A quick trip through geometrical shape comparison

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Topics in Mathematics
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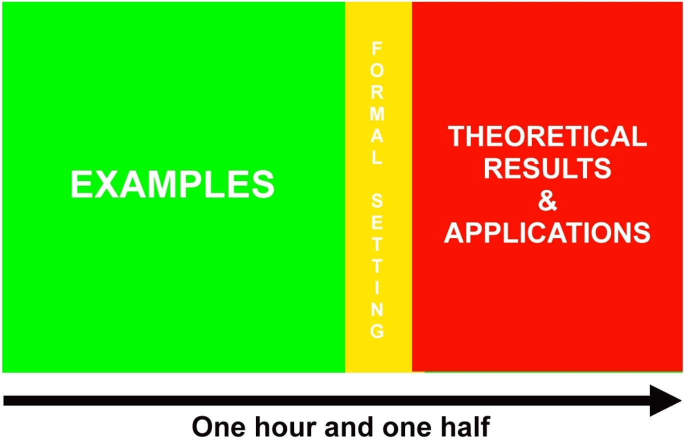
Plan of the talk



Outline

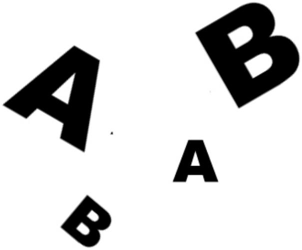
1. **The "trivial" problem of comparing shapes**
2. **Trying to put some geometrical order in the chaos**
3. **Some theoretical results and applications**

1. **The "trivial" problem of comparing shapes**
2. Trying to put some geometrical order in the chaos
3. Some theoretical results and applications

What is shape?

A "trivial" question: What is shape?

SIMPLE ANSWER: *Shape is what is left after removing scale and rotation.*

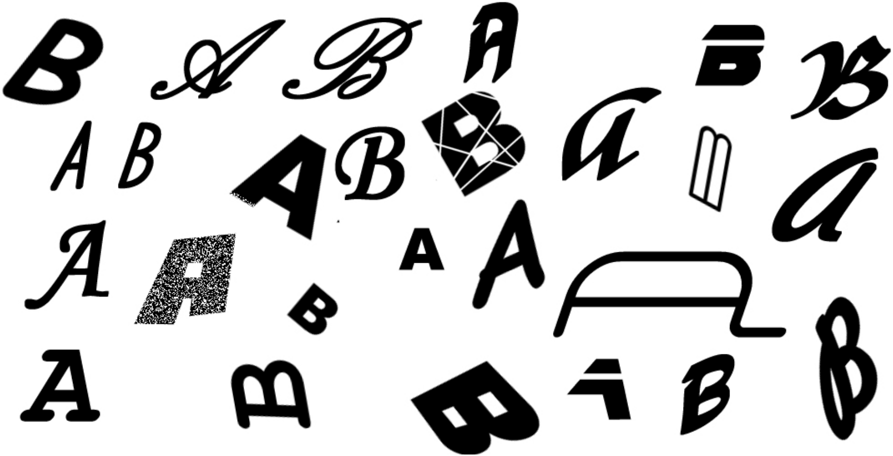


What is shape?

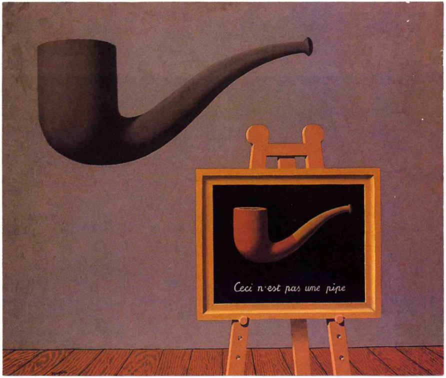
"There is always an easy solution to every human problem– neat, plausible, and wrong."

Henry Louis Mencken (1880-1956)

What is shape?



What is shape?



What is a pipe?

What is shape?

What is shape?

What is shape?

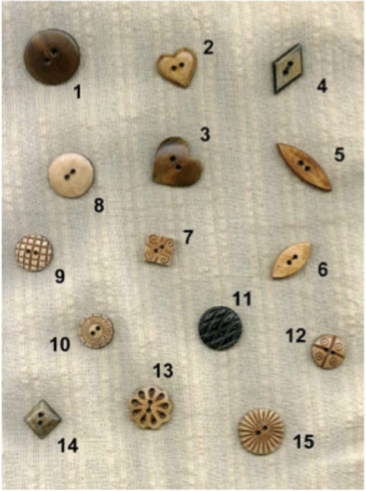
What is shape?

What is shape?

Pipe and passport of René and Georgette Magritte-Berger

What is shape?

What is shape?



Buttons

What is shape?

Guitars

What is shape?

