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OPTIMAL MATCHING BETWEEN REDUCED SIZE FUNCTIONS

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*Michele d'Amico, **Patrizio Frosini,

§Claudia Landi

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Dipartimento di Scienze e Metodi dell'Ingegneria

Via G. R. Fogliani, 1 - Tel. 0522/522610 - Fax 0522/522609

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*ARCES,
Università degli Studi di Bologna

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OPTIMAL MATCHING BETWEEN REDUCED SIZE FUNCTIONS*

MICHELE D'AMICO†, PATRIZIO FROSINI‡, AND CLAUDIA LANDI§

Abstract. We study a dissimilarity measure between shapes, expressed by the natural pseudo-distance between *size pairs*, where a shape is viewed as a topological space endowed with a real-valued continuous function. Measuring dissimilarity amounts to minimizing the change in the functions due to the application of homeomorphisms between topological spaces, with respect to the L^∞-norm. A new class of shape descriptors, called reduced size functions, is introduced. They are compared by solving an optimal matching problem between countable point sets. In this way we obtain a distance between reduced size functions that is shown to be stable, and to furnish a lower bound for the dissimilarity measure between shapes.

Key words. Shape comparison, shape representation, reduced size function, natural pseudo-distance

AMS subject classifications. Primary: 68T10; Secondary: 54C30, 54D05

1. Introduction. Shape comparison is a central problem in shape recognition, shape classification and shape retrieval. The shape comparison problem is usually dealt with by defining a suitable distance providing a measure of resemblance between shapes (see, e.g. [20], and, for a review of the literature, [19]).

In the last ten years *natural pseudo-distances* and *size functions* have been studied as geometrical-topological tools for comparing shapes, each shape viewed as a topological space $\mathcal{M}$ endowed with a real-valued continuous function φ. Every pair $(\mathcal{M}, \varphi)$ is called a *size pair*, while φ is said to be a *measuring function* [12]. The role of the function φ is to take into account only the shape properties of $\mathcal{M}$ that are relevant to the shape comparison problem at hand, while disregarding the irrelevant ones, as well as to impose the desired invariance properties.

The main idea in the definition of natural pseudo-distance between size pairs is to minimize the change in the measuring functions due to the application of homeomorphisms between topological spaces, with respect to the L^∞-norm.

When a distance between shapes is defined by a minimization process (in energy minimization methods, for example, as well as in our case), one naturally looks for the optimal transformation. In the computation of natural pseudo-distance between size pairs, this troublesome problem can be avoided by applying size functions. Indeed, size functions can reduce the comparison of shapes to a finite dimensional problem, i.e. the comparison of certain finite subsets of the real plane (cf. Fig. 2.2). This reduction allows us to study the space of all homeomorphisms between the considered topological spaces, without actually computing them (cf. [5]).

This line of research has led to a formal setting, which has turned out to be useful both from a theoretical and an applicative point of view (see, e.g., [1], [2], [3], [4], [6], [9], [11], [13], [14], [15], [17]), but it has also pointed out the usefulness of alternative definitions, providing a more agile approach to the theory.

The first contribution of this paper is the introduction of the concept of *reduced size function*. The new definition slightly changes that of size function in order to

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†ARCES, Università di Bologna, via Toffano 2/2, I-40135 Bologna, Italia (damico@dm.unibo.it).

‡Dipartimento di Matematica, Università di Bologna, P.zza di Porta San Donato 5, I-40127 Bologna, Italia, and ARCES (frosini@dm.unibo.it).

§DISMI, Università di Modena e Reggio Emilia, I-42100 Reggio Emilia, Italia (clandi@unimore.it)

obtain both theoretical and computational advantages (see Rem. 2.9). However, the main properties of size functions are maintained.

Our main contribution consists of showing that reduced size functions verify a property of perturbation robustness, immediately implying a lower bound for the change of measuring functions under the action of homeomorphisms between topological spaces. This new result is the motivation behind this paper, since it makes available a new geometrical-topological tool for studying the existence of optimal homeomorphisms between size pairs.

The organization of this paper is as follows. In Section 2 we introduce the concept of reduced size function and its main properties. In Section 3 the definition of matching distance between reduced size functions is given. In Section 4 the stability theorem is proved, together with some other useful results. The connection with natural pseudo-distances between size pairs is shown in Section 5, together with the proof of the existence of an optimal matching between reduced size functions.

