New pseudo-distances for the size function space

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ABSTRACT

A method to construct new pseudo-distances for the size function space based on the formal series representation of size functions is introduced. These new pseudo-distances allow to measure quantitatively the differences in shapes by comparing size functions. Some experiments on digital images are shown.

Keywords: size functions, formal series, pseudo-distances

1. INTRODUCTION

According to Size Theory[2,5,6,7,8,10,13,14] a shape can be described by integer functions of two real variables, called size functions. Because of the existence of effective numerical methods for size function computation[3,4], this theory has proven to be very useful for the recognition of objects which have similar but not necessarily identical shapes. For example, size functions have been successfully tested in the task of recognizing different types of leukocytes[1], leaves of different species of plants[15] and hand gestures[11]. In this framework a method to compare size functions is needed as a measure of the similarity of two shapes. The aim of this paper is to introduce a method for defining new pseudo-distances for size functions.

A simple way to compare two size functions ℓ_1 and ℓ_2 is to compute their distance in the space L^2 with respect to a compact set $K \subset \mathbb{R}^2$: we measure how much ℓ_1 differs from ℓ_2 in K by computing the square root of $\int_K (\ell_1 - \ell_2)^2$. However such a distance has two main drawbacks: the first is that working with functions requires a large amount of information storage; the second is that this distance does not allow to highlight some features of the size function considered as important by the observer and to discard others (for example, it is difficult to reject that part of information due to noise). For these reasons we suggest another approach.

In another paper[9] we prove that size functions can be represented as countable collections of points and lines of $\mathbb{R}^2$ endowed with multiplicities, i.e. formal series with support in a set of points and lines of $\mathbb{R}^2$. By using this result we can reduce the problem of comparing size functions to the one of comparing sets of points and lines in the real plane: this gives a method to construct new pseudo-distances between size functions. These pseudo-distances turn out to be much more flexible than the ones used until now.

After recalling some definitions in Section 2, we give (Section 3) the main theorem about the representation of size functions as formal series. In Section 4 we introduce some methods to obtain new pseudo-distances from this representation. In order to better illustrate these results, in Section 5 some experiments on images will be exhibited.

2. PRELIMINARY DEFINITIONS

We shall consider as shapes to study subsets $\mathcal{X}$ of a Euclidean space. Every continuous function φ from $\mathcal{X}$ to the real numbers will be called a *measuring function.* Suppose a pair $(\mathcal{X}, \varphi)$ is given.

For every real number y we shall say that two points $P, Q \in \mathcal{X}$ are $\langle \varphi \leq y \rangle$*-homotopic* if and only if either $P = Q$ or in $\mathcal{X}$ there exists a continuous path $\gamma : [0,1] \to \mathcal{X}$ joining P and Q such that $\varphi(\gamma(t)) \leq y$ for every $t \in [0,1]$.

For every real number x we shall define $\mathcal{X}\langle \varphi \leq x \rangle \stackrel{\text{def}}{=} \{P \in \mathcal{X} | \varphi(P) \leq x\}$. It is easy to see that for every $x, y \in \mathbb{R}$ the $\langle \varphi \leq y \rangle$-homotopy is an equivalence relation on $\mathcal{X}\langle \varphi \leq x \rangle$.

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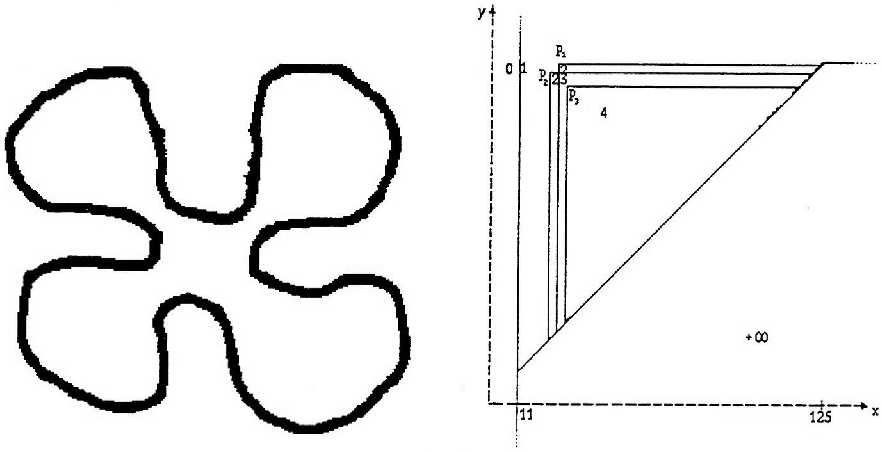