Cups

The key role of the observer

Every comparison of properties involves the presence of

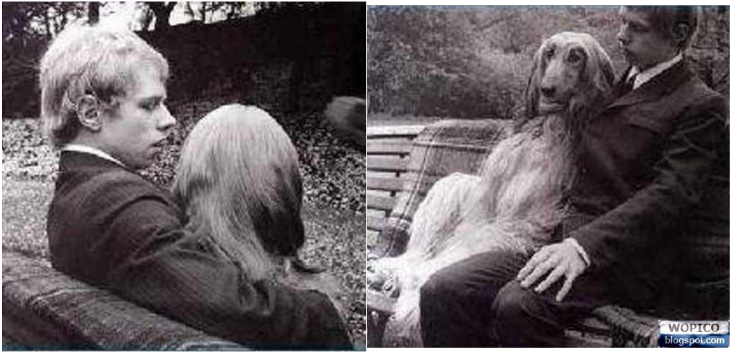
- an observer perceiving the properties
- a methodology to compare the properties

It follows that shape comparison is affected by subjectivity.

Let us give some examples illustrating this fact.

The key role of the observer

Truth often depends on the observer:



The key role of the observer

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Truth often depends on the observer:

Julian Beever

The key role of the observer

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Julian Beever

The key role of the observer

Truth often depends on the observer:

Julian Beever

The key role of the observer

Truth often depends on the observer:

Magnifying glass or cup of coffee?

The key role of the observer

Truth often depends on the observer:

A surfacing submarine?

The key role of the observer

Truth often depends on the observer:

Bump

The key role of the observer

Truth often depends on the observer:

A yellow frisbee is floating in the air

The key role of the observer

Truth often depends on the observer:

What is going on here?

The key role of the observer

Truth often depends on the observer:

Coffee or owl?

The key role of the observer

Truth often depends on the observer:

Duck or rabbit?

The key role of the observer

Truth often depends on the observer:

The black shapes are NOT the camels, the narrow stripes below the shapes are. The black shapes are the shadows of the camels, as this photo was taken from overhead.

The key role of the observer

Truth often depends on the observer:

How many rabbits?

The key role of the observer

Truth often depends on the observer:

Michael Jantzen, "Deconstructing the Houses".

The key role of the observer

Truth often depends on the observer:

Crooked House (Krzywy Domek) by Szotyńscy and Zaleski, Sopot, Poland.

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The key role of the observer

Truth often depends on the observer:

"Glued in Florence" by Christiaan Triebert

The key role of the observer

Truth often depends on the observer:

The key role of the observer

The concept of shape is subjective and relative. It is based on the act of perceiving, depending on the chosen observer. Persistent perceptions are fundamental in order to approach this concept.

- “Science is nothing but perception.” *Plato*
- “Reality is merely an illusion, albeit a very persistent one.” *Albert Einstein*

1. The "trivial" problem of comparing shapes

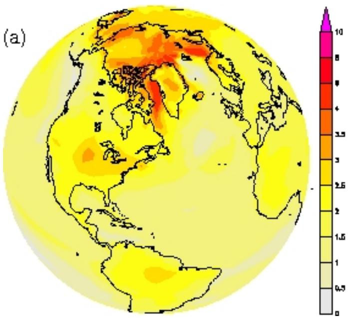
2. **Trying to put some geometrical order in the chaos**

3. Some theoretical results and applications

Our formal setting

- In shape comparison objects are not accessible directly, but only via measurements made by an observer.
- The comparison of two shapes is usually based on a family F of "measuring functions", which are defined on a set M (*set of measurements*) and take values in a set V (*set of measurement values*). Each function in F represents a measurement obtained via a measuring instrument.
- In most cases, the family F of measuring functions is invariant with respect to a given group G of transformations, that depends on the type of measurement we are considering.
- A G-invariant pseudo-metric d_F is available for the set F, so that we can quantify the difference between the measuring functions in F. *(pseudo-metric = metric without the property* $d(x, y) = 0 \implies x = y$*)*

Example 1



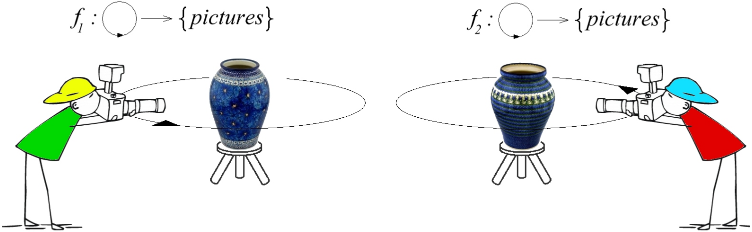
- $M = S^2$ (the globe's surface), $V = \mathbb{R}$ (the set of temperatures)
- Every $f \in F$ is a function associating each point of S^2 with its temperature.
- G is the set of rigid motions of S^2 (we observe that $F \circ G = F$)
- We can set $d_F(f_1, f_2) = \inf_{g \in G} \sup_{x \in M} |f_1(x) - f_2(g(x))|$

Example 2

- M is the vase surface, $V = \mathbb{R}^3$ (the set of colors)
- Every $f \in F$ is a function associating each point of M with its color.
- G is the set of rotations of M around the z-axis (we observe that $F \circ G = F$)
- We can set $d_F(f_1, f_2) = \inf_{g \in G} \sup_{x \in M} \|f_1(x) - f_2(g(x))\|$

Example 3

A different way of "measuring" the vases...



