Atti Sem. Mat. Fis. Univ. Modena, XLVII, 271-292 (1999)

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Metric Homotopies (**).

Abstract. – *In this paper the problem of defining some tools in order to capture the intuitive concept of «shape of an object» is studied and some approaches and solutions are considered together with some of their properties.*

Introduction.

The ordinary tools of (algebraic) topology do not yet allow an automatic and satisfactory treatment of «shape» in the intuitive, ordinary sense, as pointed out for example in [J]: everyone considers different the shapes of a dumbbell and of a sphere although the manifolds that they represent are diffeomorphic. Nevertheless, homotopy is one of the most successful theories in the study of topological spaces, and particularly of manifolds. This paper is intended to try to bridge the gap, by considering length and other real functions as additional ingredients in the definition of some equivalence relations for loops and ordered k-tuples of points. The objects treated here are compact submanifolds of Euclidean spaces. There aren't any compelling reasons for treating embedded manifolds and not simply Riemannian manifolds but we prefer to pro-

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(**) Nota giunta in Redazione il 25-VI-1996.
Work performed under the auspices of the GNSAGA-CNR, within the Project «Geometria reale e complessa» of the MURST, with the support of INdAM and under the supervision of Prof. M. Ferri.
1991 Mathematics Subject Classification: Primary 57R40; Secondary 53C22.

ceed so for two motives. The first one is that the intuitive concept of shape is not independent of the considered immersion in the environment space: no one would say that the surface of a bowling ball and the one of a sphere with some protuberances have the same shape, also in the case that they are isometric. Anyway, not all the tools proposed in this paper will be so powerful to take the immersion into consideration. The second reason is that this procedure simplifies the exposition. The nature of the equivalence classes is studied: their finiteness in the most interesting cases and their topological characteristics. We point out that the knowledge of the number of such equivalence classes (that is the functions f_1^L, f_m^V, f_k^D that will be defined in this paper respectively for the relation of L-, V- and D-homotopy) for two manifolds allows a quantitative evaluation of the difference between the shapes of such manifolds measured by using a suitable distance (which will be described in a forthcoming paper). However the purpose of this paper is not that of illustrating the usefulness of this approach to the problem but that of defining the necessary concepts and of studying their properties by inserting the treatment in a general context. A longer term goal is to endow the sets of equivalence classes with some sort of structure, to be compared with the classical ones (homotopy groups, regions of constant sign curvature ...). Section 1 contains the general definitions. Section 2 treats the L-homotopy of loops and the related function f_1^L. Section 3 is the core of the paper: it provides theorems on the L-homotopy of sufficiently close loops, on the finiteness of the set of $y-L$-homotopy classes, on the existence of minimal loops in each class, on the relation of L-homotopy with geodesics (much in the spirit of [Kl, 2.1.3], [Ha, section 37]) and on the L-homotopy of the even-dimensional orientable manifolds with sectional curvature everywhere positive. Section 4 uses discrete sets of points instead of loops, substituting length with either volume or diameter. Section 5 points out results and examples concerning this last approach to the problem.

1. – The general approach to the problem.

In order to construct a method to distinguish the shapes of two manifolds in accordance with the intuitive concept of shape we can proceed in the following way. First of all we can consider a topological space $\mathfrak{I}$ linked with the studied manifold $\mathfrak{M}$ in a way that we choose arbitrarily (in this paper we shall illustrate some possible choices) and define on $\mathfrak{I}$ a continuous real function φ. Then for every $x \in \mathbb{R}$ we can define $\mathfrak{I}_x$ as the set $\{\mu \in \mathfrak{I} : \varphi(\mu) \leq x\}$. Furthermore for every $y \in \mathbb{R}$ we can define the equivalence relation $\mathcal{R}_y$ on $\mathfrak{I}$ by saying that for $\mu, \nu \in \mathfrak{I}$ we have $\mu \mathcal{R}_y \nu$ if and only if either $\mu = \nu$ or a continuous function $\mathrm{H}_{\mu\nu}: [0, 1] \to \mathfrak{I}$ exists such that $\mathrm{H}_{\mu\nu}(0) = \mu$, $\mathrm{H}_{\mu\nu}(1) = \nu$ and $\varphi(\mathrm{H}_{\mu\nu}(\tau)) \leqslant y$ for $\tau \in [0, 1]$. Finally we can define $f(\mathcal{M}, x, y)$ as the number of equivalence classes in which $\mathfrak{I}_x$ is divided by $\mathcal{R}_y$. Obviously the «complexity» of $f(\mathcal{M}, \cdot, \cdot)$ is linked with the «complexity» of the shape of $\mathcal{M}$ and this is the reason for which we are interested in such a function (in this paper we shall also study some properties of the function f and expose some examples). In the following sections we shall use the above-mentioned method in order to define the concepts of L-homotopy, V-homotopy and D-homotopy. In L-homotopy $\mathfrak{I}$ will be the set of all the piecewise C^1 loops in a manifold $\mathcal{M}$ with the topology induced by an opportune distance d and φ will be the length function. In V-homotopy $\mathfrak{I}$ will be the set of all the ordered $(m+1)$-tuples of points of a manifold $\mathcal{M} \subset \mathrm{E}^m$ with the topology induced by an opportune distance d_m and φ will be the oriented volume function applied to the convex hulls of the considered $(m+1)$-tuples. In D-homotopy $\mathfrak{I}$ will be the set of all the ordered $(k+1)$-tuples ($k \in \mathbb{N}$) of points of a manifold $\mathcal{M} \subset \mathrm{E}^m$ with the topology induced by an opportune distance d_k and φ will be the diameter function applied to the convex hulls of the considered $(k+1)$-tuples.

REMARK 1.1. As a matter of fact we could define the sets $\mathfrak{I}_x$ and the relations $\mathcal{R}_y$ in any way, without considering a topology on $\mathfrak{I}$ and the continuous functions φ and H and therefore we could apply such a methodology also to study a set devoid of a topological structure.

2. – L-homotopy: definitions.

DEFINITION 2.1. Let $k \in \mathbb{N}$, $k \geqslant 1$. An n-dimensional topological submanifold $\mathcal{M}_n$ of E^m is said to be *piecewise* C^k (or C^k *triangulable*) if there exist an n-dimensional PL manifold $\mathcal{V}_n = |K_n|$ (K_n combinatorial manifold) and a homeomorphism $\varphi: \mathcal{V}_n \to \mathcal{M}_n$ such that the restriction of φ to $|\bar{s}|$ for every n-simplex s of K_n is a diffeomorphism of class C^k. φ will be said to be a *piecewise* C^k *diffeomorphism*. In this case if g is a function from $\mathcal{M}_n$ to another C^k manifold $\mathcal{N}_m$ such that $g \circ \varphi$ is a continuous function from $\mathcal{V}_n$ to $\mathcal{N}_m$ and the restriction of $g \circ \varphi$ to $|\bar{s}|$ for every n-simplex s of K_n is a C^k function then g will be said a *piecewise* C^k *function*.

NOTE. Because of a well-known theorem every C^∞ compact manifold is also piecewise C^∞ (see [C] and [F]).

DEFINITION 2.2. Let $\mathcal{M}$ be a piecewise C^∞ n-submanifold $(n > 0)$ of $\mathbb{E}^m$ and let $\mathbb{P}(\mathcal{M})$ be the set of all piecewise C^1 (and therefore rectifiable) loops as maps from $\mathbb{I} = [0, 1]$ to $\mathcal{M}$. We shall denote respectively by $d\pi/dt^+$ and $d\pi/dt^-$ the right and the left derivative of the loop π with respect to the parameter t and by $\mathcal{L}(\pi) = \int_0^1 \left\| \frac{d\pi}{dt^+}(t) \right\| dt$ the length of π. Then for every real number y we can define an equivalence relation on $\mathbb{P}(\mathcal{M})$ and all its subsets: for every pair (α, β) in $\mathbb{P}(\mathcal{M})^2$ we shall say that α and β are *$y - L$-homotopic* if either they are the same loop ar a function $H(t, \tau)$ exists such that:

i) $H \in C^0(\mathbb{I} \times \mathbb{I}, \mathcal{M})$,

ii) $H(t, 0) = \alpha(t)$ and $H(t, 1) = \beta(t)$ for every $t \in \mathbb{I}$,

iii) $H(0, \tau) = H(1, \tau)$ for every $\tau \in \mathbb{I}$,

iv) for every fixed $\tau \in \mathbb{I}$, defining $\pi_\tau(t) = H(t, \tau)$ for $t \in \mathbb{I}$, we have that $\pi_\tau \in \mathbb{P}(\mathcal{M})$ and $\mathcal{L}(\pi_\tau) \leqslant y$.

