On the Use of Size Functions for Shape Analysis

A. Verri C. Uras
Dipartimento di Fisica
dell'Università di Genova
Genova, Italy

P. Frosini M. Ferri
Dipartimento di Matematica
dell'Università di Bologna
Bologna, Italy

Abstract

According to a recent mathematical theory a shape can be represented by functions, named size functions, *which convey information on both the topological and metric properties of the viewed shape. In this paper the relevance of the theory of size functions to computer vision is investigated. An algorithm for the computation of the size functions is presented and many theoretical properties of the theory are demonstrated on real images. It is shown that the representation of shape in terms of size functions* (i) *can be tailored to suit the invariance of the problem at hand and* (ii) *is stable against small qualitative and quantitative changes of the viewed shape. A distance between size functions is used as a similarity measure between the representation of two different shapes. The obtained results indicate that size functions are likely to be very useful for object recognition. In particular, they seem to be well suited for the recognition of natural and articulated objects.*

1 Introduction

A.. intriguing property of the human visual system is the capability of recognizing objects independent of their apparent shape in images. The changes in the visual shape can be due to different factors. In the case of rigid and manufactured objects, for example, these changes are due to the object orientation and distance from the viewer. In the case of natural objects, these changes may also be due to the qualitative and quantitative differences between objects which belong to the same "category". Most of the techniques which have been proposed for shape analysis and object recognition appear to be appropriate for some particular and interesting cases, like polyhedral rigid objects, planar curves, or character recognition, but do not seem to be sufficiently flexible to deal with the general problem.

In a recent series of mathematical papers, studying shape through integer-valued functions, called *size functions* [1, 2, 3, 4, 5], has been proposed. The new mathematical idea underlying the concept of a size function is that of setting *metric* bounds to the classical notion of *homotopy*, i.e., of continuous deformation. Thus, size functions convey information about both the qualitative and quantitative structure of the viewed shape. The aim of this paper is to assess the potential of the theory of size functions in computer vision. An algorithm for the computation of size functions is presented and the many theoretical properties of the size functions are checked and illustrated on real images. It is shown that the representation of shape through size functions can be tailored to suit the quantitative and qualitative invariant properties of the shape to be studied. Therefore, size functions seem to be suitable for the description and recognition of objects which have similar but not necessarily identical shape (like natural, articulated, and nonrigid objects).

The paper is organized as follow: In Section 2 the approach to shape description through the theory of size functions is introduced through a simple example. An algorithm for the computation of a size function is described in Section 3. Section 4 deals with the invariant properties which may be incorporated in the theory and illustrates the tolerance of the scheme against changes of different sort on real images. Finally, Section 5 summarizes the obtained results.

2 Overview of the approach

In this Section the concepts of the theory of the size functions are illustrated over a simple example. Then, the shape representation which can be obtained from a size function is discussed. Finally, the main theoretical properties of the theory are listed.

2.1 Topological and metric obstructions

Fig. 1*a* shows an example of shape, the letter "w" in sign language performed by one of the authors. By

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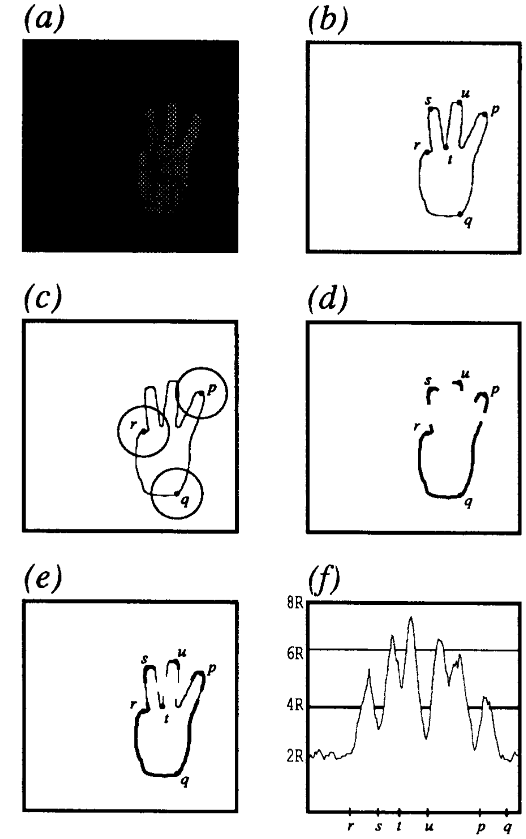