- $M = S^1$, $V = \{\textit{the set of pictures}\}$
- Every $f \in F$ is a function associating each point of S^1 with a picture.
- G is the set of rotations around the z-axis, and $F \circ G = F$
- We can set $d_F(f_1, f_2) = \inf_{g \in G} \sup_{x \in M} \|f_1(x) - f_2(g(x))\|$

Our shape pseudo-distance d_F (formal definition)

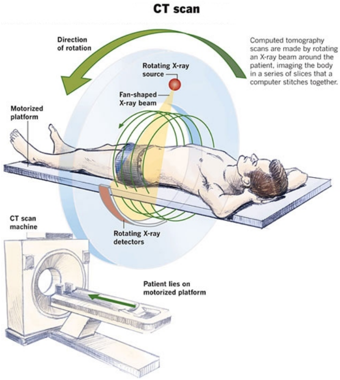
Assume that the following objects are given:

- A set M. Each point $x \in M$ represents a measurement.
- A set V. Each point $v \in V$ represents the value taken by a measurement.
- A set F of functions from M to V. Each function $f \in F$ describes a possible set of results for all measurements in M.
- A group G acting on M, such that F is invariant with respect to G (i.e., for every $f \in F$ and every $g \in G$ we have that $f \circ g \in F$).
- A pseudo-metric d_F defined on the set F, that is invariant under the action of the group G (in other words, if $f_1, f_2 \in F$ and $g \in G$ then $f_2 \circ g \in F$ and $d_F(f_1, f_2) = d_F(f_1, f_2 \circ g)$).

We call each pair (F, d_F) a (pseudo-)metric shape space.

An interesting case

It often happens that M is a topological space and V is a metric space, endowed with a metric d_V. In this case the functions in F are assumed to be continuous, and the group G is assumed to be a subgroup of the group of all self-homeomorphisms of M. As an example, let us think of a CT scanning.



An interesting case

In this example

- $M = S^1$ represents the topological space of all directions that are orthogonal to a given axis;
- $V = \mathbb{R}$ represents the metric space of all possible quantities of matter encountered by the X-ray beam in the considered direction.
- Every $f \in F$ is a function taking each direction in S^1 to the quantity of matter encountered by the X-ray beam along that direction.
- G is the group of the rotations of S^1 ($F \circ G = F$).

We can set

$$d_F(f_1, f_2) = \inf_{g \in G} \sup_{x \in M} |f_1(x) - f_2(g(x))|$$

for $f_1, f_2 \in F$.

Other interesting cases

If $V = \mathbb{R}^k$ we can use the shape pseudo-distance

$$d_F(f_1, f_2) = \inf_{g \in G} \sup_{x \in M} \| f_1(x) - f_2(g(x)) \|_\infty$$

for $f_1, f_2 \in F$. The functional $\sup_{x \in M} \| f_1(x) - f_2(g(x)) \|_\infty$ quantifies the change in the measurement induced by the transformation g.

The pseudo-metric d_F is produced by the attempt of minimizing this functional, varying the transformation g in the group G, and is called natural pseudo-distance.

More generally, if V is a metric space endowed with the metric d_V, we can set

$$d_F(f_1, f_2) = \inf_{g \in G} \sup_{x \in M} d_V\left(f_1(x), f_2(g(x))\right)$$

for $f_1, f_2 \in F$.

Other interesting cases

If $V = \mathbb{R}$ and M is a compact subset of $\mathbb{R}^m$, we can set

$$d_F(f_1, f_2) = \inf_{g \in G} \left(\int_{x \in M} |f_1(x) - f_2(g(x))|^p \ dx \right)^{\frac{1}{p}}$$

after fixing $p \geq 1$.
The functional $\left(\int_{x \in M} |f_1(x) - f_2(g(x))|^p \ dx\right)^{\frac{1}{p}}$ quantifies the change in the measurement induced by the transformation g.

If G is the group of all isometries of M, d_F is a pseudo-metric that is invariant under the action of G.

Usual assumptions

In the rest of this talk we will assume that M is a topological space and V is a metric space.