In such a case we shall write $\alpha \overset{L}{\underset{y}{=}} \beta$ and say that H is a *$y - L$-homotopy* between α and β. In the case $y \geqslant \mathcal{L}(\alpha)$ the symbol $\alpha \overset{L}{\underset{y}{=}} \beta$ means, in plain words, that we can «transform with continuity α into β without exceeding the length y».

On $\mathbb{P}(\mathcal{M})$ we can define a distance: for every pair $(\alpha, \beta) \in \mathbb{P}(\mathcal{M})^2$ we set $d(\alpha, \beta) = \sup_{t \in \mathbb{I}} \|\alpha(t) - \beta(t)\|$.

REMARK 2.1. In this paper the distance on $\mathcal{M}$ will be always the one induced by the Euclidean metric in $\mathbb{E}^m$ (obviously the topology that we obtain this way is equal to the one induced by the geodetic distance). Furthermore we point out that $d(\alpha, \beta)$ indicates the sup not of the geodetic but the Euclidean distances between corresponding points in the loops α and β. Moreover we observe that the function length is not continuous with respect to the topology induced by the distance d. In spite of this fact we shall use the distance d: the reason is that Lemma 3.2 in the following section does not need a stronger distance in order to work.

DEFINITION 2.3. Let x and y be real numbers. Let us denote by $\mathbb{P}(\mathcal{M}, x)$ the subset of $\mathbb{P}(\mathcal{M})$ containing the loops of length less than or equal to x. Let us denote by $f_1^L(\mathcal{M}, x, y)$ the number of equivalence classes into which $\mathbb{P}(\mathcal{M}, x)$ is divided by the relation of $y - L$-homotopy if such a number is finite, $+\infty$ otherwise.

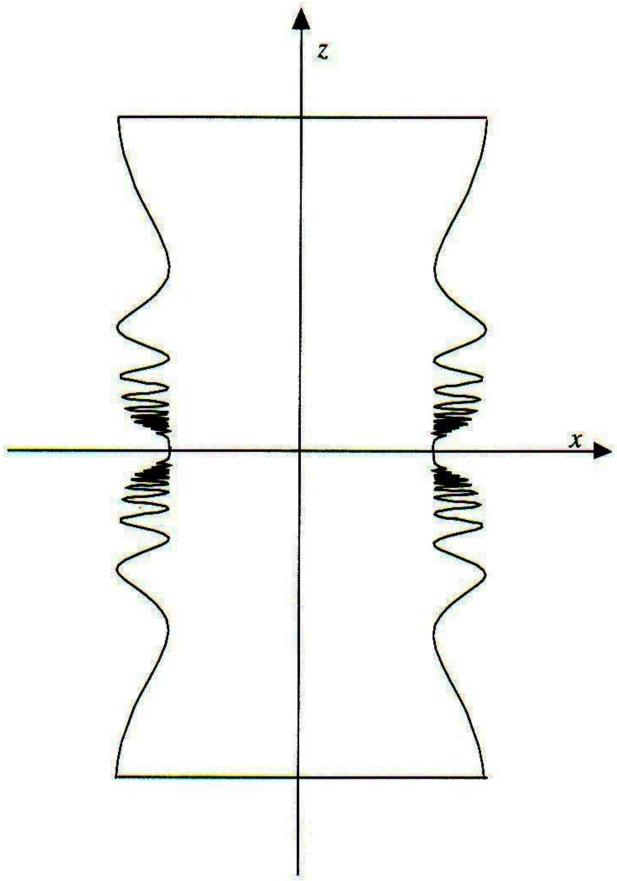

Fig. 2.1.

NOTE. The usefulness of the index «1» in the symbol «$f_1^L(\mathcal{M}, x, y)$» will be clear in section 3, Remark 3.4.

REMARK 2.2. Of course if $x \geq 0$ and $x > y$ then $f_1^L(\mathcal{M}, x, y) = +\infty$ and if $x < 0$ then $f_1^L(\mathcal{M}, x, y) = 0$. The values f_1^L are interesting for $0 \leq x \leq y$. However it is possible that $f_1^L(\mathcal{M}, x, y) = +\infty$ also in the case $0 \leq x = y$, as the following example shows.

EXAMPLE 2.1. Let us consider the piecewise C^∞ 2-manifold $\mathcal{M}_1 \subset \mathbb{E}^3$ obtained by the rotation around the z-axis of the curve represented in figure 2.1. The equation of the right hand side curve (the left hand side one is symmetrical) is the following:

$$x = \frac{1}{4} + \frac{1}{10} \cdot \exp\left(-\frac{1}{10^3 \cdot z^2}\right) \cdot \sin^2\left(\frac{1}{z}\right) \quad \text{for } z \neq 0 \text{ and } x = \frac{1}{4} \text{ for } z = 0 .$$

It is immediate to verify that $f_1^L(\mathcal{M}, \pi/2, \pi/2) = +\infty$.

REMARK 2.3. We prefer to study the length function and not the energy function of paths: this choice simplifies some results connecting the function f_1^L to the possibility of measuring how different the shapes of two manifolds are (See the Introduction).

3. – L-homotopy: basic results.

Theorem 3.1 will show that the situation is not as bad as it seems with regard to the possibility that f_1^L has $+\infty$ as a value. In order to prove it we must use the following Lemmas (3.1) and (3.2) and Corollary (3.1). Furthermore in this section we will point out other useful results. The following theorems are well-known under hypotheses similar to the ones we are interested in and their proofs are given here only for seek of completeness although it is possible to adapt to our purposes methods and results exposed, for instance, in [Kl] or M]. The reason for which we don't proceed this way is that the classical results concern the function «energy» and not the function «length» (we have already explained why we prefer to use the latter). Moreover they deal with the minima, not in every single equivalence class in $\mathbb{P}(\mathcal{M}, x)$ with respect to $y-L$-homotopy, but globally either in $\mathbb{P}(\mathcal{M}, x)$ or in the homotopy classes of $\mathbb{P}(\mathcal{M}, x)$ (clearly we are interested in the first case). So, all things considered, it is simpler to use special proofs.

LEMMA 3.1. *Let $\mathcal{M}$ be a piecewise C^∞ n-submanifold of E^m. If $g\colon \mathrm{I} \to \mathrm{I}$ is a piecewise C^1 function with $g(0)=0$ and $g(1)=1$, $\pi_1 \in \mathbb{P}(\mathcal{M}, x)$ and $\pi_2 = \pi_1 \circ g$ then $\pi_2 \in \mathbb{P}(\mathcal{M}, x)$ and $\pi_1 \underset{y}{\overset{L}{=}} \pi_2$ for every $y \geq x$.*

PROOF. It is trivial to verify that $\pi_2 \in \mathbb{P}(\mathcal{M}, x)$. Considering the $x-L$-homotopy $H(t, \tau) = \pi_1(t\cdot(1-\tau) + g(t)\cdot\tau)$ we have that $\pi_1 \underset{x}{\overset{L}{=}} \pi_2$. Therefore $\pi_1 \underset{y}{\overset{L}{=}} \pi_2$ for every $y \geq x$. ∎

COROLLARY 3.1. *For every $\pi \in \mathbb{P}(\mathcal{M}, x)$ with $\mathcal{M}$ piecewise C^∞ n-submanifold of E^m there exists another loop $\tilde{\pi} \in \mathbb{P}(\mathcal{M}, x)$ that is $x-L$-homotopic to π and such that $\left\| \frac{d\tilde{\pi}}{d\lambda^+}(\lambda) \right\| < x+1$ for $\lambda \in [0,1)$ and $\left\| \frac{d\tilde{\pi}}{d\lambda^-}(\lambda) \right\| < x+1$ for $\lambda \in (0,1]$ (λ parameter of $\tilde{\pi}$).*

PROOF. We can consider the function $\lambda_\pi:[0,1]\to[0,1]$ so defined:

$$\lambda_\pi(t) = \frac{t + \int_0^t \left\| \frac{d\pi}{du^+}(u) \right\| du}{\mathcal{L}(\pi)+1}.$$

This function is continuous and invertible (because it is a monotone increasing function). Moreover, by calling $t_1, \ldots, t_{k-1}$ the points where λ_π is not differentiable and setting $t_0 = 0$ and $t_k = 1$, we have that the restriction of λ_π to every closed interval $[t_j, t_{j+1}]$ for $0 \leqslant j \leqslant k-1$ is C^1 with positive derivative. So its inverse function $t_\pi(\lambda)$ is also continuous (because the domain of λ_π is compact) and its restriction to every closed interval $[\lambda_\pi(t_j), \lambda_\pi(t_{j+1})]$ for $0 \leqslant j \leqslant k-1$ is C^1 with positive derivative. Therefore t_π is a piecewise C^1 diffeomorphism. Moreover $t_\pi(0) = 0$ and $t_\pi(1) = 1$ and by Lemma 3.1 we have $\pi \overset{L}{\underset{x}{=}} \tilde{\pi}$ if we define $\tilde{\pi} = \pi \circ t_\pi$. We can verify that $\left\| \frac{d\tilde{\pi}}{d\lambda^+}(\lambda) \right\| < \mathcal{L}(\pi) + 1 \leqslant x + 1$ for every $\lambda \in [0,1)$ $\Bigg($in fact we observe that if $\bar{t} \in [0,1]$ then $\left\| \frac{d\tilde{\pi}}{d\lambda^+}(\lambda_\pi(\bar{t})) \right\| =$
$= \left\| \frac{d\pi}{dt^+}(\bar{t}) \right\| \Big/ \left| \frac{d\lambda_\pi}{dt^+}(\bar{t}) \right| = \left\| \frac{d\pi}{dt^+}(\bar{t}) \right\| \cdot (\mathcal{L}(\pi)+1) \Big/ \left(1 + \left\| \frac{d\pi}{dt^+}(\bar{t}) \right\| \right) <$
$< \mathcal{L}(\pi) + 1 \Bigg)$.