Figure 1: Topological and metric obstructions. (a) An image of the sign "w" performed by one of the authors. (b) The outline α of the sign of (a) obtained by means of standard edge detection and contour following techniques. (c) Computation of the measuring function L at the points p, q, and r. (d) The thick edges mark the points with $L \leq 4R$. (e) The thin edges mark the points with $4R \leq L \leq 6R$. (f) Plot of the measuring function L over the curve α.

using standard edge detection and contour following techniques, the contour α of Fig. 1b, which corresponds to the outline of the hand of Fig. 1a, can be easily obtained. Let us introduce the key concepts of measuring function and size function on the curve α.

As a preliminary step, let us define a transformation H which brings a point of α onto some other point of α without leaving the curve. The transformation H induces an equivalence relation on the points of α, where two points v and w are said to be ***H-equivalent*** if there exists a continuous trajectory on α which brings v onto w, or if v and w belong to the same arcwise-connected component of α. For example, the points p, q, r, s, t, and u of Fig. 1b are all pairwise H-equivalent. Since, independent of the shape of α, all the points of α fall into one and the same equivalence class, the purely topological concept of H-equivalence is clearly not sufficient to characterize the shape of α. Intuitively, this reflects the absence of "topological obstructions" between the points of α.

In the theory of size functions, this problem is overcome by means of the notion of *measuring* function [1]. The purpose of a measuring function is to generate "metric obstructions" for the transformation H. Let us illustrate the notion of measuring function through a particular example. For each point v of α, let $L = L(v)$ be the length of the portion of α which lies within the circle $c(v)$ of radius R and center v. Fig. 1c shows how to compute L at the points p, q, and r. Let $R = D/5$, where D is the diameter of α. It is clear that $L(p)$, $L(q)$, and $L(r)$ can be computed as the sum of the length of the (possibly many) arcs of α which lie within the circles $c(p)$, $c(q)$, and $c(r)$ of Fig. 1c respectively. The function L, which is defined on the contour α, is an example of measuring function.

Let us now modify the definition of H by means of L. Two points v and w of α are said to be $H(L \leq y)$-*equivalent* if $v = w$ or a trajectory exists on α from v to w along which L never exceeds y. Let us call a trajectory along which L never exceeds y an $(L \leq y)$-*trajectory*. Intuitively, the points of α with $L > y$ can be thought of as metric obstructions for the $(L \leq y)$-trajectories from v to w. This fact is illustrated in Fig. 1d where the points with $L > 4R$ (the metric obstructions) have not been drawn. From the gaps in Fig. 1d, it is easy to conclude that, between p, q, r, s, and u, the points q and r are the only pair of $H(L \leq 4R)$-equivalent points.

The notion of $H(L \leq y)$-equivalence is essential for the definition of *size* function [1]. The size function $l_L(\alpha; x, y)$, for $x < y$, and x and $y \in \Re^2$, is defined as the number of equivalence classes in which the set of points with $L \leq x$ is divided by the $H(L \leq y)$-equivalence relation. Let us compute the size function l_L at the point (x, y) with $x = 4R$ and $y = 6R$. The set of points of α with $L \leq 4R$ are the thick edges of Fig. 1e. Thus, the size function $l_L(\alpha; 4R, 6R)$ is the number of equivalence classes in which the thick edges of Fig. 1e are divided by the $H(L \leq 6R)$-equivalence relation. This amounts to look for $(L \leq 6R)$-trajectories between all the possible pairs of thick edges. The thin edges of Fig. 1e, which are the points with $4R < L \leq 6R$, represent the "extra" space which has been made available to the $(L \leq 6R)$-trajectories.

It is easy to see that the size function $l_L(\alpha; 4R, 6R)$ equals the number of connected components of the curve of Fig. 1e (ignoring the difference between thick and thin edges) which contains at least one point with $L \leq 4R$, i.e., a thick edge. Since of the three connected components of the set of points with $L \leq 6R$, the one which contains the point t consists only of thin edges, it follows that $l_L = 2$. Note that the points p, q, r, and u, which were not $H(L \leq 4)$-equivalent, now belong to the same equivalence class.