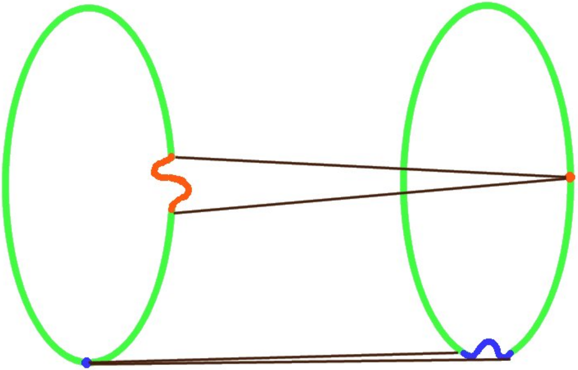
These assumptions allow us to require that if two measurements are close to each other (in some reasonable sense), then the values obtained by these measurements are close to each other, too.

The functions in F will be assumed to be continuous. The group G will be assumed to be a subgroup of the group $Homeo(M)$ of all self-homeomorphisms of M.

Usual assumptions

Why do we just consider self-homeomorphisms of M?

Why couldn't we use, e.g., relations on M?



A result that suggests not to use relations in our setting

The following result highlights a problem about doing that:

Non-existence Theorem

Let M be a Riemannian manifold. Let us endow $Homeo(M)$ with the uniform convergence metric d_{UC}: $d_{UC}(h,h') = \max_{x\in M} d_M(h(x),h'(x))$ for every $h,h' \in Homeo(M)$, where d_M is the geodesic distance on M. Then $(Homeo(M), d_{UC})$ cannot be embedded in any compact metric space (K,d_K) endowed with an internal binary operation $\bullet$ that extends the usual composition $\circ$ between homeomorphisms in $Homeo(M)$ and commutes with the passage to the limit in K. In particular, $Homeo(M)$ cannot be embedded in such a way into the set of binary relations on M.

P. Frosini, C. Landi, *No embedding of the automorphisms of a topological space into a compact metric space endows them with a composition that passes to the limit*, Applied Mathematics Letters, 24 (2011), n. 10, 1654–1657.

A result that suggests not to use relations in our setting

In plain words, the previous theorem shows that, if our space of measurements M is a Riemannian manifold, no reasonable embedding of the set of all self-homeomorphisms of M into another compact metric space (K, d_K) exists. In particular, there does not exist any reasonable embedding into the space of binary relations on M.

This is due to the fact that any such embedding couldn't preserve the usual composition between homeomorphisms in $Homeo(M)$ and commute with the passage to the limit in K.

Remark

The previous theorem can be extended to topological spaces that are far more general than manifolds. It is sufficient that they contain a subset U that is homeomorphic to an n-dimensional open ball for some $n \geq 1$.

Some theoretical results about the natural pseudo-distance

Until now, most of the results about the natural pseudo-distance have been proven for the case when M is a closed manifold, $G = Homeo(M)$, and the measuring functions take their values in $V = \mathbb{R}$. Here we recall three of these results:

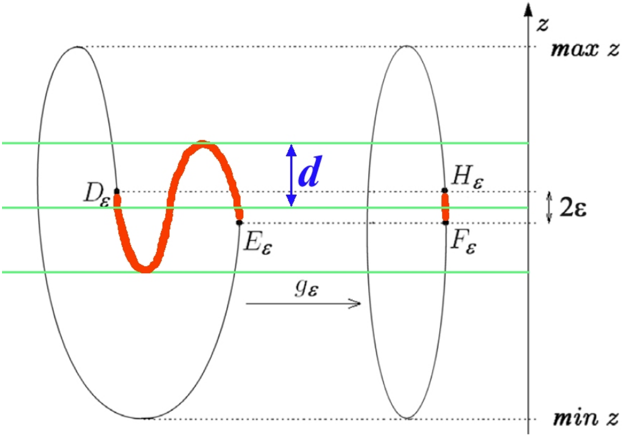
Theorem (for curves)

Assume that M is a closed curve of class C^1 and that $f_1, f_2 : M \to \mathbb{R}$ are two functions of class C^1. Then, if $d = \inf_{g \in Homeo(M)} \max_{x \in M} |f_1(x) - f_2(g(x))|$, at least one of the following properties holds:

- *d equals the distance between a critical value of f_1 and a critical value of f_2;*
- *d equals half the distance between two critical values of f_1;*
- *d equals half the distance between two critical values of f_2.*

Some theoretical results about the natural pseudo-distance

For example, in this case the natural pseudo-distance d equals half the distance between two critical values of f_1:



Some theoretical results about the natural pseudo-distance

Theorem (for surfaces)

Assume that M is a closed surface of class C^1 and that $f_1, f_2 : M \to \mathbb{R}$ are two functions of class C^1. Then, if $d = \inf_{g \in Homeo(M)} \max_{x \in M} |f_1(x) - f_2(g(x))|$, at least one of the following properties holds:

- *d equals the distance between a critical value of f_1 and a critical value of f_2;*
- *d equals half the distance between two critical values of f_1;*
- *d equals half the distance between two critical values of f_2;*
- *d equals one third of the distance between a critical value of f_1 and a critical value of f_2.*

(OPEN PROBLEM: *Is the last case possible?*)

Some theoretical results about the natural pseudo-distance

Theorem (for manifolds)

Assume that M is a closed manifold of class C^1 and that $f_1, f_2 : M \to \mathbb{R}$ are two functions of class C^1. Then, if $d = \inf_{g \in Homeo(M)} \max_{x \in M} |f_1(x) - f_2(g(x))|$, at least one of the following properties holds:

- *a positive odd integer m exists, such that $m \cdot d$ equals the distance between a critical value of f_1 and a critical value of f_2;*
- *a positive even integer m exists, such that $m \cdot d$ equals the distance between two critical values either of f_1 or of f_2.*

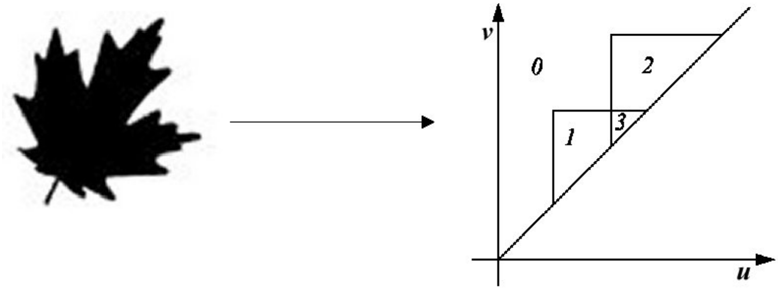
(OPEN PROBLEM: *Is the case $m \geq 3$ possible?*)

Some references about the natural pseudo-distance for topological spaces

- P. Frosini, M. Mulazzani, *Size homotopy groups for computation of natural size distances*, Bulletin of the Belgian Mathematical Society - Simon Stevin, 6 (1999), 455-464.
- P. Donatini, P. Frosini, *Natural pseudodistances between closed topological spaces*, Forum Mathematicum, 16 (2004), n. 5, 695-715.
- P. Donatini, P. Frosini, *Natural pseudodistances between closed surfaces*, Journal of the European Mathematical Society, 9 (2007), 331-353.
- P. Donatini, P. Frosini, *Natural pseudodistances between closed curves*, Forum Mathematicum, 21 (2009), n. 6, 981-999.

Natural pseudo-distance and size functions

- The natural pseudo-distance is usually difficult to compute.
- Lower bounds for the natural pseudo-distance can be obtained by computing the size functions.



Definition of size function

Given a topological space M and a continuous function $f = (f_1, \dots, f_k) : M \to \mathbb{R}^k$,

Lower level sets

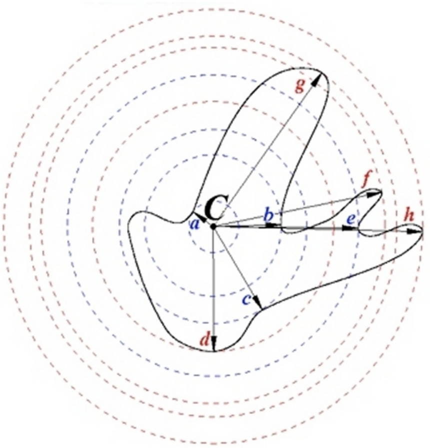
For every $u \in \mathbb{R}^k$, $M\langle f \preceq u \rangle = \{x \in M : f(x) \preceq u\}$.
($u = (u_1, \dots, u_k) \preceq v = (v_1, \dots, v_k)$ means $u_j \leq v_j$ for every index j.)

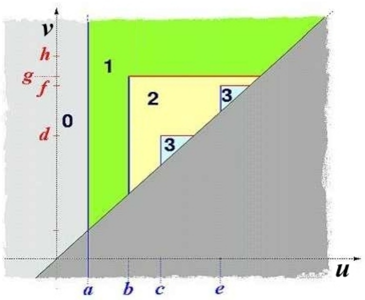
Definition (Frosini 1991)

The Size Function of (M, f) is the function ℓ that takes each pair (u, v) with $u \prec v$ to the number $\ell(u, v)$ of connected components of the set $M\langle f \preceq v \rangle$ that contain at least one point of the set $M\langle f \preceq u \rangle$.

Example of a size function, in the case that the measuring function has only one component

Here the measuring function equals the distance from C.