Likewise, we have that if $\lambda \in (0,1]$ then

$$\left\| \frac{d\tilde{\pi}}{d\lambda^-}(\lambda) \right\| < \mathcal{L}(\pi) + 1 \leqslant x + 1. \quad \blacksquare$$

LEMMA 3.2. *For every closed (i.e. compact and without boundary) C^∞ submanifold $\mathfrak{M}$ of $\mathbb{E}^m$ and every pair (ε, x) with $\varepsilon > 0$, $x \geqslant 0$, a $\delta = \delta(\mathfrak{M}, \varepsilon, x) > 0$ exists such that if two loops α and β in $\mathrm{P}(\mathfrak{M}, x)$ are distant less than δ with respect to the distance $d(\alpha,\beta) = \sup_{t\in \mathrm{I}} \|\alpha(t) - \beta(t)\|$ defined on $\mathrm{P}(\mathfrak{M})$ then they are $y - L$-homotopic for every $y \geqslant x + \varepsilon$.*

REMARK 3.1. Before proving the lemma, if we consider an n-dimensional *PL* manifold $\mathcal{V}_n = |K_n|$ (K_n combinatorial manifold), we observe that if $\pi_1, \pi_2 \mathrm{P}(\mathcal{V}_n, x)$ and for every $t \in \mathrm{I}$ $\pi_1(t)$ and $\pi_2(t)$ belong to the space $|\bar{s}|$ of the same n-simplex s of K_n then π_1 and π_2 are $x - L$-homotopic (and therefore also $x-L$-homotopic for every $y>x$) because the function $H_{\pi_1\pi_2}(t,\tau)=\pi_1(t)\cdot(1-\tau)+\pi_2(t)\cdot\tau$ is an $x-L$-homotopy between π_1 and π_2 in $\mathcal{V}_n$ (in fact $\mathcal{L}(H_{\pi_1\pi_2}(t,\tau))\leqslant\mathcal{L}(\pi_1)\cdot(1-\tau)+\mathcal{L}(\pi_2)\cdot\tau\leqslant x$ for every $\tau\in\mathrm{I}$). therefore Lemma 3.2 is not anti-intuitive, in spite of the fact that the distance d does not involve the derivatives of the paths.

PROOF OF LEMMA 3.2. Because of the compactness of $\mathcal{M}$ a $\delta_1>0$ exists such that if two points P, Q of $\mathcal{M}$ are distant less than δ_1 then there exists one and only one minimal geodesic $\gamma(P,Q)\colon\mathrm{I}\to\mathcal{M}$ with $\gamma(P,Q)(0)=P$ and $\gamma(P,Q)(1)=Q$ which depends smoothly on its end-points P, Q (See [M], Ch. II, section 10, Corollary 10.8, p. 62). Therefore if $d(\alpha,\beta)<\delta_1$ we can define $H_{\alpha\beta}\colon\mathrm{I}\times\mathrm{I}\to\mathcal{M}$ as $H_{\alpha\beta}(t,\tau)=$ $=\gamma(\alpha(t),\beta(t))(\tau)$ for $t,\tau\in\mathrm{I}$ and say, since α and β are piecewise C^1, that $H_{\alpha\beta}$ is a piecewise C^1 function. In particular $H_{\alpha\beta}$ is continuous and, defining $\pi_\tau(t)=H_{\alpha\beta}(t,\tau)$ for $(t,\tau)\in\mathrm{I}\times\mathrm{I}$, it results $\pi_\tau\in\mathrm{P}(\mathcal{M})$ for every fixed $\tau\in\mathrm{I}$. Now let us study the length of π_τ for every $\tau\in\mathrm{I}$. For every $\eta>0$ we have:

i) Since $H_{\alpha\beta}$ is piecewise C^1 on the compact $\mathrm{I}\times\mathrm{I}$ we can choose a finite set $\{t_0,t_1,\dots,t_k\}$ for which

$$\mathcal{L}(\pi_\tau)\leqslant(1+\eta)\cdot\sum_{i=0}^{k-1}\|H_{\alpha\beta}(t_{i+1},\tau)-H_{\alpha\beta}(t_i,\tau)\|$$

for every $\tau\in\mathrm{I}$.

ii) Because of the properties of the geodesics a δ_2 exists with $0<\delta_2\leqslant\delta_1$ such that if $d(\alpha,\beta)<\delta_2$ then

$$\|H_{\alpha\beta}(t_{i+1},\tau)-H_{\alpha\beta}(t_i,\tau)\|\leqslant$$
$$\leqslant(1+\eta)\cdot\|(\alpha(t_{i+1})-\alpha(t_i))\cdot(1-\tau)+(\beta(t_{i+1})-\beta(t_i))\cdot\tau\|$$

for every $\tau\in\mathrm{I}$. So, if $d(\alpha,\beta)<\delta_2$ then by i) and ii) we have

$$\mathcal{L}(\pi_\tau)\leqslant(1+\eta)^2\cdot\sum_{i=0}^{k-1}\|(\alpha(t_{i+1})-\alpha(t_i))\cdot(1-\tau)+(\beta(t_{i+1})-\beta(t_i))\tau\|\leqslant$$
$$\leqslant(1+\eta)^2\cdot\left((1-\tau)\cdot\sum_{i=0}^{k-1}\|\alpha(t_{i+1})-\alpha(t_i)\|+\tau\cdot\sum_{i=0}^{k-1}\|\beta(t_{i+1})-\beta(t_i)\|\right)\leqslant$$
$$\leqslant(1+\eta)^2\cdot((1-\tau)\cdot\mathcal{L}(\alpha)+\tau\cdot\mathcal{L}(\beta))\leqslant(1+\eta)^2\cdot x\,.$$

For $\eta\leqslant\sqrt{1+\varepsilon/x}-1$ we obtain that $\mathcal{L}(\pi_\tau)\leqslant x+\varepsilon$ and therefore

$H_{\alpha\beta}$ is an $(x+\varepsilon)-L$-homotopy. For $\delta(\mathcal{M}, \varepsilon, x) = \delta_2$ the thesis is proved. ■

REMARK 3.2. Lemma 3.2 implies that for $y > x$ every equivalence class in $\mathbb{P}(\mathcal{M}, x)/\overset{L}{\underset{y}{=}}$ is a closed and open set in $\mathbb{P}(\mathcal{M}, x)$ with respect to the topology induced by the distance $d(\alpha, \beta) = \sup_{t \in \mathbb{I}} \|\alpha(t) - \beta(t)\|$.

Moreover we observe that if C is an equivalence class in $\mathbb{P}(\mathcal{M}, x)/\overset{L}{\underset{x}{=}}$ and for every positive natural number k C_k is the equivalence class in $\mathbb{P}(\mathcal{M}, x)/\overset{L}{\underset{(x+1/k)}{=}}$ that contains C then $C = \bigcap_{k=1}^{\infty} C_k$ and so C is a closed set because all the sets C_k are closed.