An equivalent representation of the connected components of the set of points with $L \leq 4R$ under the $(L \leq 6R)$-equivalence relation is shown in Fig. 1f, in which L is plotted against the curve $\alpha = \alpha(a)$ with $a \in [0, 1]$ and $\alpha(0) = \alpha(1)$. The thick horizontal line of Fig. 1f makes it clear that the set of points with $L \leq 4R$ consists of four connected components (the leftmost and rightmost component belong to the same component because $\alpha(0) = \alpha(1)$). The thin horizontal line shows that these components reduce to two when $L \leq 6R$.

Before discussing the main properties of the notion of size function, let us show how shape information is represented by means of a size function.

2.2 Shape representation

The size function $l_L = l_L(\alpha; x, y)$ is an integer-valued function of the two real variables x and y. Let us first show that all the relevant information is contained in a region of finite area of the plane (x, y). Let us divide the plane (x, y) in four regions, A, B, C, and T. The region A consists of all the points at the left of the vertical line $x = L'$, B of all the points with $L' < x \leq y$ and $y > L''$, C of all the points with $x > L'$ and $y < x$, and T of all the points of the triangle with $L' \leq y \leq L''$ and $L' \leq x \leq y$, where L' and L'' are the minimum and the maximum of L over α respectively. Let us now show that in A, B, and C the size function is independent of the shape of the curve α. First, since the set of points with $L < L'$ is the empty set, we have that, for all the points in A, $l_L = 0$. Then, since for $L > L''$ there are no metric obstructions, we have that, for all the points in B, $l_L = 1$. Finally, since for $y < x$ each point of α identifies a different equivalence class, for all the points in C, $l_L = +\infty$.

Thus, all the relevant information is contained in the triangular region T enclosed by the regions A, B, and C. Moreover, it is evident that a size function is always piecewise constant, nondecreasing along the x-axis, and nonincreasing along the y-axis. These properties follow easily from the definition of size function.

2.3 Basic properties

The notions of measuring function and size function have a number of interesting properties. Let us briefly summarize these properties, some of which will be illustrated in greater detail in Section 4. In what follows α denotes a generic curve of the image plane. First, there is a wide range of possible choices for a measuring function. In principle, any continuous real function defined on a curve α can serve as a measuring function. For example, along with the function L of the previous Section, the curvature, the distance of a point of α from a certain point, like the center of mass, or the y-coordinate of a point of α with respect to some system of reference are equally good choices of measuring functions. In addition, a measuring function need not be defined on single points of α. For example, a measuring function can be defined on the pairs of points of α. The only difference is the fact that the measuring function cannot be visualized any longer through a one-dimensional plot and that the notion of $H(L \leq y)$-*equivalence* must be redefined on $\alpha \times \alpha$, the Cartesian product of the curve with itself. Examples of measuring functions defined on $\alpha \times \alpha$ are the Euclidean distance between a pair of points of α, and the ratio between this distance and the length of the shortest arc of α which joins the pair of points.

A fundamental property of the proposed scheme is that the representation of shape through a size function inherits the invariance properties (if any) of the underlying measuring function. These properties may include Euclidean invariance (like invariance for scaling, and translation and rotation over the image plane), or invariance for affine, projective, and perspective transformation. Clearly, the function L of Section 2 is invariant for translation and rotation. The scaling invariance can always be obtained by scaling the maximum of the measuring function to a fixed value. In general, the invariance properties of the problem at hand may be used to constrain the search for the appropriate measuring function.

A further property of the representation of shape through size functions is robustness against quantitative and qualitative changes. This property derives from the facts that the proposed representation combines topological (i.e., qualitative) aspects of shape with metric (i.e., quantitative) aspects of shape in a redundant fashion, since a size function is piecewise constant. Intuitively, small qualitative and quantitative changes give rise to differences in the size functions over correspondingly small areas of the triangular region of interest. Consequently, the representation of shape in terms of size functions is likely to be suitable for the recognition of objects which are qualitatively and quantitatively similar but not identical.

Let us conclude this Section with a theoretical remark. The theory of size functions is not restricted to the case of curves. In principle the shape of a surface of arbitrary dimension can be represented through a size function [1]. This paper and the present research have been restricted to the analysis of curves on the image plane. The extension to the two dimensional case, in which the surface is simply the pattern of the grey values, is currently under development.