2. Reduced size functions. In this section we introduce reduced size functions, that is, a notion derived from size functions ([12]) allowing a simplified treatment of the theory.

In what follows, $\mathcal{M}$ denotes a compact connected and locally connected topological space. Δ denotes the diagonal $\{(x,y) \in \mathbb{R}^2 : x = y\}$; Δ^+ denotes the open half-plane $\{(x,y) \in \mathbb{R}^2 : x < y\}$ above the diagonal; $\bar{\Delta}^+$ denotes the closed half-plane $\{(x,y) \in \mathbb{R}^2 : x \leq y\}$ above the diagonal.

We shall call any pair $(\mathcal{M}, \varphi)$, where $\varphi : \mathcal{M} \to \mathbb{R}$ is a continuous function, a *size pair*. The function φ is said to be a *measuring function*.

Assume a size pair $(\mathcal{M}, \varphi)$ is given. For every $x \in \mathbb{R}$, let $\mathcal{M}\langle \varphi \leq x \rangle$ denote the lower level set $\{P \in \mathcal{M} : \varphi(P) \leq x\}$.

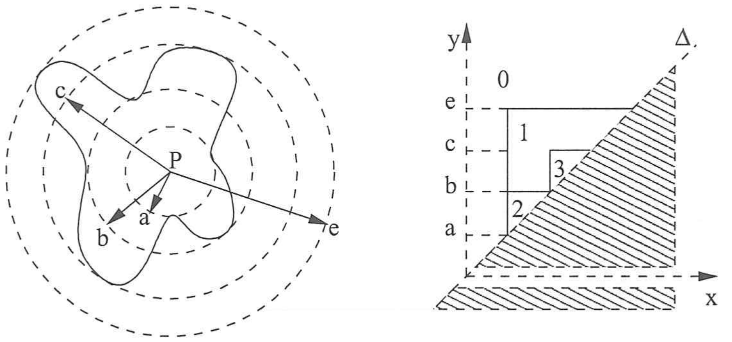
DEFINITION 2.1. *For every real number y, we shall say that two points $P, Q \in \mathcal{M}$ are* $\langle \varphi \leq y \rangle$-connected *if and only if a connected subset C of $\mathcal{M}\langle \varphi \leq y \rangle$ exists, containing both P and Q.*

DEFINITION 2.2. (Reduced size function) *We shall call* reduced size function *associated with the size pair $(\mathcal{M}, \varphi)$ the function $\ell^*_{(\mathcal{M},\varphi)} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell^*_{(\mathcal{M},\varphi)}(x,y)$ equal to the number of equivalence classes into which the set $\mathcal{M}\langle \varphi \leq x \rangle$ is divided by the relation of $\langle \varphi \leq y \rangle$-connectedness, if $y < \max \varphi$, and* 0 *otherwise.*

In other words, if $y < \max \varphi$, $\ell^*_{(\mathcal{M},\varphi)}(x,y)$ counts the number of connected components in $\mathcal{M}\langle \varphi \leq y \rangle$ that contain at least one point of $\mathcal{M}\langle \varphi \leq x \rangle$. The finiteness of this number is an easily obtainable consequence of the compactness and local-connectedness of $\mathcal{M}$.

An example of reduced size function is illustrated in Fig. 2.1. In this example we consider the size pair $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is the curve represented by a continuous line in Fig. 2.1 (a), and φ is the function "distance from the point P". The reduced size function associated with $(\mathcal{M}, \varphi)$ is shown in Fig. 2.1 (b). Here, the domain of the reduced size function is divided by solid lines, representing the discontinuity points of the reduced size function. These discontinuity points divide Δ^+ into regions on which the reduced size function is constant. The value displayed in each region is the value taken by the reduced size function in that region. As for the values taken on the discontinuity lines, they are easily obtained by observing that reduced size functions are right-continuous, both in the variable x and in the variable y.

Most properties of classical size functions continue to hold for reduced size functions. Here we report a few of them which will be useful in the sequel. Proofs are omitted, since they are completely analogous to those for classical size functions.



(a) (b)

FIG. 2.1. *(b) Reduced size function of the size pair $(\mathcal{M},\varphi)$, where $\mathcal{M}$ is the curve represented by a continuous line in (a), and φ is the function "distance from the point P".*

The following result, expressing a relation between two reduced size functions corresponding to two spaces, $\mathcal{M}$ and $\mathcal{W}$, that can be matched without changing the measuring functions more that h, is analogous to Th. 3.2 in [11].