THEOREM 3.1. *If $\mathcal{M}$ is a closed C^∞ submanifold of $\mathbb{E}^m$ and $x < y$ then $f_1^L(\mathcal{M}, x, y) < +\infty$.*

PROOF (INDIRECT). Let us suppose $f_1^L(\mathcal{M}, x, y) = +\infty$. Then $x \geqslant 0$ and there is an infinite set S of loops in $\mathbb{P}(\mathcal{M}, x)$ pairwise non-$y-L$-homotopic. By using Corollary 3.1 (that is replacing every loop π with $\tilde{\pi}$) we can suppose that all the loops in S have right and left derivative of norm less than $x+1$ and therefore that the loops in S are equiuniformly continuous. Therefore (by recalling also that $\mathcal{M}$ is compact) we can use the generalized Arzelà's theorem with respect to the distance $d(\alpha, \beta) = \sup_{t \in \mathbb{I}} \|\alpha(t) - \beta(t)\|$ (See [KF], Ch. II, Section 18, Theorem 7) and say that there is a sequence of loops of S that converges in $C^\infty(\mathbb{I}, \mathcal{M})$ with respect to the topology induced by the above-mentioned distance. This fact contradicts Lemma 3.2. ■

THEOREM 3.2. *Let $\mathcal{M}$ be a closed C^∞ submanifold of $\mathbb{E}^m$. Then, for chosen $x, y \in \mathbb{R}$ with $x \geqslant 0$, every equivalence class of $\mathbb{P}(\mathcal{M}, x)$ with respect to $y-L$-homotopy contains at least one loop $\hat{\pi}$ which is a global minimum point in the considered class for the function $\mathcal{L}(\pi)$ (length of π).*

PROOF. If $x > y$ the thesis is trivial because every equivalence class contains only one loop. On the opposite, if $x \leqslant y$, we shall prove the existence of a global minimum point for $\mathcal{L}$ in the considered class by constructing a sequence of loops converging to such a point. We proceed so: for every $y-L$-homotopy class C of $\mathbb{P}(\mathcal{M}, x)$ we can choose in it a sequence $(\pi_i)_{i \in \mathbb{N}}$ of loops such that for every $i \in \mathbb{N}$ π_i is $y-L$-homotopic to π_{i+1} and $\lim_{i \to \infty} \mathcal{L}(\pi_i) = \omega$, where ω is the inf of the lengths of the loops in the considered class. Unfortunately this is not the «right» sequence yet because it may not converge to a piecewise C^1 loop but now we shall use it to define another sequence $(\widehat{\pi}_i)_{i\in\mathbb{N}}$ (whose loops will be obtained replacing «little arcs» of the π_i's with geodetic arcs) which will converge to a global minimum point for $\mathcal{L}$ in C. First of all by Corollary 3.1 we can replace every π_i with $\tilde{\pi}_i$: however, for the sake of simplicity, we shall continue to use the symbol t for the parameter and the symbol π_i for the loop.

If $y > x$ let us proceed in the following way. Let us consider $\delta = \delta(\mathcal{M}, (y-x)/2, x)$ such as in Lemma 3.2 and the points $t_j = (j\cdot\delta/2)/(x+1)$ $(j\in\mathbb{N})$ such that $0 \leqslant t_j \leqslant 1$. Moreover, if t_h is the greatest of the considered points t_j and is not 1, set $t_{h+1} = 1$: let T be the set of the points t_j included in $[0, 1]$. For every loop π_i and every $\mathrm{I}_j = [t_j, t_{j+1}]$ with t_j, $t_{j+1} \in T$ we choose in $\mathcal{M}$ a C^∞ path $\beta_i^j : [0, 1] \to \mathcal{M}$ of minimum length such that $\beta_i^j(0) = \pi_i(t_j)$ and $\beta_i^j(1) = \pi_i(t_{j+1})$: it will exist because of the compactness of $\mathcal{M}$ (see [B], Ch. VII, Section 7, Lemma 7.8 and Theorem 7.7). If β_i^j is not constant we shall suppose that it is parameterized by the parameter $\tau_{ij} = s_{ij}/\mathcal{L}(\beta_i^j)$ where s_{ij} and $\mathcal{L}(\beta_i^j)$ are respectively the curvilinear abscissa and the length of β_i^j: so it will be a geodesic. For the properties of paths of minimum length and geodesics see [B]. Then for every $i\in\mathbb{N}$ if $t_j, t_{j+1}\in T$ we define $\widehat{\pi}_i(t) = \beta_i^j((t-t_j)/(t_{j+1}-t_j))$ for $t \in \mathrm{I}_j$. Since it results $\|\widehat{\pi}_i(t) - \pi_i(t)\| \leqslant \delta$ for $t \in \mathrm{I}$ (in fact, because of our choice of δ and the chosen parameterization of the loops π_i, it results $\|\pi_i(t) - \pi_i(t_j)\| \leqslant \delta/2$ and $\|\widehat{\pi}_i(t) - \widehat{\pi}_i(t_j)\| \leqslant \delta/2$, and moreover we have $\widehat{\pi}_i(t_j) = \pi_i(t_j)$ for every $t_j \in T$, $t \in \mathrm{I}_j$), we can say that the loops $\widehat{\pi}_i(t)$ are all $y - L$-homotopic by using Lemma 3.2.

Finally, if $y = x$ we can observe that there are two possibilities: either all the loops π_i have the least length in their equivalence class in $P(\mathcal{M}, x)$ with respect to $y - L$-homotopy or not. In the first case the thesis of theorem 3.2 is trivial, in the second one we can repeat the same above-showed argument after replacing x with $\mathcal{L}(\pi_k) < x$ for a great enough k and the sequence $(\pi_i)_{i\in\mathbb{N}}$ with the sequence $(\pi_{i+k})_{i\in\mathbb{N}}$. So we can suppose that we are not in the trivial case and that the loops $\widehat{\pi}_i(t)$ are all $y - L$-homotopic. Moreover we have $\lim_{i\to\infty} \mathcal{L}(\widehat{\pi}_i) = \lim_{i\to\infty} \mathcal{L}(\pi_i) = \omega$, because for every i $\mathcal{L}(\widehat{\pi}_i) \leqslant \mathcal{L}(\pi_i)$. Now we can extract from $(\widehat{\pi}_i)_{i\in\mathbb{N}}$ a subsequence $(\widehat{\pi}_{i_k})_{k\in\mathbb{N}}$ such that the corresponding sequences $(\beta_{i_k}^j(t_j))_{k\in\mathbb{N}}$ and $((d/d\tau_{i_kj}^+)\, \beta_{i_k}^j(t_j))_{k\in\mathbb{N}}$ converge for every $t_j \in T$. In fact we know that geodesics are smooth paths (see [B], Ch. VII, Section 5) and therefore that the functions $\beta_{i_k}^j$ are right-derivable at every $t_j \in T$ with $0 \leqslant t_j < 1$. The subsequence exists becaues $\mathcal{M}$ is compact and because $\|(d/d\tau_{i_kj}^+)\, \beta_{i_k}^j(t_j)\| = \mathcal{L}(\beta_{i_k}^j) \leqslant x$ for every $k\in\mathbb{N}$ and $t_j \in T$ with $0 \leqslant$

$\leq t_j < 1$ (for $\beta^j_{i_k}$ non-constant the reason is in the chosen parameterization τ_{ij}) and therefore the above-mentioned sequences are both bounded. For the sake of simplicity we shall continue to call $(\widehat{\pi}_i)_{i\in\mathbb{N}}$ the extracted subsequence. Now let us consider the loop $\widehat{\pi}$ so defined: for every $t_j \in T$ with $0 \leq t_j < 1$ let $\beta^j : [0, 1] \to \mathcal{M}$ be the C^∞ path that is solution of the system of differential equations that defines geodesics with the initial conditions $\beta^j(0) = \lim_{i\to\infty} \beta^j_i(t_j)$ and $(d/d\tau_j^+)\beta^j(0) = \lim_{i\to\infty} (d/d\tau_{ij}^+)\beta^j_i(t_j)$ (τ_j is the parameter of the solution β^j). By using the theorem about the dependence of the solution of a system of differential equations on the initial conditions (see [Hu] and [B]) we can prove that for every j with $t_j \in T$ and $0 \leq t_j < 1$ $(\beta^j_i)_{i\in\mathbb{N}}$ converges uniformly to β^j in $\mathbb{I}_j$ and $((d/d\tau_{ij}^+)\beta^j_i)_{i\in\mathbb{N}}$ converges uniformly to $(d/d\tau_j^+)\beta^j$ in $\mathbb{I}_j$. Therefore by setting $\widehat{\pi}(t) = \beta^j((t-t_j)/(t_{j+1}-t_j))$ for $t \in \mathbb{I}_j$ we have that $(\widehat{\pi}_i)_{i\in\mathbb{N}}$ converges uniformly to $\widehat{\pi}$ in $\mathbb{I}$, that $\widehat{\pi} \in \mathbb{P}(\mathcal{M})$ and that $\mathcal{L}(\widehat{\pi}) = \lim_{i\to\infty} \mathcal{L}(\widehat{\pi}_i) = \omega$. So, by Lemma 3.2 (which implies the closure in $\mathbb{P}(\mathcal{M}, x)$ of the class of $y-L$-homotopy of the paths $\widehat{\pi}_i$) $\widehat{\pi}$ is $y-L$-homotopic to every loop in the sequence $(\widehat{\pi}_i)_{i\in\mathbb{N}}$ and $\widehat{\pi}$ is the wanted global minimum point for $\mathcal{L}$ in C. ∎