3 An Algorithm for the computation of size functions

This Section describes the implementation of an algorithm for the discrete computation of a size function of a curve of the image plane.

For the sake of simplicity, let us illustrate the implementation in the particular case in which the measuring function φ is defined on single points of a curve α (that is, $k = 1$) with $\varphi \geq 0$ (a more general description can be found in ref. [6]). In addition, let $B(p)_\delta$ be the open circle of center p and radius δ, and l_φ and $\bar{l}_\varphi$ the size function in the continuous and discrete cases respectively. The algorithm consists of four steps.

1. Sample (or approximate) the curve α at a finite number N of points p^i, $i = 1, ..., N$, so that *(i)* $\alpha \subset \cup_{i=1}^{N} B(p^i)_\delta$ and *(ii)* the set $B(p^i)_\delta \cap \alpha$ is nonempty and connected for $i = 1, ..., N$ (see Fig. 2*a*).

2. Define the graph G whose vertices are the points p^i and whose edges link vertices which correspond to adjacent points on α. Compute $\varphi(p^i)$ at each point p^i, $i = 1, ..., N$ (see Fig. 2*b*).

3. Compute the maximum $\varphi^{\max}$ of $\varphi(p^i)$, $i = 1, ..., N$ and fix a $\Delta \geq \epsilon_\varphi(\delta)$, where $\epsilon_\varphi(\delta)$ is the modulus of continuity of φ at δ.

4. For $y = 0$ to $y \leq \varphi^{\max}$

   (a) Define the subgraph $G_{\varphi \leq y}$ of G induced by the set of vertices of G for which $\varphi \leq y$ (Fig. 2*c*).

   (b) For $x = 0$ to until $x \leq y$

      i. Let $\bar{l}_\varphi(\alpha; x, y)$ be the number of connected components of $G_{\varphi \leq y}$ which contain at least a vertex p^i such that $\varphi(p^i) \leq x$ (Fig. 2*d*).

      ii. $x \rightarrow x + \Delta$.

   (c) $y \rightarrow y + \Delta$.

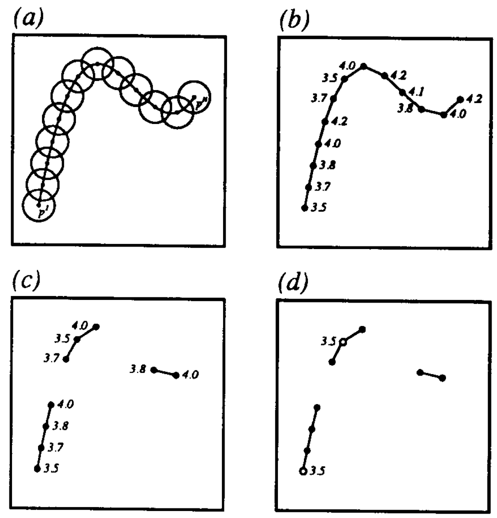

Figure 2: The algorithm for the discrete computation of a size function. (*a*) Curve sampling and covering. (*b*) The graph associated with the sampled curve. The numbers, which correspond to hypothetical values of the measuring function φ at each sampled point, are associated with the corresponding vertex. (*c*) Subgraph of the graph of (*b*) induced by the set of vertexes with $\varphi \leq 4.0$. (*d*) The vertexes of the subgraph of (*c*) with $\varphi \leq 3.6$ are shown as open circles. Therefore, the value of the size function of the sampled curve of (*a*) at the point $(3.6, 4)$ equals 2.

The conditions (i) and (ii) of the first step ensure that the curve α is covered in such a way that each open circle contains exactly one connected arc of α. It is evident that the size function which can be computed in the discrete will be the same for all the continuous curves for which the open circles of the first step satisfy (i) and (ii). The graph G, in the second step, is a discrete representation of α such that a path on G between the vertices p^i and p^j is the discrete counterpart of a trajectory between points of the two arcs $B(p^i)_\delta \cap \alpha$ and $B(p^j)_\delta \cap \alpha$. The third step determines the minimal resolution at which $\bar{l}_\varphi$ is worth computing and the thickness of the "white" stripes, or the regions of uncertainty in the value of the size function in the continuous case. In the final step $\bar{l}_\varphi$ is computed over a grid of equally spaced points within the triangular region $T_\varphi(\alpha) = \{(x,y) : 0 \le y \le \varphi^{\max}, 0 \le x \le y\}$.

