions "particular facet of its environment", "changing external conditions" and "different circumstances" may not make sense. If we agree that complete knowledge of the universe that we are studying is not possible and if we decide to rely on the judgment of an observer with bounded capabilities, we *cannot* consider any data that are not accessible to the observer. In a deterministic universe, the phenomenon of contradiction appears to be strictly connected to the existence of bounds of knowledge. For example, let us consider the most classical case of contradiction – that is, an individual asserting two incompatible statements. When we are involved as observers in this event, we usually guess that there must be differences between the situations producing the different answers (e.g., psychological differences). The point is that if we do not perceive these differences (since we cannot access them as observers), it is almost useless to claim their existence, at least from a practical point of view.

- **Objection e:** *"What you call contradiction should be more properly called adaptation for survival."*
  Answer: These concepts are quite different. First of all, there are contradictory behaviors that are harmful for survival (changing one's own behavioral rules without any change in the environment is often dangerous, as can easily be verified by the example of a driver who decides to assign a personal meaning to the colors of traffic lights). More interestingly, "adaptation for survival" is not necessarily a contradiction, since the observer can find such an adaptation quite reasonable. Adaptation for survival may, however, be perceived as contradictory when the observer is not able to understand the reason for such a change. From this point of view, the claim made in this paper is not that intelligence implies adaptation, but that intelligence necessarily implies a kind of adaptation that is perceived as unreasonable by the observer.

- **Objection f:** *"Why do you use the concept of cellular automata in your approach?"*
  Answer: As we state in the paper, cellular automata can emulate a universal Turing machine and are very simple at a local level. Moreover, they naturally adapt to describing evolution in time and space. Although we could express the same ideas in another context, the concept of cellular automata makes it particularly straightforward and easy. Another reason motivating our choice is the possibility of easily including the observer in cellular automata, allowing interaction between an entity and the corresponding observer. This line of research has not been explored in this work, but we plan to do so in a forthcoming paper.

- **Objection g:** *"The notion of intelligence cannot be illustrated by something as trivial as the game of Life."*
  Answer: In principle, the game of *Life* can emulate any Turing machine and hence all algorithms we can implement on a computer can also be implemented in *Life* (cf. Section 4.1). Saying that intelligence cannot be represented in the functioning of a cellular automaton implies, from a theoretical point of view, the assertion that computers cannot emulate intelligence. This might well be the case, but if so, the proof is lacking, as far as the author knows.

- **Objection h:** *"Intelligence and contradiction are not concepts depending on the existence of an observer. The validity of a mathematical proof does not depend on the existence of a reader of such a proof."*
  Answer: We will certainly not try to attack an idealistic approach to knowledge. However, independently of own epistemological attitude, reality and science (and A.I. in particular) are full of contradictions and controversial judgments. On the other hand, the history of mathematics is rife with examples of statements and proofs that were revealed to be erroneous many years after their first appearance, while the search for a concept of absolute truth seems naïve after Gödel and Turing, at least according to scientific methodology. To

assert that when sufficient data and computational ability are available neither controversial statements nor mistakes will ever appear implies a vision of science that totally leaves out the process of research and discovery. Rejecting concepts such as contradiction and incomprehensibility might (perhaps) be acceptable in some philosophical thought experiment, but it would seem foolhardy in the attempt to study the real processes of intelligence. Studying intelligence after eliminating all references to inconsistency, madness, misunderstanding (and so on) would be similar to studying biology after eliminating any references to death.

- **Objection i:** *"What is the point of this paper? What is the point of proving the link between intelligence and contradiction?"*

Answer: The point of this paper is, in the first place, to construct a mathematical framework where the concepts of intelligence and contradiction can be represented and formally treated. In the second place, it is to suggest a possible link between these two concepts, which emerges as a straightforward consequence of our definitions. Knowing whether such a link really exists seems important, both from a theoretical and a practical point of view. Attempts to avoid contradiction might be dangerous, both in software engineering and in Artificial Intelligence. A general approach to the problem appears to be useful. In any case, the main purpose of this paper is to define an issue and its relevance in scientific terms, not to fully work out the corresponding answer. The results found herein should be seen only as the necessary and quite straightforward consequence of a particular mathematical model.

8. Conclusions

In this paper we have proposed some formal definitions of the concepts of observer, entity, intelligence and contradiction. On this basis we have proved that any sufficiently intelligent entity $\mathcal{E}$ *must* be contradictory for any observer with bounded capabilities, under the assumptions that the lifetime of $\mathcal{E}$ is finite and that the environment is deterministic.

In practice, we know that the more intelligent a living being is, the more difficult it is to predict its behavior by means of deterministic rules. This is another way of expressing the previous statement.

We have also performed some computational experiments showing that our theoretical conclusions are supported by empirical evidence.

Our attempt to define a mathematical model in which we can study the relations between contradiction and intelligence is obviously only a subjective proposal. However, a systematic approach to problems involving the active role of contradiction in intelligent beings seems at this point to be essential to the study of complex systems.

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