THEOREM 3.3. *Let $\mathcal{M}$ be a C^∞ n-submanifold without boundary of $\mathbb{E}^m$. If $0 \leq x \leq y$ and $\alpha \in \mathbb{P}(\mathcal{M}, x)$ has the minimum length in its equivalence class of $\mathbb{P}(\mathcal{M}, x)/\overset{L}{\underset{y}{=}}$ then α is $x-L$-homotopic to a closed geodesic obtained by a reparameterization of α.*

PROOF. If α is constant then it is already a closed geodesic. Let us examine the case of α non-constant. We know that, by setting

$$g(t) = \frac{1}{\mathcal{L}(\alpha)} \cdot \int_0^t \left\| \frac{d\alpha}{du^+}(u) \right\| du$$

(i.e a multiple of curvilinear abscissa), a loop $\beta \in \mathbb{P}(\mathcal{M}, x)$ exists such that $\beta \circ g = \alpha$. It results $\alpha \overset{L}{\underset{x}{=}} \beta$ by Lemma 3.1 because g is piecewise C^1, $g(0) = 0$ and $g(1) = 1$. Now let us prove that β is a geodesic. First of all we observe that because of the compactness of $\mathcal{M}$ a $\delta > 0$ exists such that if two points P, Q whichever of $\mathcal{M}$ are distant less than δ with respect to the distance induced by the Euclidean distance on $\mathbb{E}^m$ then there exists one and only one minimal geodesic $\gamma(P, Q): \mathbb{I} \to \mathcal{M}$ with $\gamma(P, Q)(0) = P$ and $\gamma(P, Q)(1) = Q$ which depends smoothly on its endpoints P, Q (cf. [M], Ch. II, Section 10, Corollary 10.8, p. 62). If β were not a geodesic then (in case by translating of a constant modulus 1 the parameter t) there would be two numbers t_1 and t_2 with $0 \leq t_1 < t_2 \leq 1$ and $\|\beta(t_1) - \beta(t)\| < \delta$ for

$t \in [t_1, t_2]$ such that

$$\int_{t_1}^{t_2} \left\| \frac{d\beta}{dt^+}(t) \right\| dt > \int_0^1 \left\| \frac{d}{dt}\gamma(\beta(t_1), \beta(t_2))(u) \right\| du$$

(see [B], Ch. VII, Section 7, Corollary 7.5).

So, by defining

$$\widehat{\beta}(t) = $$
$$= \begin{cases} \gamma(\beta(t_1), \beta(t_2))((t-t_1)/(t_2-t_1)) & \text{if } t_1 \leqslant t \leqslant t_2, \\ \beta(t) & \text{it either } 0 \leqslant t < t_1 \text{ or } t_2 < t \leqslant t_1 \end{cases}$$

we could say that $\widehat{\beta} \in \mathbb{P}(\mathcal{M}, x)$. Moreover, by defining

$$H(t, \tau) =$$
$$= \begin{cases} \gamma(\beta(t_1), \beta(t_1 + \tau\cdot(t_2 - t_1)))((t-t_1)/(\tau\cdot(t_2-t_1))) \\ \qquad\qquad \text{if } t_1 \leqslant t \leqslant t_1 + \tau\cdot(t_2 - t_1) \text{ and } 0 < \tau \leqslant 1, \\ \beta(t) \quad \text{it either } 0 \leqslant t < t_1 \text{ or } t_1 + \tau\cdot(t_2-t_1) < t \leqslant 1 \text{ or } \tau = 0 \end{cases}$$

we would have that H is an $x - L$-homotopy between β and $\widehat{\beta}$. This fact would be a contradiction because $\mathcal{L}(\widehat{\beta}) < \mathcal{L}(\beta)$ and we have supposed that β is a global minimum point for the function $\mathcal{L}$ in its equivalence class in $\mathbb{P}(\mathcal{M}, x) / \overset{L}{\underset{y}{=}}$. So β is the wanted geodesic. ■

NOTE. We point out that for $y = x$ the existence of a $y - L$-homotopy between β and $\widehat{\beta}$ shown in the previous proof cannot be simply derived from Lemma 3.2 by using the fact that $d(\beta, \widehat{\beta})$ is small, because that lemma does not work for $y = x$. So we had to produce a slightly more complicated proof.

PROPOSITION 3.1. *Let $\mathcal{M}$ be a closed C^∞ n-submanifold of $\mathbb{E}^m$. If $\mathbb{G}(\mathcal{M}, x)$ is the set of the closed geodesics in $\mathcal{M}$ with length less than or equal to x, then for $0 \leqslant x \leqslant y$ $\mathbb{P}(\mathcal{M}, x) / \overset{L}{\underset{y}{=}}$ and $\mathbb{G}(\mathcal{M}, x) \overset{L}{\underset{y}{=}}$ have the same cardinality.*

PROOF. By theorems 3.2 and 3.3. ■

REMARK 3.3. By Proposition 3.1, in order to study the function $f_1^L(\mathcal{M}, x, y)$ we may study the closed geodesics in $\mathcal{M}$. Therefore it is interesting to consider results about the number of closed geodesics on a manifold. For some examples we refer to [T] and [BH]. In [S] Souček has proved that the set Γ of lengths of all closed geodesics on a real-analytic compact Riemann manifold is always a discrete set. By Proposition 3.1 this fact proves that if $\mathcal{M}$ is real-analytic and compact then the set of the discontinuity points of $f_1^L(\mathcal{M}, x, y)$, as a function of x, is discrete for every fixed y.

Now as an example of the usefulness of the results mentioned in this section we can compute $f_1^L(\mathcal{M}, x, y)$ when $\mathcal{M}$ is orientable, even-dimensional and has everywhere positive Gauss curvature with respect to every bidimensional direction.

PROPOSITION 3.2. *Let $\mathcal{M}$ be a smooth, orientable, closed, even-dimensional and connected submanifold of $\mathbb{E}^m$ with Gauss curvature everywhere positive with respect to every bidimensional direction. Then we have that*

$$f_1^L(\mathcal{M}, x, y) = \begin{cases} 0 & \text{if } x < 0\,, \\ 1 & \text{if } 0 \leq x \leq y\,, \\ +\infty & \text{if } x \geq 0 \text{ and } x > y\,. \end{cases}$$

PROOF. Because of Remark 2.2 we have to consider only the case $0 \leq x \leq y$. Since for $0 \leq x \leq y$ every equivalence class in $\mathbb{P}(\mathcal{M}, x) / \overset{L}{\underset{y}{=}}$ contains at least a global minimum point that is a geodesic (see theorems 3.2 and 3.3), by using the Synge's Lemma (see [GKM], Section 7.5) we can say that such a minimum point is a constant path. Since two constant paths on a connected manifold are always $0-L$-homotopic, we have that in the case $0 \leq x \leq y$ $\mathbb{P}(\mathcal{M}, x) / \overset{L}{\underset{y}{=}}$ contains only one equivalence class and therefore $f_1^L(\mathcal{M}, x, y) = 1$. ■

REMARK 3.4. The concept of L-homotopy can be generalized by substituting loops with piecewise C^1 functions from S^k to the manifold $\mathcal{M}$ ($k \geq 1$ fixed) and the function length with the more general function volume. This procedure leads in a natural way to the function $f_k^L(\mathcal{M}, x, y)$ (which generalizes the function $f_1^L(\mathcal{M}, x, y)$). We skip the obvious definitions.

4. – V-homotopy and D-homotopy.