PROPOSITION 2.3. *Let $(\mathcal{M},\varphi)$ and $(\mathcal{W},\psi)$ be two size pairs. If $f:\mathcal{M}\to\mathcal{W}$ is a homeomorphism such that* $\max_{P\in\mathcal{M}}|\varphi(P)-\psi(f(P))|\le h$, *then for every $(\bar{x},\bar{y})\in\Delta^+$ we have*

$$\ell^*_{(\mathcal{M},\varphi)}(\bar{x}-h,\bar{y}+h)\le \ell^*_{(\mathcal{W},\psi)}(\bar{x},\bar{y}).$$

The next proposition, analogous to Prop. 6 in [13], gives some constraints on the presence of discontinuity points for reduced size functions.

PROPOSITION 2.4. *Let $(\mathcal{M},\varphi)$ be a size pair. For every point $\bar{p}=(\bar{x},\bar{y})\in\Delta^+$, a real number $\epsilon>0$ exists such that the open set*

$$W_\epsilon(\bar{p}):=\{(x,y)\in\mathbb{R}^2:|\bar{x}-x|<\epsilon,|\bar{y}-y|<\epsilon,x\ne\bar{x},y\ne\bar{y}\}$$

is contained in Δ^+, and does not contain any discontinuity point for $\ell^*_{(\mathcal{M},\varphi)}$.

For reduced size functions it is possible to define cornerpoints exactly in the same way as for classical size functions ([13]):

DEFINITION 2.5. (Cornerpoint) *For every point $p=(x,y)\in\Delta^+$, and for every pair of real positive numbers α, β with $x+\alpha<y-\beta$, let us define the number $\mu_{\alpha,\beta}(p)$ as*

$$\begin{aligned}&\ell^*_{(\mathcal{M},\varphi)}(x+\alpha,y-\beta)-\ell^*_{(\mathcal{M},\varphi)}(x+\alpha,y+\beta)\\&-\ell^*_{(\mathcal{M},\varphi)}(x-\alpha,y-\beta)+\ell^*_{(\mathcal{M},\varphi)}(x-\alpha,y+\beta).\end{aligned}$$

The finite number $\mu(p):=\min\{\mu_{\alpha,\beta}(p):\alpha,\beta>0,x+\alpha<y-\beta\}$ will be called multiplicity of p *for $\ell^*_{(\mathcal{M},\varphi)}$. Moreover, we shall call* cornerpoint *for $\ell^*_{(\mathcal{M},\varphi)}$ any point $p\in\Delta^+$ such that the number $\mu(p)$ is strictly positive.*

As an example, the cornerpoints for the reduced size function of Fig. 2.2 are the points p (with multiplicity 3), q and r (with multiplicity 1). The point s is not a cornerpoint since its multiplicity is equal to 0.

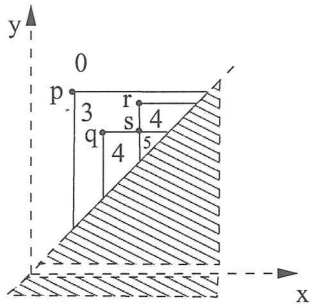


FIG. 2.2. *Cornerpoints of a reduced size function: p, q and r are the only cornerpoints, and have multiplicity equal to 3 (p), and 1 (q and r); s is not a cornerpoint since its multiplicity is equal to 0.*

An analog of Prop. 8 in [13], stating that cornerpoints create discontinuity points spreading downwards and towards the right to Δ, also holds for reduced size functions.

PROPOSITION 2.6. *If $\bar{p} = (\bar{x}, \bar{y})$ is a cornerpoint for $\ell^*_{(\mathcal{M},\varphi)}$, then the following statements hold:*

i) *If $\bar{x} \leq x < \bar{y}$, then $\bar{y}$ is a discontinuity point for $\ell^*_{(\mathcal{M},\varphi)}(x, \cdot)$;*

ii) *If $\bar{x} < y < \bar{y}$, then $\bar{x}$ is a discontinuity point for $\ell^*_{(\mathcal{M},\varphi)}(\cdot, y)$.*

We shall make use of an analog of Prop. 10 in [13], which can be re-formulated for reduced size functions as follows.