Figure 1. A subset of the plane (in black) and its size function computed w.r.t. the measuring function distance from the barycentre.

DEFINITION 21. *Consider the function $\ell_{(\mathcal{X},\varphi)}$ from $\mathbb{R}^2$ to the extended set $\mathbb{N} \cup \{+\infty\}$ of the natural numbers defined by setting $\ell_{(\mathcal{X},\varphi)}(x,y)$ equal to the number of equivalence classes into which $\mathcal{X}\langle\varphi \leq x\rangle$ is divided by the relation of $\langle\varphi \leq y\rangle$-homotopy. We shall call $\ell_{(\mathcal{X},\varphi)}$ the size function associated to the pair $(\mathcal{X},\varphi)$.*

To illustrate how size functions work, in Figure 1 we show a planar shape and its size function calculated with respect to the measuring function distance from the barycentre. Discontinuities of the size function divide the domain into regions and the number displayed in each region is equal to the value taken by the size function at that point. From this example it can be seen that details of the shape are represented in the region near the diagonal; the further from the diagonal, the coarser is the information on the viewed shape.

The possibility of choosing the measuring function between all continuous functions on $\mathcal{X}$ is what makes size functions suitable for the general recognition problem and not only for particular ones. For instance the choice made in the example shown in Figure 1 allows us to detect the presence of four big bumps in the studied shape.

For the main results about Size Theory we refer to previous papers [5,6,8,12]. Nevertheless it will be useful to point out some simple properties of size functions.

From the very definition of size function one has immediately that every size function is non-decreasing in the variable x and non-increasing in the variable y.

Moreover it is easy to see that if $\mathcal{X}$ is compact then $\ell_{(\mathcal{X},\varphi)}(x,y) = 0$ for $x < \min_{P\in\mathcal{X}} \varphi(P)$ and for every $x, y \geq \max_{P\in\mathcal{X}} \varphi(P)$ $\ell_{(\mathcal{X},\varphi)}(x,y)$ is equal to the number of arcwise connected components of $\mathcal{X}$; furthermore if for $y < x$ there exists an infinite number of points $P \in \mathcal{X}$ such that $y < \varphi(P) < x$ then $\ell_{(\mathcal{X},\varphi)}(x,y) = +\infty$.

We can summarize the above remarks by saying that in most situations occurring in Image Analysis, outside a compact region of $\mathbb{R}^2$ above the diagonal a size function does not provide us with relevant information about the shape we are studying.

On the other hand if $\mathcal{X}$ is "good" enough the associated size functions are not trivial; indeed it can be shown that if $\mathcal{X}$ is a finite union of compact arcwise connected and locally arcwise connected subsets of a Euclidean space then $\ell_{(\mathcal{X},\varphi)}(x,y) < +\infty$ for $x < y$. Therefore in the sequel we shall assume that $\mathcal{X}$ satisfies these properties.

Finally note that a discrete analogue of the concepts just described has been developed [3]. This allows us to obtain algorithms for the computation of size functions.

3. BASIC RESULTS

It must be observed that when we study a shape from the size functions viewpoint what we really look at is the structure of the set of discontinuity points of the size function and its values in the regions cut out by such a set.