L-homotopy is a natural development of homotopy but is a non-simple tool to use. In fact it is difficult to compute the equivalence classes

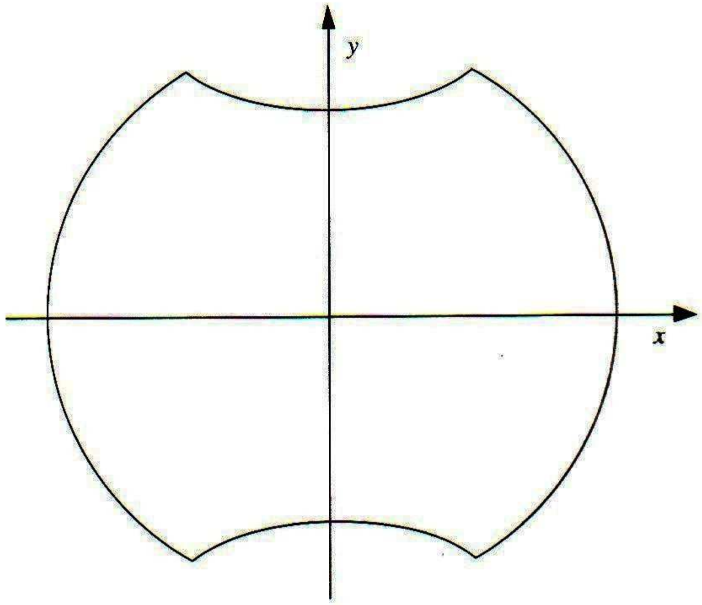

Fig. 4.1.

with respect to L-homotopy even though we can use, for example, results like Proposition 3.1. However it is possible to define other relations similar to L-homotopy but simpler to study in the concrete, although less naturally correlated to classical homotopy theory.

In this paragraph we shall present two of these relations, which we shall call V-homotopy and D-homotopy, but before we define them we shall expose their motivation. Let us consider the surface $\mathcal{N}$ obtained by the rotation around the y-axis of the curve represented in fig. 4.1.: we have a sphere with two hollows. It is impossible «to grasp» $\mathcal{N}$ by a loop because $\mathcal{N}$ is isometric to S^2 and so, for Proposition 3.2, every loop of length λ on $\mathcal{N}$ (as it happens in S^2) is $\lambda - L$-homotopic to a constant loop, but it is possible «to grasp» $\mathcal{N}$ by two of our (or a robot's) fingers in such a way that the tips of our fingers cannot be joined if first they don't move apart. On the contrary we cannot «grasp» S^2 by two fingers. So $\mathcal{N}$ and S^2 are undistinguishable for L-homotopy but not «for our fingers»: L-homotopy does not consider the immersion as a part of the concept of «shape» and this may be a fault (compare with what has been pointed out in the introduction of this paper). Moreover we observe that the procedure of «grasping by our fingers» can be used also to study 1-manifolds while L-homotopy is trivial and useless in this case. These are the reasons for which we want to formalize the above-mentioned idea of grasping an object «by using our fingers». The first formalization that we expose is V-homotopy.

DEFINITION 4.1. Let $\mathcal{M}$ be a piecewise C^∞ n-submanifold $(n > 0)$ of E^m and let us consider the set $\mathcal{M}^{m+1}$ of all ordered $(m+1)$-tuples of points in $\mathcal{M}$. then for every real number y we can define an equivalence relation on $\mathcal{M}^{m+1}$ and all its subsets: for every pair $((P_0, \dots, P_m), (Q_0, \dots, Q_m))$ in $\mathcal{M}^{m+1} \times \mathcal{M}^{m+1}$ we shall say that $(P_0, \dots, P_m)$ and $(Q_0, \dots, Q_m)$ are *$y - V$-homotopic* if either they are the same ordered $(m+1)$-tuple or a function $H(i, \tau)$ exists such that:

i) $H \in C^0(\{0, \dots, m\} \times \mathbb{I}, \mathcal{M})$ where $\mathbb{I} = [0, 1]$ (the topology on $\{0, \dots, m\} \times \mathbb{I} \subset \mathbb{E}^2$ is that induced by the Euclidean topology on $\mathbb{E}^2$),

ii) $H(i, 0) = P_i$ and $H(i, 1) = Q_i$ for every $i \in \{0, \dots, m\}$,

iii) for every $\tau \in \mathbb{I}$ $vol((H(0, \tau,), H(1, \tau), \dots, H(i, \tau), \dots, H(m, \tau))) \leqslant y$, vol being the function that takes every ordered $(m+1)$-tuple into the value that the form $(1/m!)\, dx_1 \wedge dx_2 \wedge \dots \wedge dx_m$ takes on the ordered m-tuple of vectors of $\mathbb{R}^m (H(1, \tau) - H(0, \tau), H(2, \tau) - H(0, \tau), \dots, H(m, \tau) - H(0, \tau))$. In such a case we shall write $(P_0, \dots, P_m) \underset{y}{\overset{V}{=}} (Q_0, \dots, Q_m)$ and say that H is a *$y - V$-homotopy* between $(P_0, \dots, P_m)$ and $(Q_0, \dots, Q_m)$. On $\mathcal{M}^{m+1}$ we can define a distance d_m: for every pair $((P_0, \dots, P_m), (Q_0, \dots, Q_m) \in \mathcal{M}^{m+1} \times \mathcal{M}^{m+1}$ we set $d_m((P_0, \dots, P_m), (Q_0, \dots, Q_m) = \max_{0 \leqslant i \leqslant m} \|P_i - Q_i\|$.

In the case $y \geqslant vol((P_0, \dots, P_m))$ the symbol $(P_0, \dots, P_m) \underset{y}{\overset{V}{=}} (Q_0, \dots, Q_m)$ means, in plain words, that we can «transform $(P_0, \dots, P_m)$ into $(Q_0, \dots, Q_m)$ without exceeding the volume y». Volume can be thought of as a real function defined on $\mathcal{M}^{m+1}$. A $y - V$-homotopy is actually just a path in $\mathcal{M}^{m+1}$, where no point of the path has a value greater than y.

DEFINITION 4.2. Let x and y be real numbers. Let us denote by $(\mathcal{M}^{m+1})^V_x$ the subset of $\mathcal{M}^{m+1}$ containing the ordered $(m+1)$-tuples on which the function vol takes a value less than or equal to x. Let us denote by $f^V_m(\mathcal{M}, x, y)$ the number of equivalence classes into which $(\mathcal{M}^{m+1})^V_x$ is divided by the relation of $y - V$-homotopy if such a number is finite, $+\infty$ otherwise.

REMARK 4.1. it is interesting to notice that some results and statements exposed in Section 3 for L-homotopy still hold for V-homotopy. In particular if we replace «$\mathrm{P}(\mathcal{M})$» by «$\mathcal{M}^{m+1}$», «$\mathrm{P}(\mathcal{M}, x)$» by «$(\mathcal{M}^{m+1})^V_x$»,

«f_1^L» by «f_m^V», the word «loop» by the expression «ordered $(m+1)$-tuple of points», the length function $\mathcal{L}$ by the function *vol*, the distance d between loops by the distance d_m between ordered $(m+1)$-tuples of points, the hypothesis $x \geqslant 0$ by the hypotesis x greater than or equal to the minimum of *vol* on $\mathcal{M}^{m+1}$ and the set $\mathrm{G}(\mathcal{M}, x)$ of the closed geodesics of $\mathcal{M}$ with length less than or equal to x by the set $\mathrm{C}_{m+1}(\mathcal{M}, x)$ of the ordered $(m+1)$-tuples of points of $\mathcal{M}$ that *vol* takes into a number less than or equal to x and that are local minima for *vol* then Remark 3.2, Lemma 3.2 (with δ not depending on x), Theorems 3.1 and 3.2 and Proposition 3.1 still hold. Actually, the corresponding proofs are simpler.

Another formalization of the idea described at the beginning of this section can be the one that we obtain by substituting in the above-mentioned definitions the concept of $(m+1)$-tuple by the one of $(k+1)$-tuple (with k generic natural number) and the function *vol* by the function Δ which takes every ordered $(k+1)$-tuple into the diameter of its convex hull (by the way we point out that another possible choice could be the k-dimensional volume). We shall call D-homotopy the corresponding theory. This procedure leads in a natural way to the function $f_k^D(\mathcal{M}, x, y)$ (which corresponds naturally to the function $f_m^V(\mathcal{M}, x, y)$). The results mentioned in Remark 4.1 can be extended also to D-homotopy. We give in the following the exact formalization of the above-mentioned concepts.