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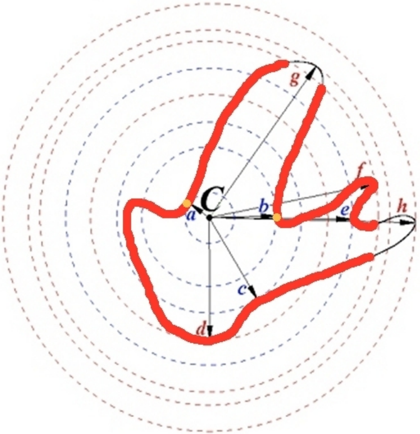

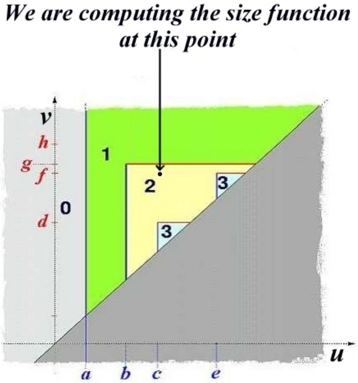

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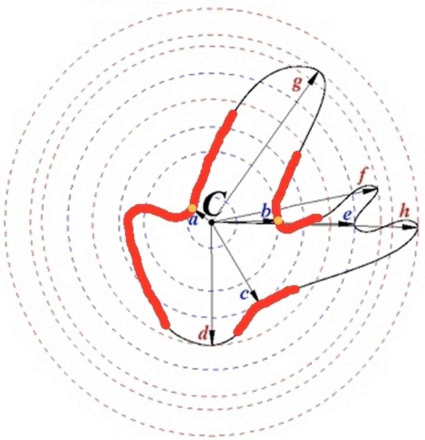

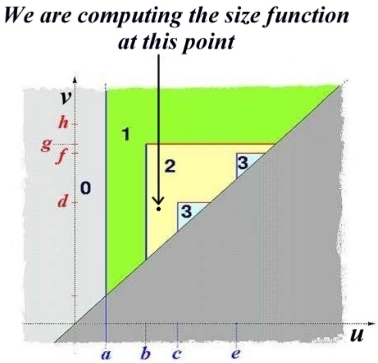

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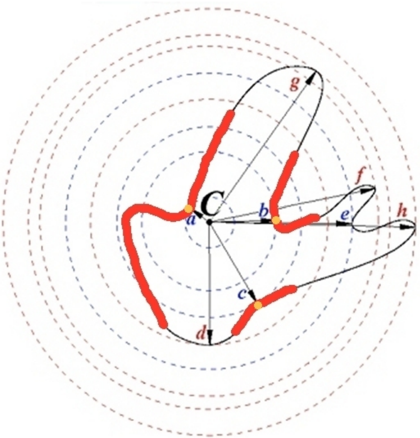

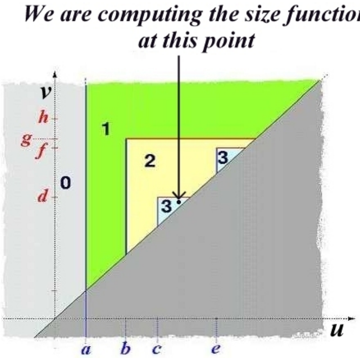

Example of a size function, in the case that the measuring function has only one component

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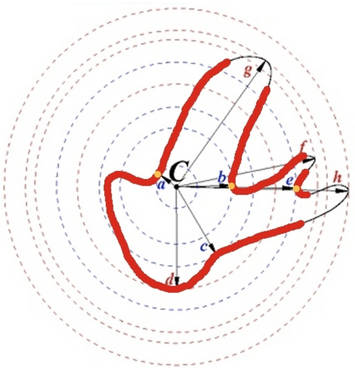
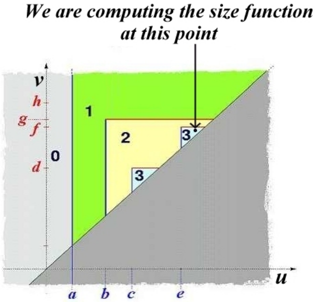
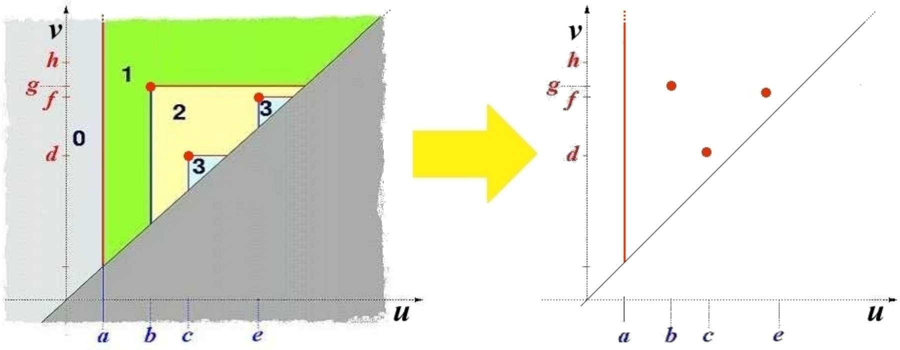