This makes sense because discontinuities of a size function must satisfy some general properties[9]. The most important is certainly that for $x < y$ discontinuities in the variable x propagate downward to the diagonal $\{(x, y) \in \mathbb{R}^2 | x = y\}$ and discontinuities in the y propagate toward right to the diagonal. In other words the set of discontinuities partitions the domain of a size function into overlapping triangular regions (possibly of infinite area) leaning against the diagonal (see Figure 1). We also have that jumps of size functions cannot decrease as discontinuities spread towards the diagonal.

These facts suggest us to give the following definitions:

DEFINITION 31. *For every point $p = (x, y)$ with $x < y$ we shall call multiplicity of p the number*

$$\lim_{\epsilon \to 0+} \ell_{(\mathcal{X},\varphi)}(x+\epsilon, y-\epsilon) - \ell_{(\mathcal{X},\varphi)}(x+\epsilon, y+\epsilon) - \ell_{(\mathcal{X},\varphi)}(x-\epsilon, y-\epsilon) + \ell_{(\mathcal{X},\varphi)}(x-\epsilon, y+\epsilon)$$

and denote it by $m^1(p)$. If the number $m^1(p)$ is strictly positive we say that p is a cornerpoint for the size function $\ell_{(\mathcal{X},\varphi)}$.

This definition just means that along an orizontal line passing through a cornerpoint the vertical jump of $\ell_{(\mathcal{M},\varphi)}$ changes in every sufficiently small neighbourhood of the cornerpoint. It can be shown that the above limit actually exists and is finite. Moreover under our hypothesis on $\mathcal{X}$ a size function has at most a countable set of cornerpoints and for any $\epsilon > 0$ we have that in the region $\mathcal{S}_\epsilon = \{(x, y) \in \mathbb{R}^2 | x < y - \epsilon\}$ cornerpoints are even in finite number. In the example shown in Figure 1 the only cornerpoints are p_1, p_2, p_3 and each one has multiplicity 1.

The idea is that of using cornerpoints to identify triangles of finite area which, as we have seen, are a major feature in a size function. We can give a similar notion for triangles of infinite area:

DEFINITION 32. *For every vertical line $r : x = k$ with $k \in \mathbb{R}$ we shall call multiplicity of r the number*

$$m^2(r) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0^+, y \to +\infty} \ell_{(\mathcal{X},\varphi)}(k+\epsilon, y) - \ell_{(\mathcal{X},\varphi)}(k-\epsilon, y).$$

When $m^2(r)$ is strictly positive the line r will be said a cornerline for the size function.

Again it can be shown that the above limit exists, is finite and that our assumptions on $\mathcal{X}$ imply that a size function can have at most a finite number of cornerlines. In the example shown in Figure 1 the only cornerline has equation $x = 11$ and has multiplicity equal to 1.

The value of a size function at a point (x, y) with $x < y$ can be calculated as the sum of the multiplicities of cornerpoints and cornerlines that identify triangles containing that point.

Let us now denote by $\mathcal{R}$ the set of vertical lines of equation $x = k$ with $k \in \mathbb{R}$; we shall call *formal series in* $\mathcal{S}_\epsilon \cup \mathcal{R}$ any object of the form $\sigma = \sum m(X)X$ with $m(X) \in \mathbb{N}$ and X varying in a subset of $\mathcal{S}_\epsilon \cup \mathcal{R}$; the set of $X \in \mathcal{S}_\epsilon \cup \mathcal{R}$ such that $m(X) \neq 0$ is called *support of* σ.

We are ready to state the main theorem of this paper:

THEOREM 33. *For every positive real number ϵ there exists an injective mapping α from the set of size functions quotiented by the relation of coincidence almost everywhere in $\mathcal{S}_\epsilon$ into the set of formal series in $\mathcal{S}_\epsilon \cup \mathcal{R}$ with finite support.*

α is defined in a natural way as the map which takes the class of a size function ℓ into the formal series $\sum m(X)X$ where X varies in the set of all cornerpoints and cornerlines for ℓ belonging to $\mathcal{S}_\epsilon \cup \mathcal{R}$ and $m(X)$ is its multiplicity. From the definition of α and properties of size functions we see that not every formal series in $\mathcal{S}_\epsilon \cup \mathcal{R}$ belongs to the image of α. Actually a formal series σ with finite support in $\mathcal{S}_\epsilon \cup \mathcal{R}$ represents a size function if and only if for every point p in the support of σ there exists a line $x = k$ also in the support of σ such that p belongs to the half-plane $\{(x, y) \in \mathbb{R}^2 | x > k\}$.