DEFINITION 4.3. Let $\mathcal{M}$ be a piecewise C^∞ n-submanifold $(n>0)$ of $\mathbb{E}^m$ and let us consider the set $\mathcal{M}^{k+1}$ of all ordered $(k+1)$-tuples of points in $\mathcal{M}(k \in \mathbb{N})$. Then for every real number y we can define an equivalence relation on $\mathcal{M}^{k+1}$ and all its subsets: for every pair $((P_0, \ldots, P_k), (Q_0, \ldots, Q_k))$ in $\mathcal{M}^{k+1} \times \mathcal{M}^{k+1}$ we shall say that $(P_0, \ldots, P_k)$, and $(Q_0, \ldots, Q_k)$ are *$y-D$-homotopic* if either they are the same ordered $(k+1)$-tuple or a function $H(i, \tau)$ exists such that:

i) $H \in C^0(\{0, \ldots, k\} \times \mathrm{I}, \mathcal{M})$ where $\mathrm{I}=[0,1]$ (the topology on $\{0, \ldots, k\} \times \mathrm{I} \subset \mathbb{E}^2$ is that induced by the Euclidean topology on $\mathbb{E}^2$),

ii) $H(i, 0)=P_i$ and $H(i, 1)=Q_i$ for every $i \in \{0, \ldots, k\}$,

iii) for every $\tau \in \mathrm{I}$ $\Delta((H(0, \tau,), H(1, \tau), \ldots, H(i, \tau), \ldots, H(k, \tau))) \leqslant$ $\leqslant y$, Δ being the function that takes every ordered $(k+1)$-tuple into the diameter of the convex hull of the $(k+1)$-tuple. In such a case we shall write $(P_0, \ldots, P_k) \overset{D}{\underset{y}{=}} (Q_0, \ldots, Q_k)$ and say that H is a *$y-D$-homotopy* between $(P_0, \ldots, P_k)$ and $(Q_0, \ldots, Q_k)$. On $\mathcal{M}^{m+1}$ we can define a distance d_k: for every pair $((P_0, \dots, P_k), (Q_0, \dots, Q_k)) \in \mathcal{M}^{k+1} \times \mathcal{M}^{k+1}$ we set $d_k((P_0, \dots, P_k), (Q_0, \dots, Q_k)) = \max_{0 \leq i \leq k} \|P_i - Q_i\|$.

Obviously the diameter we spoke about in the previous definition is the one inherited from the embedding of $\mathcal{M}$ in E^m. In the case $y \geq \Delta((P_0, \dots, P_k))$ the symbol $(P_0, \dots, P_k) \overset{D}{\underset{y}{=}} (Q_0, \dots, Q_k)$ means, in plain words, that we can «transform $(P_0, \dots, P_k)$ into $(Q_0, \dots, Q_k)$ without exceeding the diameter y». Diameter can be thought of as a real function defined on $\mathcal{M}^{k+1}$. A $y - D$-homotopy is actually just a path in $\mathcal{M}^{k+1}$, where no point of the path has a value greater than y.

DEFINITION 4.4. Let x and y be real numbers. Let us denote by $(\mathcal{M}^{k+1})^D_x$ the subset of $\mathcal{M}^{k+1}$ containing the ordered $(k+1)$-tuples on which the function Δ takes a value less than or equal to x. Let us denote by $f^D_k(\mathcal{M}, x, y)$ the number of equivalence classes into which $(\mathcal{M}^{k+1})^D_x$ is divided by the relation of $y - D$-homotopy if such a number is finite, $+\infty$ otherwise.

REMARK 4.2. It is easy to prove that if a manifold $\mathcal{N} \subset \mathrm{E}^m$ is obtained by a manifold $\mathcal{M} \subset \mathrm{E}^m$ by a direct similarity of ratio h then we have $f^L_k(\mathcal{N}, h^k x, h^k y) = f^L_k(\mathcal{M}, x, y)$, $f^V_m(\mathcal{N}, h^m x, h^m y) = f^V(\mathcal{M}, x, y)$, $f^D_k(\mathcal{N}, hx, hy) = f^D_k(\mathcal{M}, x, y)$, for every value of the numerical variables. This yields an obvious sufficient condition for two manifolds to be non-similar.

REMARK 4.3. Let $\mathcal{M}$ be a closed, connected and piecewise C^∞ n-submanifold of E^m. Let μ and M be respectively the minimum and the maximum of the function *vol* on $\mathcal{M}^{m+1}$ (obviously $M = -\mu$) and let $\mathcal{D}$ be the diameter of $\mathcal{M}$. We point out that, because of the given definitions, the following statements hold: if $x > y$ and $(y, x) \cap [\mu, M] \neq \emptyset$ then $f^V_m(\mathcal{M}, x, y) = +\infty$, if $y \geq M$ and $x \geq y$ then $f^V_m(\mathcal{M}, x, y) = 1$ and if $x < \mu$ then $f^V_m(\mathcal{M}, x, y) = 0$. Moreover, in D-homotopy, if $x > y$ and $(y, x] \cap [0, \mathcal{D}] \neq \emptyset$ then $f^D_k(\mathcal{M}, x, y) = +\infty$, if either $y \geq \mathcal{D}$ and $x \geq 0$ or $y \geq x = 0$ then $f^D_k(\mathcal{M}, x, y) = 1$ and if $x < 0$ then $f^D_k(\mathcal{M}, x, y) = 0$. So the values of f^V_m and f^D_k are interesting respectively for either $\mu < x \leq y$ or $\mu = x$ and for $0 < x \leq y$. By considering suitable manifolds with infinitely many undulations we could give examples in which f^V_m and f^D_k take $+\infty$ as a value also for $x = y$ (cf. Example 2.1 for L-homotopy).

5. – Results and examples about V-homotopy and D-homotopy.

In the following we shall give some results and examples that will point out some properties of V-homotopy (and incidentally some of D-homotopy and some differences between these two approaches to the problem of distinguishing shapes).

PROPOSITION 5.1. *Let* $\mathcal{M}$ *be a piecewise* C^∞ *submanifold of* E^m. *Then* $f_{k+1}^D(\mathcal{M}, x, y) \geq f_k^D(\mathcal{M}, x, y)$ *for every* $k \in \mathrm{N}$ *and* $x, y \in \mathrm{R}$.

PROOF. It follows immediately from definition of D-homotopy, by observing that $(P_0, P_1, \ldots, P_k) \overset{D}{\underset{y}{=}} (Q_0, Q_1, \ldots, Q_k) \in \mathcal{M}^{k+1}$ if and only if $(P_0, P_0, P_1, \ldots, P_k) \overset{D}{\underset{y}{=}} (Q_0, Q_0, Q_1, \ldots, Q_k) \in \mathcal{M}^{k+2}$. ■

So if the D-homotopy of a manifold $\mathcal{M}$ is not trivial by considering ordered $(k+1)$-tuples then it is not trivial either by considering ordered $(k+2)$-tuples.

PROPOSITION 5.2. *Let* $\mathcal{M}$ *be a polyhedron in* E^m. *Every ordered* $(m+$ $+1)$*-tuple* $(P_0, \ldots, P_m) \in \mathcal{M}^{m+1}$ *is* $vol((P_0, \ldots, P_m - V))$*-homotopic to an ordered* $(m+1)$*-tuple of vertices of* $\mathcal{M}$.

PROOF. The thesis follows by considering that vol is a linear function (and therefore a concave function) in every variable and that so it cannot have a strict local minimum in a point of the interior of a face of $\mathcal{M}$. This condition allows us to construct (in an obvious way) the wanted $vol((P_0, \ldots, P_m)) - V$-homotopy. ■

REMARK 5.1. Proposition 5.2 simplifies the computation of V-homotopy for a polyhedron $\mathcal{M}$ by reducing everything to the study of ordered $(m+1)$-tuples of vertices of $\mathcal{M}$. The analogous for D-homotopy of such statement is not true because the function Δ is not concave in its variables. For instance if $\mathcal{M}$ is the boundary of an equilateral triangle we see that every ordered triple (P_0, P_1, P_2) of central points of the sides of $\mathcal{M}$ is a local minimum point for Δ that is not $\Delta((P_0, P_1, P_2)) - D$-homotopic to an ordered triple of vertices of $\mathcal{M}$.

LEMMA 5.1. *Let* $\mathcal{M}$ *be a piecewise* C^∞, *closed and connected* n*-submanifold of* E^m. *If in an ordered* $(m+1)$*-tuple* $(P_0, \ldots, P_m) \in \mathcal{M}^{m+1}$ *we have* $P_i = P_j$ *for a pair* (i, j) *of distinct indices then for every point* Q *of* $\mathcal{M}$ *is results* $(P_0, \ldots, P_m) \overset{V}{\underset{0}{=}} (Q, \ldots, Q) \in \mathcal{M}^{m+1}$

PROOF. Let us consider $m+1$ piecewise C^1 curves $\gamma_0, \gamma_1, \ldots, \gamma_m$:

$[0, 1] \to \mathfrak{M}$ such that $\gamma_i = \gamma_j$, $\gamma_k(0) = P_k$ and $\gamma_k(1) = Q$ for every k (they exist because $\mathfrak{M}$ is connected for hypothesis). Then the function $H(k, \tau) = \gamma_k(\tau)$ is a $0 - V$-homotopy between $(P_0, \ldots, P_m)$ and $(Q, \ldots, Q)$ because $vol((H(0, \tau), \ldots, H(m, \tau))) = 0$ for every $\tau \in [0, 1]$ (since $H(0, \tau), \ldots, H(m, \tau)$ belong to a linear manifold of dimension not greater than $m - 1$). ■