$\longrightarrow$sizeshow.jar+cerchio.avi

We observe that each size function can be described by giving a set of points (vertices of triangles in figure)



A matching distance can be used to compare size functions

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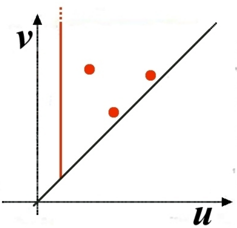

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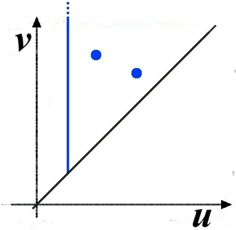

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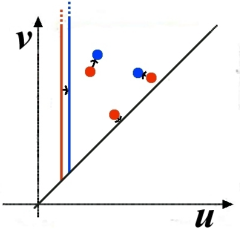
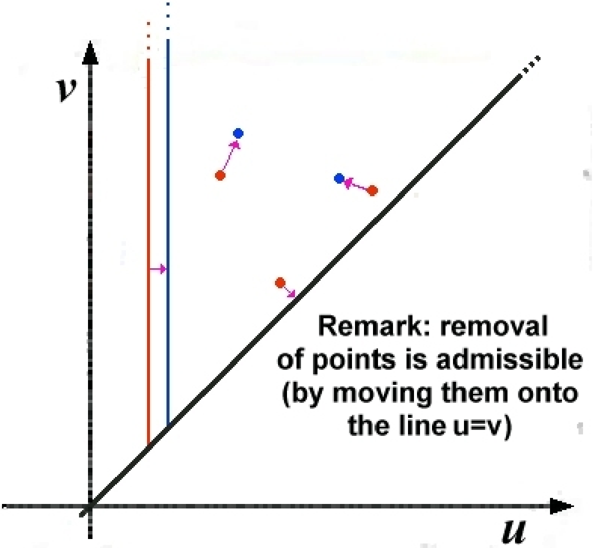
= ***cost of this motion with respect to the max-norm***

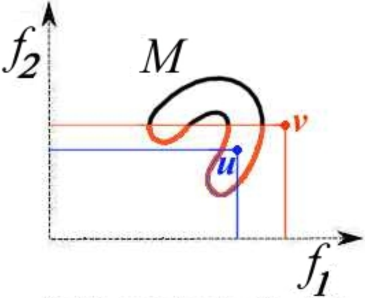


A matching distance can be used to compare size functions

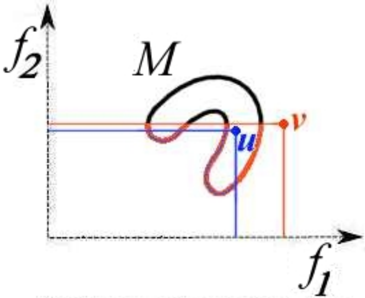


Example of a size function, in the case that the measuring function has two components

Here the measuring function $f = (f_1, f_2)$ is given by the coordinates of each point of the curve M.



In this case the size function takes the value 1.



In this case the size function takes the value 2.

Some references about size functions

- A. Verri, C. Uras, P. Frosini, M. Ferri, *On the use of size functions for shape analysis*, Biological Cybernetics, 70 (1993), 99–107.
- A. Verri, C. Uras, *Metric-topological approach to shape representation and recognition*, Image and Vision Computing, 14, n. 3 (1996), 189–207
- C. Uras, A. Verri, *Computing size functions from edge maps*, International Journal of Computer Vision, 23, n. 2 (1997), 169–183.
- S. Biasotti, L. De Floriani, B. Falcidieno, P. Frosini, D. Giorgi, C. Landi, L. Papaleo, M. Spagnuolo, *Describing shapes by geometrical-topological properties of real functions*, ACM Computing Surveys, vol. 40 (2008), n. 4, 12:1–12:87.

Persistent homology groups and size homotopy groups

Size functions have been generalized by Edelsbrunner and al. to homology in higher degree (i.e., counting the number of holes instead of the number of connected components). This theory is called Persistent Homology:

H. Edelsbrunner, D. Letscher, A. Zomorodian, *Topological persistence and simplification*, Discrete & Computational Geometry, vol. 28, no. 4, 511–533 (2002).

Size functions have been also generalized to size homotopy groups:

P. Frosini, M. Mulazzani, *Size homotopy groups for computation of natural size distances*, Bulletin of the Belgian Mathematical Society, vol. 6, no. 3, 455–464 (1999).