THEOREM 2.7. (Representation Theorem) *For every $(\bar{x}, \bar{y}) \in \Delta^+$ we have*

$$\ell^*_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y}) = \sum_{x \leq \bar{x}, y > \bar{y}} \mu((x, y)).$$

This can be checked in the example of Fig. 2.2. The points where the reduced size function takes value 0 are exactly those for which there is no cornerpoint lying to the left and above them. The points where the reduced size function takes value 3 are those for which only the cornerpoint p (with multiplicity 3) lies to the left and above them. The points where the reduced size function takes value 5 are those for which only the cornerpoints p, q, r (with multiplicity 3, 1 and 1, respectively) are to the left and above them. Similar remarks hold for the points where the reduced size function takes value 4.

As a consequence of Th. 2.7, we obtain that cornerpoints, together with their multiplicities, uniquely determine reduced size functions.

The following result is the analog of Cor. 3 in [13].

PROPOSITION 2.8. *For each strictly positive real number ϵ, reduced size functions have at most a finite number of cornerpoints in $\{(x, y) \in \mathbb{R}^2 : x + \epsilon < y\}$.*

Therefore, if the set of cornerpoints of a reduced size function has an accumulation point, it necessarily belongs to the diagonal Δ. An example of reduced size function with cornerpoints accumulating onto the diagonal is shown in Fig. 2.3.

REMARK 2.9. *By comparing Th. 2.7 and the analogous result expressed by Prop. 10 in [13], one can observe that the former is stated more straightforwardly. As a consequence of this simplification, all the statements in this paper that follow from Th. 2.7 are less cumbersome than they would be if we applied size functions instead of reduced size functions. This is the main motivation for introducing the notion of reduced size function.*

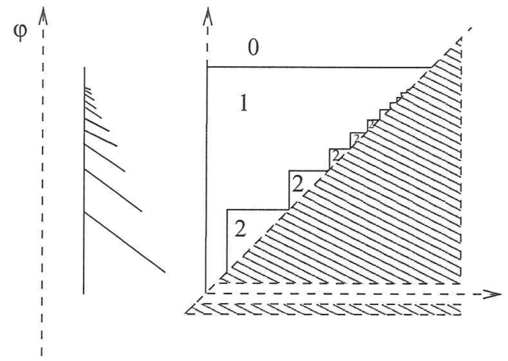


FIG. 2.3. *A reduced size function with cornerpoints accumulating onto the diagonal: $\mathcal{M}$ is the space depicted on the left and φ measures the height of the point.*

3. Matching distance. In this section we define a matching distance between reduced size functions. The idea is to compare reduced size functions by measuring the cost of transporting the cornerpoints of one reduced size function to those of the other one, with the property that the longest of the transportations should be as short as possible. Since, in general, the number of cornerpoints of the two reduced size function is different, we also enable the cornerpoints to be transported onto the points of Δ. When the number of cornerpoints is finite, the matching distance may be related to the bottleneck transportation problem (cf., e.g., [8], [16]). In our case, however, the number of cornerpoints may be countably infinite. Nevertheless, we prove the existence of an optimal matching.

Of course the matching distance is not the only metric between reduced size functions that we could think of. Other metrics for size functions have been considered in the past ([7], [10]). However, the matching distance seems to us to be of particular interest since, as we shall see, it allows for a connection with the natural pseudo-distance between size pairs. Moreover, it has already been experimentally tested successfully in [1].

DEFINITION 3.1. (Representative sequence) *Let ℓ^* be a reduced size function. We shall call* representative sequence for ℓ^* *any sequence of points $a : \mathbb{N} \to \bar{\Delta}^+$, (briefly denoted by (a_i)), with the following properties:*

1. *For each $i \in \mathbb{N}$, either a_i is a cornerpoint for ℓ^*, or a_i belongs to Δ.*
2. *If p is a cornerpoint for ℓ^* with multiplicity $\mu(p)$, then there are exactly $\mu(p)$ indexes $i_1, i_2, \ldots, i_{\mu(p)}$ such that $a_{i_k} = p$ for $1 \le k \le \mu(p)$.*
3. *The set of indexes i with $a_i \in \Delta$ is countably infinite.*

Throughout the rest of the paper, d will denote the following pseudo-distance on $\bar{\Delta}^+$:

$$d\left((x,y),(x',y')\right) := \min\left\{\max\left\{|x-x'|,|y-y'|\right\}, \max\left\{\frac{y-x}{2}, \frac{y'-x'}{2}\right\}\right\}.$$

In other words, the pseudo-distance d between two points p and p' compares the cost of moving p to p' and the cost of moving p and p' onto the diagonal and takes the smaller. Costs are computed using the distance induced by the max-norm. In particular, the pseudo-distance d between two points p and p' on the diagonal is always