Therefore size functions can be transformed in a more compact and manageable object.

4. NEW PSEUDO-DISTANCES FOR SIZE FUNCTIONS

It follows from Theorem 3.3 that we can reduce the problem of comparing size functions to the problem of comparing formal series. This has two main advantages: it is clear that formal series allow a concise and algebraic representation of size functions; moreover, since each term of a formal series contains a certain amount of information, by discarding some terms we are able to highlight certain features of the size function under study instead of others.

For these reasons we shall give three pseudo-metrics on the set of size functions induced by distances and pseudo-distances defined on the set of formal series with finite support in $\mathcal{S}_\epsilon \cup \mathcal{R}$. Let us recall that the definition of pseudo-distance differs from that of distance just for one of its axioms: the axiom "$\mathrm{d}(x, y) = 0$ if and only if $x = y$" holding for distances is replaced by "$\mathrm{d}(x, x) = 0$" in the definition of pseudo-distance. We allow pseudo-distances to take the value $+\infty$.

In order to define pseudo-distances between size functions let us choose two sets $I \subseteq \mathcal{S}_\epsilon$ and $J \subseteq \mathcal{R}$. Now let us consider two size functions ℓ_1 and ℓ_2. The map α of Theorem 3.3 takes these two size functions in two formal series σ_1 and σ_2 with finite support in $\mathcal{S}_\epsilon \cup \mathcal{R}$. Let us denote by $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ the formal series obtained from σ_1 and σ_2 respectively by deleting the terms lying out of $I \cup J$. It follows that any distance or pseudo-distance between $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ induces a pseudo-distance between ℓ_1 and ℓ_2. We shall use the same symbols to denote both the (pseudo-)distances between $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ and the induced pseudo-distances between ℓ_1 and ℓ_2.

In what follows we shall show three methods to measure the distance between $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$. We suppose that $\tilde{\sigma}_1 = \sum_{X \in I_1 \cup J_1} m(X)X$ and $\tilde{\sigma}_2 = \sum_{Y \in I_2 \cup J_2} n(Y)Y$ where I_1 and I_2 denote two finite subsets of I while J_1 and J_2 denote two finite subsets of J.

Deformation distance. We can consider as many copies of each point of I_1 (resp. I_2) as its multiplicity so to form a new set of distinct points $\tilde{I}_1$ (resp. $\tilde{I}_2$). Repeat the same for lines in J_1 and J_2 to obtain the sets $\tilde{J}_1$ and $\tilde{J}_2$. Let F be the set of all injective functions f from a subset D_f of $\tilde{I}_1$ into $\tilde{I}_2$ and let G be the set of all bijective functions g from $\tilde{J}_1$ to $\tilde{J}_2$. For technical reasons we allow the possibility that D_f is empty; therefore F contains also the function which takes the empty set to itself. Moreover if $\tilde{J}_1 = \tilde{J}_2 = \emptyset$ we take G equal to the set containing only the function from the empty set to the empty set.

For each pair $(f, g) \in F \times G$ we can now compute the value $v(f, g)$ as follows: we start with $v(f, g) = 0$ and then for each $p \in D_f$ we increase $v(f, g)$ by the distance of p from $f(p)$; analogously for each $r \in \tilde{J}_1$ we add to $v(f, g)$ the distance between r and $g(r)$; finally for each point $p \in \tilde{I}_1 \setminus D_f$ we increase $v(f, g)$ by the distance of p from the diagonal $\Delta = \{(x, y) \in \mathbb{R}^2 | x = y\}$ and for each point $q \in \tilde{I}_2 \setminus f(D_f)$ we increase $v(f, g)$ by the distance of q from the diagonal Δ.