Stability of the matching distance between size functions

An important property of the matching distance is that size functions (and the ranks of persistent homology groups) are stable with respect to it. This stability can be expressed by means of the inequality

$$D_{match} \leq d_F$$

where d_F denotes the natural pseudo-distance.

Therefore, the matching distance between size functions gives a lower bound for the natural pseudo-distance.

In plain words, the stability of D_{match} means that a small change of the measuring function induces a small change of the size function.

Comparing shapes

Stability allows us to use the matching distance to compare shapes:

	0.0000	0.0181	0.1411	0.1470	0.1325	0.1287	0.1171	0.1187
	0.0000	0.0003	0.0025	0.0026	0.0023	0.0022	0.0020	0.0021
	0.0181	0.0000	0.1419	0.1478	0.1304	0.1265	0.1171	0.1187
	0.0003	0.0000	0.0026	0.0026	0.0023	0.0022	0.0020	0.0021
	0.1411	0.1419	0.0000	0.0137	0.1583	0.1370	0.1127	0.1017
	0.0025	0.0025	0.0000	0.0002	0.0028	0.0024	0.0020	0.0018
	0.1470	0.1478	0.0137	0.0000	0.1533	0.1381	0.1137	0.1021
	0.0026	0.0026	0.0002	0.0000	0.0027	0.0024	0.0020	0.0018
	0.1325	0.1304	0.1583	0.1533	0.0000	0.0921	0.0588	0.1000
	0.0023	0.0023	0.0028	0.0027	0.0000	0.0014	0.0016	0.0017
	0.1287	0.1265	0.1370	0.1381	0.0921	0.0000	0.1069	0.1048
	0.0022	0.0022	0.0024	0.0024	0.0014	0.0000	0.0019	0.0018
	0.1171	0.1171	0.1127	0.1137	0.0588	0.1069	0.0000	0.0350
	0.0020	0.0020	0.0020	0.0020	0.0016	0.0019	0.0000	0.0006
	0.1187	0.1187	0.1017	0.1021	0.1000	0.1048	0.0350	0.0000
	0.0021	0.0021	0.0018	0.0018	0.0017	0.0018	0.0006	0.0000

Upper lines refer to the 2-dimensional matching distance associated with a suitable function $f = (f_1, f_2)$ taking values in $\mathbb{R}^2$.
Lower lines refer to the maximum between the 1-dimensional matching distances associated with f_1 and f_2, respectively.

The idea of G-invariant size function

Classical size functions and persistent homology are not tailored on the group G. In some sense, they are tailored on the group $Homeo(M)$ of all self-homeomorphisms of M.

In order to obtain better lower bounds for the natural pseudo-distance we need to adapt persistent homology, and to consider G-invariant persistent homology.

Roughly speaking, the main idea consists in defining size functions and persistent homology by means of a set of chains that is invariant under the action of G.

We skip the details of this procedure, that produces better lower bounds for the natural pseudo-distance.

Uniqueness of models with respect to size functions: the case of curves

We have seen that size functions allow to obtain lower bounds for the natural pseudo-distance. What about upper bounds? This problem is far more difficult.
However, the following statement holds:

Theorem

Let $f = (f_1, f_2), f' = (f'_1, f'_2) : S^1 \to \mathbb{R}^2$ be "generic" functions from S^1 to $\mathbb{R}^2$. If the size functions of the four pairs of measuring functions $(\pm f_1, \pm f_2), (\pm f'_1, \pm f'_2)$ (with corresponding signs) coincide, then there exists a C^1-diffeomorphism $h : S^1 \to S^1$ such that $f' \circ h = f$. Moreover, it is unique.

P. Frosini, C. Landi,
Uniqueness of models in persistent homology: the case of curves,
Inverse Problems, 27 (2011), 124005.

Conclusions

In this talk we have illustrated a general metric scheme for geometrical shape comparison. The main idea of this model is that in shape comparison objects are not accessible directly, but only via measurements made by an observer. It follows that the comparison of two shapes is usually based on a family F of functions, which are defined on a topological space M and take values in a metric space (V, d). Each function in F represents a measurement obtained via a measuring instrument and, for this reason, it is called a "measuring function". In most cases, the set F of measuring functions is invariant with respect to a given group G of transformations, that depends on the type of measurement we are considering.

After endowing F with a natural pseudo-metric, we have presented some examples and results, motivating the use of this theoretical framework.

Closing salutation

"Il catalogo delle forme è sterminato: finché ogni forma non avrà trovato la sua città, nuove città continueranno a nascere. Dove le forme esauriscono le loro variazioni e si disfano, comincia la fine delle città."

"The catalogue of forms is endless: until every shape has found its city, new cities will continue to be born. When the forms exhaust their variety and come apart, the end of cities begins."

Italo Calvino, *Le città invisibili*

THANKS FOR YOUR ATTENTION