The computational load of the algorithm depends on the choice of the measuring function. If the measuring function is defined on single points of the contour, the computation takes less than a second on a SPARC workstation. The computational time may go up to several seconds for a measuring function which is defined on pairs of points of the contour.

4 Invariant properties

In this Section the invariance properties of the size functions mentioned in Section 2 are demonstrated on real images. In order to quantitatively determine whether, or not, similar shapes are given a similar representation and different shapes are actually distinguishable, a distance between size functions needs to be defined.

4.1 A distance between size functions

There are many ways in which a distance between size functions can be defined. Probably the only common requirement to the possible definitions is that the scale-invariant property must be preserved (i.e., the distance between the size functions of the same shape at different scales must always vanish). Let us introduce the simple distance function which will be used throughout the rest of the paper.

Let φ be a measuring function, α_1 and α_2 two curves, and $\varphi^{\max}(\alpha_i)$ the maximum of φ on α_i, for $i = 1,2$. Without loss of generality it can be assumed that $\varphi \ge 0$. Let us scale φ by defining $\hat{\varphi} = \varphi/\varphi^{\max}(\alpha_i)$ on α_i, for $i = 1,2$. Then, a scale-invariant distance D between the size functions $l_{\hat{\varphi}}(\alpha_1)$ and $l_{\hat{\varphi}}(\alpha_2)$ can be defined as [7]

$$D = 2\int_0^1 dy \int_0^y dx |l_{\hat{\varphi}}(\alpha_1; x,y) - l_{\hat{\varphi}}(\alpha_2; x,y)|.$$

The distance D is simply the L^1 norm of the difference over the triangular region with $0 \le x \le y$ and $0 \le y \le 1$. Similarly, in the discrete case, if $\bar{l}_{\hat{\varphi}}(\alpha_1)$ and $\bar{l}_{\hat{\varphi}}(\alpha_2)$ are computed at the same fixed resolution R and regarded as triangular matrices

$\bar{l}_{\hat{\varphi}}(\alpha_1)_{i,j}$ and $\bar{l}_{\hat{\varphi}}(\alpha_2)_{i,j}$ with $i = 1,...,R-1$ and $j = 1,...,R-i$, then the distance D can be redefined as

$$D = \frac{2}{R(R-1)} \sum_{i=1}^{R-1} \sum_{j=1}^{R-i} | \bar{l}_{\hat{\varphi}}(\alpha_1)_{i,j} - \bar{l}_{\hat{\varphi}}(\alpha_2)_{i,j} | \quad (1)$$

where the normalization factor is chosen so that $D = 1$ if, on average, the triangular matrices $\bar{l}_{\hat{\varphi}}(\alpha_1)$ and $\bar{l}_{\hat{\varphi}}(\alpha_2)$ differ by 1 at each entry. The entries on the diagonal of the triangular matrices do not appear in the sum in the right hand side of Eq. 1 because they may be severely affected by noise.

4.2 Euclidean invariance

Let us now make some quantitative estimates of the invariance of the representation of shape which can be obtained through a size function. Fig. 3*a* shows the image of an ivy leaf. The size function induced by the measuring function D_c, that is, "the distance of a point of the outline from the center of mass of the outline", is shown in Fig. 3*b*. In principle, the function D_c is clearly invariant for translation and rotation of the shape over the image plane. Fig. 3*c* shows an image of the same leaf translated and rotated while the camera was kept in a fixed position. The size function associated with the outline of the image of Fig. 3*c* is shown in Fig. 3*d*. It can easily be seen that the size functions of Fig. 3*a* and *c* are very similar. Correspondingly, the distance D between the size functions of Fig. 3*b* and *d*, computed by means of Eq. 1, is equal to .06.

The property of scale invariance is illustrated in Fig. 3*e* and *f*. In the image of Fig. 3*e*, the camera was viewing at the same ivy leaf from a further viewpoint. It is evident that the size function of Fig. 3*f*, which was obtained from the outline of the image of Fig. 3*e*, is very similar to the size function of Fig. 3*b*. In this case, the distance between the size functions of Fig. 3*b* and *f*, computed by means of Eq. 1, is .04.