Thus we can define the following distance between $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$:

$$\mathrm{dist}_1(\tilde{\sigma}_1, \tilde{\sigma}_2) \stackrel{\mathrm{def}}{=} \begin{cases} +\infty & \text{if } G = \emptyset \\ \min_{(f,g) \in F \times G} v(f, g) & \text{otherwise} \end{cases}.$$

In plain words what we actually do to calculate such a distance is to measure the reciprocal distances of pairs of points and pairs of lines of the two formal series under study, allowing to destroy some points by sending them onto the diagonal. Then we choose the matching which minimize the sum of these distances. Thus we obtain a distance between formal series which takes into account multiplicities and is small when formal series are similar. In Figure 2 we show how it works, when $I \cup J = \mathcal{S}_\epsilon \cup \mathcal{R}$ and ϵ is very small, on two formal series $r + a + b + c$ and $r' + a' + c'$ obtained as images of two size functions by the map α of Theorem 3.3. The set of discontinuity points of the two size functions are represented by continuous and dotted lines respectively. One sees that among all the possible choices for (f, g) the one that realizes the minimum for $v(f, g)$ is such that $f(a) = a', f(c) = c', g(r) = r'$ whereas b is sent onto the diagonal.

Hausdorff pseudo-distance. Another method to compare formal series with finite support one can think of is based on the concept of Hausdorff distance: with the same notations as before and by denoting with $d(\cdot, \cdot)$ the usual distance between lines one sets

$$\mathrm{dist}_2(\tilde{\sigma}_1, \tilde{\sigma}_2) = \max\left\{\max_{p \in I_1} \min_{q \in I_2} \|p - q\|, \max_{q \in I_2} \min_{p \in I_1} \|p - q\|\right\} + \max\left\{\max_{r \in J_1} \min_{s \in J_2} d(r, s), \max_{s \in J_2} \min_{r \in J_1} d(r, s)\right\}.$$

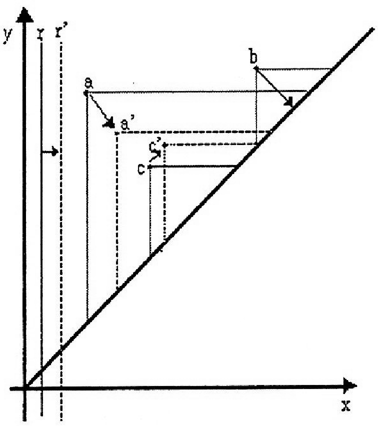

Figure 2. How to compute the pseudo-distance dist_1 between two size functions.

Anyway in doing so we forget the information about multiplicities.

L^p pseudo-distance. A further way to define a pseudo-distance between formal series is based on the idea of associating to each element X of $I \cup J$ an arbitrary L^p function $f_X : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with support in $\{(x, y) \in \mathbb{R}^2 | x < y\}$. Thus we can associate to $\tilde{\sigma}_1$ the L^p function $\sum_{X \in I_1 \cup J_1} m(X) f_X$ and to $\tilde{\sigma}_2$ the L^p function $\sum_{Y \in I_2 \cup J_2} n(Y) f_Y$. Now we can set

$$\mathrm{dist}_3(\tilde{\sigma}_1, \tilde{\sigma}_2) = \left\| \sum_{X \in I_1 \cup J_1} m(X) f_X - \sum_{Y \in I_2 \cup J_2} m(Y) f_Y \right\|_p ,$$

where $\| \cdot \|_p$ denotes the usual L^p-norm.