PROPOSITION 5.3. *If $\mathfrak{M}$ is a piecewise C^∞ boundary of a convex subset of $\mathbb{E}^m$ then for $0 \leq x \leq y$ it results $f_m^V(\mathfrak{M}, x, y) = 1$.*

PROOF. Let us consider an ordered $(m+1)$-tuple whichever $(P_0, P_1, \ldots, P_m) \in (\mathfrak{M}^{m+1})_x^V$ with $x \geq 0$. Let H be the so defined function from $\{0, 1, \ldots, m\} \times [0, 1]$ to $\mathfrak{M}$: $H(0, \tau) = \gamma(\tau)$ and $H(i, \tau) = P_i$ for $i \neq 0$, where γ is a piecewise smooth curve on $\mathfrak{M}$ such that $\gamma(0) = P_0$, $\gamma(1) = P_1$ and the distance between $\gamma(\tau)$ and the linear hull Π of the points $P_1, \ldots, P_m$ is never increasing in τ. Such a curve exists because $\mathfrak{M}$ is the piecewise C^∞ boundary of a convex set. Two cases are possible: either $vol((P_0, P_1, \ldots, P_m)) \geq 0$ or not. In the first case we have that $vol((H(0, \tau), H(1, \tau), \ldots, H(m, \tau)))$ is a non-increasing function in τ, in the second case it results $vol((H(0, \tau), H(1, \tau), \ldots, H(m, \tau))) \leq 0$ for every $\tau \in [0, 1]$. In every case, since $x \geq 0$, H is an $x - V$-homotopy between the two $(m+1)$-tuples $(P_0, P_1, P_2, \ldots, P_m)$ and $(P_1, P_1, P_2, \ldots, P_m)$.

Therefore, chosen a point $Q \in \mathfrak{M}$ whichever, by Lemma 5.1 we obtain $(P_0, P_1, \ldots, P_m) \overset{V}{\underset{x}{=}} (Q, Q, \ldots Q)$. So the ordered $(m+1)$-tuples of $(\mathfrak{M}^{m+1})_x^V$ are all $y - V$-homotopic to $(Q, Q, \ldots, Q)$ for $0 \leq x \leq y$: this means that $f_m^V(\mathfrak{M}, x, y) = 1$ for every x with $0 \leq x \leq y$. ■

REMARK 5.2. Proposition 5.3 does not say that f_m^V is always trivial for a piecewise smooth boundary of a convex. In fact the function f_m^V gives information also for negative values of its variables (on the contrary of f_1^L and f_k^D): as an example we refer to the values of $f_2^V(\mathfrak{J}, x, y)$ computed in Proposition 5.5. The analogous for D-homotopy of Proposition 5.3 is not true. In order to understand this it is sufficient to observe that an ordered triple (P_0, P_1, P_2) constituted by the central points in the sides of an equilateral triangle $\mathfrak{J}$ of side 1 cannot be changed inside the perimeter into the triple (P_0, P_0, P_0) without the diameter taking the value $\sqrt{3}/2$. So for $y < \sqrt{3}/2$ we have $f_2^D(\mathfrak{J}, 1/2, y) > 1$.

PROPOSITION 5.4. *Let $\mathfrak{M}$ be a polyhedron in $\mathbb{E}^m$ with r vertices and let $x, y \in \mathbb{R}$, $x \leq y$. If $r < m+1$ then $f_m^V(\mathfrak{M}, x, y) = 1$ for $x \geq 0$ and $f_m^V(\mathfrak{M}, x, y) = 0$ for $x < 0$. If $r \geq m+1$ then $f_m^V(\mathfrak{M}, x, y) \leq$*

$\leq r!/(r-m-1)!+1$ *for* $x \geq 0$ *and* $f_m^V(\mathcal{M}, x, y) \leq 1/2 \cdot r!/(r-m-1)!$ *for* $x < 0$.

PROOF. The first statement follows from the fact if $r < m+1$ then the dimension of the linear hull of every $(P_0, P_1, \ldots, P_m) \in \mathcal{M}^{m+1}$ is not greater than $m-1$ and therefore it results $vol((P_0, P_1, \ldots, P_m)) = 0$. The second statement follows from Proposition 5.2 and Lemma 5.1. In fact for $x \geq 0$ we can upperly bound $f_m^V(\mathcal{M}, x, y)$ by the number of ordered $(m+1)$-tuples of pairwise distinct vertices of $\mathcal{M}$ (that is $r!/(r-m-1)!$) plus 1 (which represents the equivalence class containing every $(m+1)$-tuple whose points are not all pairwise distinct). On the other hand in case $x < 0$ we can upperly bound $f_m^V(\mathcal{M}, x, y)$ by $1/2 \cdot r!/(r-m-1)!$ because it is a number greater than or equal to the number of ordered $(m+1)$-tuples on which the function *vol* takes a negative value, since for every ordered $(m+1)$-tuple $(P_0, P_1, P_2, \ldots, P_m)$ of points of $\mathcal{M}$ we have $vol((P_1, P_0, P_2, \ldots, P_m)) = -vol((P_0, P_1, P_2, \ldots, P_m))$. ■

Now, in order to illustrate what we have pointed out in this section we want to compute the function $f_2^V(\mathfrak{J}, x, y)$ where $\mathfrak{J}$ is the boundary of an equilateral triangle of side 1 embedded in $\mathbb{E}^2$. Obviously $\mathfrak{J}$ can be considered embedded also in $\mathbb{E}^m$ for $m > 2$ but in this case $f_m^V(\mathfrak{J}, x, y)$ is trivial and not interesting.

PROPOSITION 5.5.

$$f_2^V(\mathfrak{J}, x, y) = \begin{cases} 0 & \text{if } x < -\dfrac{\sqrt{3}}{4}, \\ 3 & \text{if } x = -\dfrac{\sqrt{3}}{4} \text{ and } y < x, \\ +\infty & \text{if } y < x \text{ and } (y, x) \cap \left[-\dfrac{\sqrt{3}}{4}, \dfrac{\sqrt{3}}{4}\right] \neq \emptyset, \\ 1 & \text{if } \dfrac{\sqrt{3}}{4} \leq y < x, \\ 3 & \text{if } -\dfrac{\sqrt{3}}{4} \leq x, \quad x \leq y \text{ and } y < -\dfrac{\sqrt{3}}{16}, \\ 1 & \text{if } -\dfrac{\sqrt{3}}{4} \leq x, \quad x \leq y \text{ and } y \geq -\dfrac{\sqrt{3}}{16}. \end{cases}$$

PROOF. The non-trivial values f_2^V (cf. Remark 4.3) are obtained by considering Proposition 5.2 and the following statement. Calling A, B and C the vertices of $\mathfrak{J}$ (listed clockwise) we claim that $-\sqrt{3}/16$ is the least number h such that $(A, B, C) \overset{V}{=} (B, C, A)$. Our assertion is proved in the following way. If $(A, B, C) \underset{h}{\overset{V}{=}} (B, C, A)$. then there exist three continuous functions $P(t)$, $Q(t)$ and $R(t)$ from $[0, 1]$ to $\mathfrak{J}$ with $P(0) = A$, $P(1) = B$, $Q(0) = B$, $Q(1) = C$, $R(0) = C$, $R(1) = A$ and $vol((P(t), Q(t), R(t))) \leq h$ for every $t \in [0, 1]$. Because of the continuity of $P(t)$, $Q(t)$ and $R(t)$ there exists at least a $\bar{t}$ such that either $P(\bar{t}) = Q(\bar{t})$ (and so $vol((P(\bar{t}), Q(\bar{t}), R(\bar{t}))) = 0$) or the line s joining $P(\bar{t})$ and $Q(\bar{t})$ is parallel to the side AC and $R(\bar{t})$ is below s (and therefore, as is easy to prove, $vol((P(\bar{t}), Q(\bar{t}), R(\bar{t}))) \geq -\sqrt{3}/16$). So in both the cases if $vol((P(t), Q(t), R(t))) \leq h$ for every $t \in [0, 1]$ then it must be $h \geq -\sqrt{3}/16$. On the other hand if we define $P(t) = (1-t)\cdot A + t\cdot B$, $Q(t) = (1-t)\cdot B + t\cdot C$, $R(t) = (1-t)\cdot C + t\cdot A$ for $0 \leq t \leq 1$ we have that the greatest value of $vol((P(t), Q(t), R(t)))$ varying t is $-\sqrt{3}/16$. This fact proves our statement. ■

It might be interesting to study connections of V- and D-homotopy with Morse Theory. The main difficulty seems to consist in the often unavoidable degeneracy of critical points of diameter and volume.

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