4.3 "Ad hoc" invariance

Articulated objects modify their shape according to the changes of some internal parameter. Figs. 4*a* and *b* show the same pair of scissors with different opening. For recognition purposes, it would be desirable to be able to represent the shape of articulated objects independently from their internal parameters. In the present framework, this problem can be solved by looking for an appropriate measuring function. For example, the measuring function D_p, that is, "the distance of a point of the outline from the pivot" is invariant for different openings (actually, due to self-occlusions, D_p can only be approximately invariant).

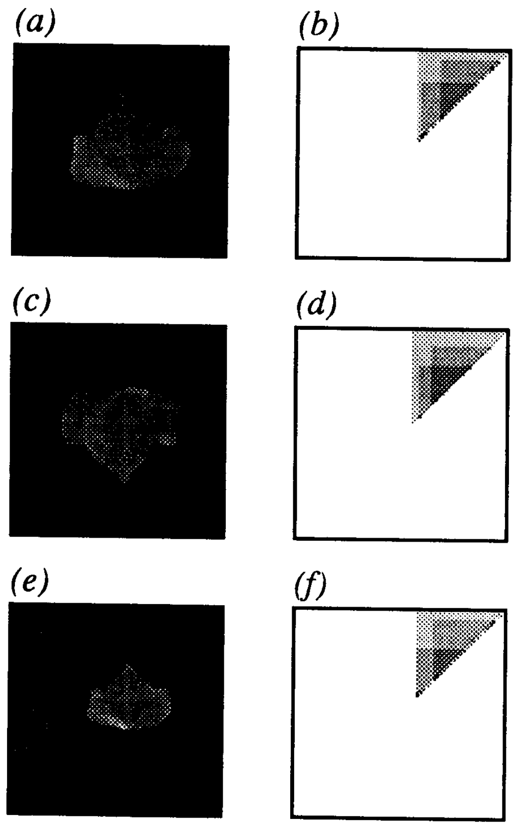

Figure 3: Invariance for Euclidean transformations. (a) An image of an ivy leaf. (b) The size function of the outline of the leaf of (a) induced by the measuring function D_c. (c) An image of the same ivy leaf translated and rotated over the supporting plane. (d) The size function of the outline of (c). (e) An image of the same ivy leaf from a further viewpoint. (f) The size function of the outline of (e). The color coding is light grey for 1, grey for 2, darker grey for 3, and black for 4 and the diagrams are scaled between 0 and the maximum of D_c over each outline.

The size functions induced by D_p and associated with the contours of Fig. 4c and d are shown in Fig. 4e and f respectively. By using Eq. 1, the distance between the size functions of Fig. 4e and f is found to be equal to .15. Notice that the difference between the two representations is mainly due to the different amount of self-occlusion of a portion of the contours of Fig. 4c and d in the neighborhood of the pivot.

4.4 Qualitative invariance

Let us conclude this Section by showing that a suitable choice of the measuring function can detect a particular aspect of shape. Fig. 5a and b show the images of two oak leaves which, qualitatively, can be thought of as having the same shape. Fig. 5c and d