Of course the pseudo-distances we have just defined depend on some choices, namely the choice of ϵ, I, J. As we have already observed in Section 2 the smaller is ϵ the more detailed is the information about the shapes under study. As for the choice of I and J, the possibility of discarding cornerpoints and cornerlines lying out of $I \cup J$ is very useful when working in presence of noise and occlusions: in these situations one can single out certain sub-structures in the size function as the relevant ones. Therefore it is desirable to be able to compare just these sub-structures instead of the whole functions.

5. CONCLUSIONS AND EXAMPLES

Let us measure the similarity between pairs of shapes by using the first pseudo-distance defined above. In Figures 3, 4 and 5 we show three sets of images: the contour of two "zed"s (Figure 3), of two "g"s (Figure 4) and that of two "s"s (Figure 5) . For each image we exhibit the corresponding size function with measuring function abscissa of the point. In each example we have set the minimum of this function equal to 0. Moreover we have chosen $I = \mathcal{S}_\epsilon$, $J = \mathcal{R}$ and ϵ close to 0.

The formal series corresponding to the size function ℓ_1 representing the "zed" in Figure 3, top, is $r + p_1 + p_2 + p_3 + p_4$ where $r : x = 0, p_1 = (0.94, 62.06), p_2 = (22.57, 31.97), p_3 = (10.34, 75.23), p_4 = (54.54, 69.59)$; for the size function ℓ_2 representing the "zed" in Figure 3, bottom, the associated formal series is $s + q_1 + q_2 + q_3 + q_4$ with $s : x = 0, q_1 = (6.15, 64.59), q_2 = (13.32, 36.90), q_3 = (17.42, 77.91), q_4 = (52.28, 77.91)$. Thus we have that $\mathrm{dist}_1(\ell_1, \ell_2) = 32.46$.

Let us now compute the pseudo-distance between the size functions ℓ_3 and ℓ_4 representing the "g"s of Figure 4. The corresponding formal series are: $r_1 + r_2 + r_3 + p$ with $r_1 : x = 0, r_2 : x = 8.06, r_3 : x = 34.95, p = (26.88, 72.60)$ for ℓ_3; $s_1 + s_2 + s_3 + q$ with $s_1 : x = 0, s_2 : x = 8.33, s_3 : x = 48.95, q = (41.66, 69.79)$ for ℓ_4. Thus the pseudo-distance between the two "g"s is $\mathrm{dist}_1(\ell_3, \ell_4) = 29.31$.

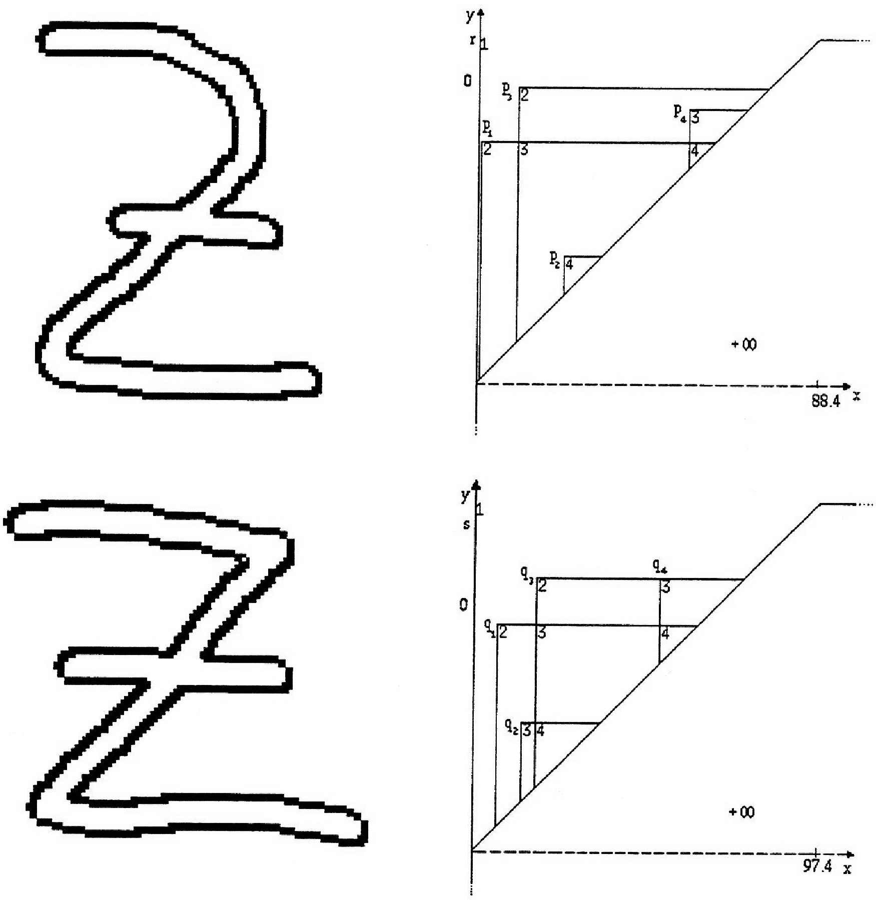

Figure 3. Contours of two "zed"s and their size functions ℓ_1 (top) and ℓ_2 (bottom) w.r.t. the measuring function abscissa of the point.

Finally let us compute the pseudo-distance between the two "s"s of Figure 5. Called ℓ_5 and ℓ_6 the corresponding size functions we have that $\alpha(\ell_5) = r + p_1 + p_2$ where $r : x = 0$, $p_1 = (6.67, 52.05)$ and $p_2 = (14.67, 60.06)$; $\alpha(\ell_6) = r + q_1 + q_2$ with $r : x = 0$, $q_1 = (8.27, 47.30)$ and $q_2 = (9.46, 50.26)$. It follows that the value of dist_1 on these two size functions is 17.58.

It is easy to compute the pseudo-distance between any "zed" and any "g": it is equal to $+\infty$ because there exists no bijection between the set of cornerlines of a "zed" and that of a "g". The same holds for the pseudo-distance between any "s" and any "g".

Now if we look at the pseudo-distances between a "zed" and an "s", we have that $\text{dist}_1(\ell_1, \ell_5) = 46.11$, $\text{dist}_1(\ell_2, \ell_5) = 63.41$, $\text{dist}_1(\ell_1, \ell_6) = 60.26$ and $\text{dist}_1(\ell_2, \ell_6) = 73.13$. Thus we can see that the pseudo-distance between any of the four possible pairs of different letters is larger than any pseudo-distance between equal letters.

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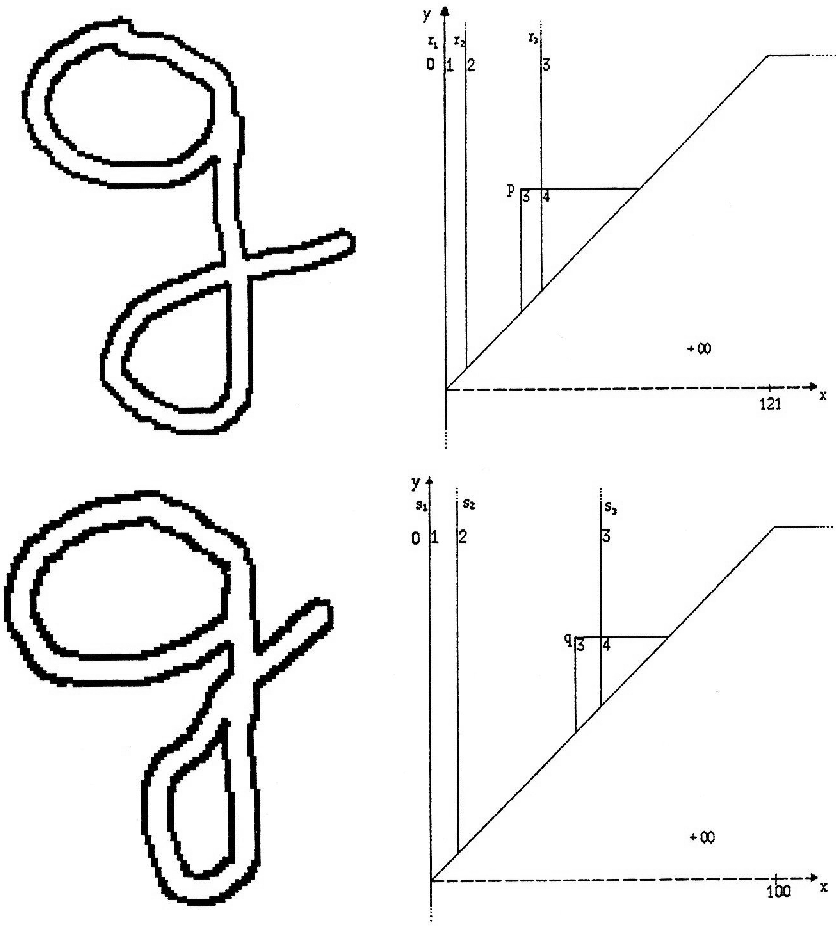