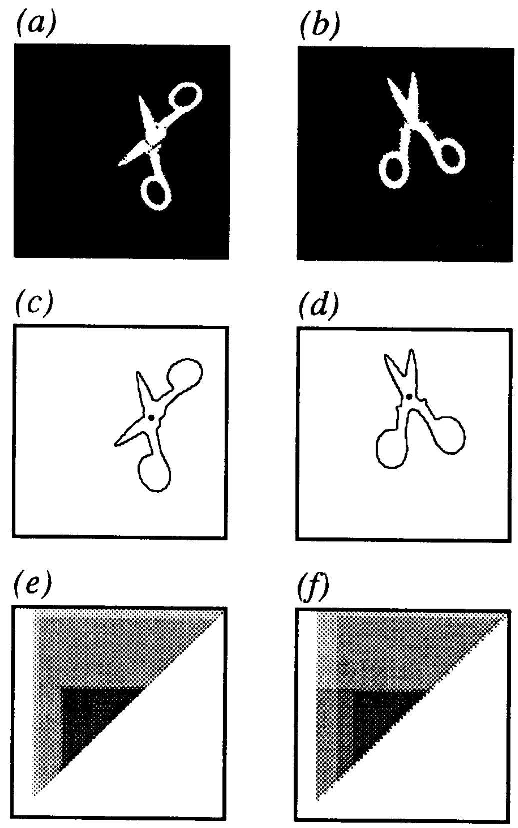

Figure 4: Invariance "ad hoc". (a) and (b) Two images of the same pair of scissors with different opening. (c) and (d) Outlines of the shapes of (a) and (b) obtained by means of the same procedures of Fig. 1. (e) and (f) Color coded representations of the size functions of (c) and (d) induced by the measuring function D_p, "distance of a point of the outline from the pivot". In both (c) and (d), the pivot was located as the midpoint of the segment whose endpoints are the intersections of the principal inertial axis with the outline. The color coding is as in Fig. 3 and the diagrams are scaled between 0 and the maximum of D_p over each outline.

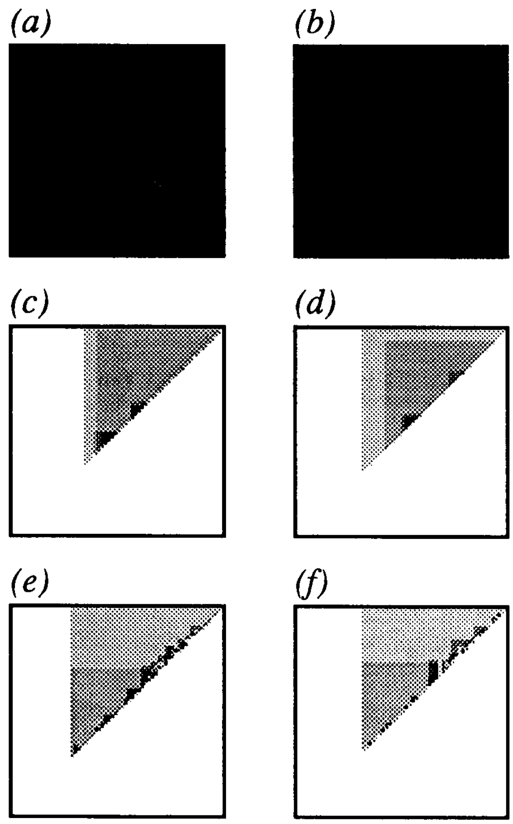

Figure 5: Choosing the appropriate measuring function. (a) and (b) Images of two oak leaves. (c) and (d) Color coded representations of the size functions of the outlines of the leaves of (a) and (b) respectively induced by the measuring function D_c. (e) and (f) Color coded representations of the size functions of the outlines of the leaves of (a) and (b) respectively induced by the measuring function R_{ec} "ratio between the Euclidean distance of a pair of points and the length of the shortest arc of the contour which joins a pair of points". The color coding is as in Fig. 3 and the diagrams are scaled between 0 and the maximum of R_{ec} over each outline.

show the size functions associated with the outline of the leaves of Fig. 5a and b respectively and induced by the measuring function D_c, the "distance of a point from the center of mass". The distance between the size functions of Fig. 5c and d computed by means of Eq. 1 is .15. Even if the distance between the two size functions is fairly small, it is clear that the two size functions are qualitatively different. A different choice of measuring function can lead to a rather different result. Fig. 5e and f show the size function associated with the outline of the leaves of Fig. 5a and b but induced by the measuring function "ratio between the Euclidean distance of a pair of points and the length of the shortest arc of the contour which joins a pair of points". While the quantitative distance between the size functions of Fig. 5e and f is still .15, it is clear that the diagrams of Fig. 5e and f are qualitatively more similar than the diagrams of Fig. 5c and d.

5 Conclusions

In this paper the potential of the theory of size functions to computer vision has been assessed. An algorithm for the computation of size functions from real images has been implemented and used to illustrate a number of theoretical properties of the theory which are likely to be useful for object recognition. Based on the presented experimental results it can be concluded that the representation of shape in terms of size functions *(i)* can be tailored to suit the invariance of the problem at hand and *(ii)* is stable against small qualitative and quantitative changes of the viewed shape. In addition a size function can be designed to highlight a particular aspect of the shape of an object, aspect which can be useful to build similar representations of shape which are similar but quantitatively or qualitatively different. A distance between size functions has been introduced to measure the similarity between the representation of two different shapes. The obtained results indicate that size functions are likely to be very useful for the recognition of objects which have similar but not identical shape.

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