Figure 4. Contours of two "g"s and their size functions ℓ_3 (top) and ℓ_4 (bottom) w.r.t. the measuring function abscissa of the point.

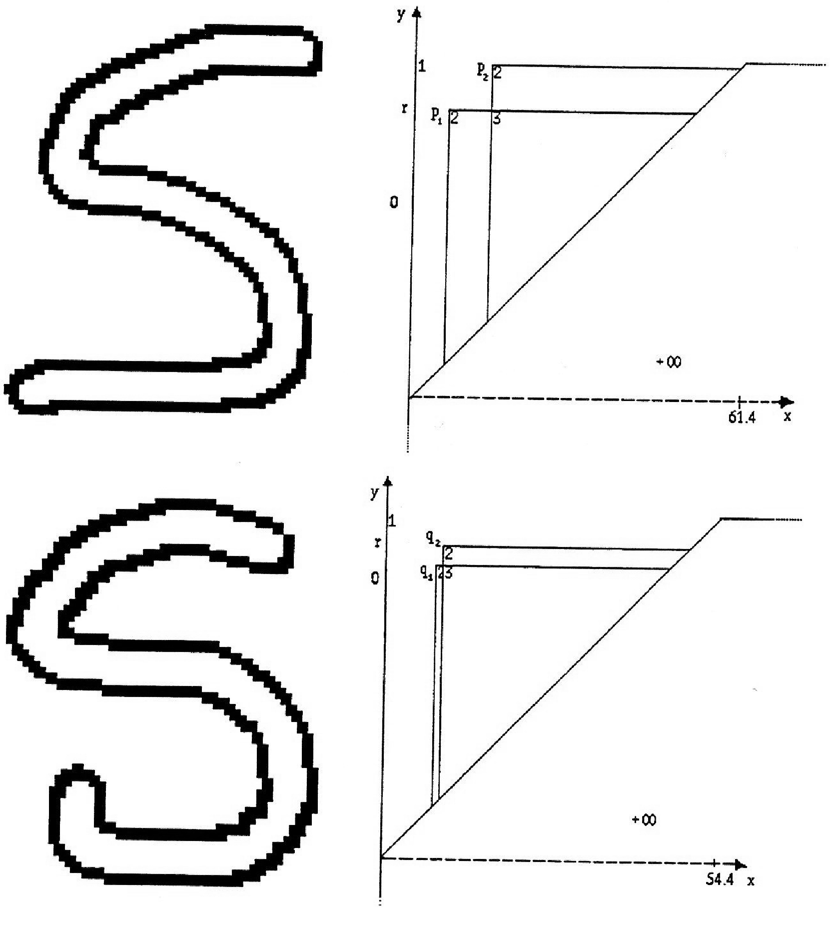

Figure 5. Contours of two "s"s and their size functions ℓ_5 (top) and ℓ_6 (bottom) w.r.t. the measuring function abscissa of the point.

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