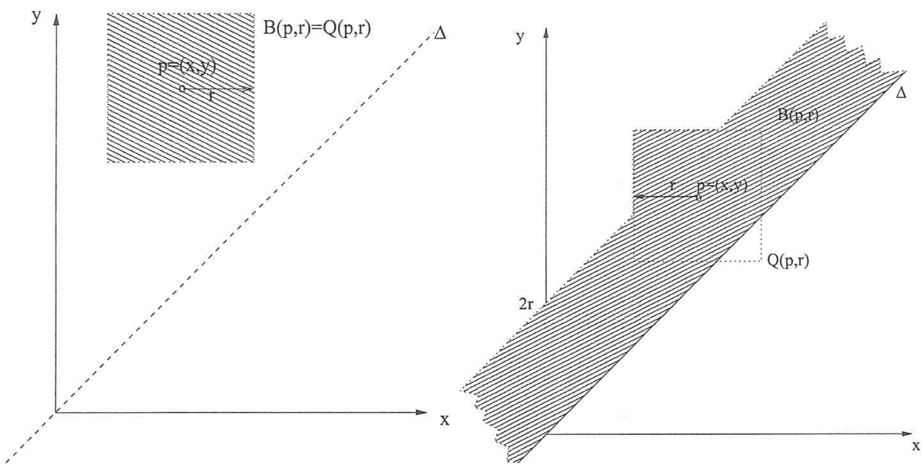


FIG. 3.1. *Disks induced by the metric d (shaded). Left:* $d(p,\Delta) \geq r$. *Right:* $d(p,\Delta) < r$.

0; the pseudo-distance d between two points p and p', with p above the diagonal and p' on the diagonal, is equal to the distance, induced by the max-norm, between p and the diagonal.

Therefore, $d(p,p')$ can be considered a measure of the minimum of the costs of moving p to p' along two different paths (i.e. the path that takes p directly to p' and the path that passes through Δ). This observation easily yields that d is actually a pseudo-distance.

It is also useful to observe what disks induced by the pseudo-distance d look like. For $r > 0$, the usual notation $B(p,r)$ will denote the open disk $\{p' \in \bar{\Delta}^+ : d(p,p') < r\}$. Thus, if p has coordinates (x,y) with $y - x \geq 2r$ (that is, $d(p,\Delta) \geq r$), then $B(p,r)$ is the open square centered at p with sides of length $2r$ parallel to the axes. Whereas, if p has coordinates (x,y) with $y - x < 2r$ (that is, $d(p,\Delta) < r$), then $B(p,r)$ is the union of the open square, centered at p, with sides of length $2r$ parallel to the axes, with the stripe $\{(x,y) \in \mathbb{R}^2 : 0 \leq y - x < 2r\}$, intersected with $\bar{\Delta}^+$(see also Fig. 3.1).

In what follows, the notation $B(p,r)$ will always refer to the open disk, centered at p with radius r, induced by the distance d. Whereas, the notation $Q(p,r)$ will refer to the open square centered at p, with sides of length $2r$ parallel to the axes (that is, the open disk centered at p with radius r, induced by the max-norm). Also, $\bar{Q}(p,r)$ will refer to the closure of $Q(p,r)$, when $r > 0$, while $\bar{Q}(p,0) := \{p\}$.

DEFINITION 3.2. (Matching distance) *Let ℓ_1^* and ℓ_2^* be two reduced size functions. If (a_i) and (b_i) are two representative sequences for ℓ_1^* and ℓ_2^* respectively, then the* matching distance *between ℓ_1^* and ℓ_2^* is the number*

$$d_{match}(\ell_1^*, \ell_2^*) := \inf_{\sigma} \sup_{i} d(a_i, b_{\sigma(i)}),$$

where i varies in $\mathbb{N}$ and σ varies among all the bijections from $\mathbb{N}$ to $\mathbb{N}$.

It is easy to see that this definition is independent from the choice of the representative sequences of points for ℓ_1^* and ℓ_2^*. In fact, if (a_i) and $(\hat{a}_i)$ are representative sequences for the same reduced size function ℓ^*, a bijection $\hat{\sigma} : \mathbb{N} \to \mathbb{N}$ exists such that $d(\hat{a}_i, a_{\hat{\sigma}(i)}) = 0$ for every index i.

Recalling that reduced size functions are uniquely determined by their cornerpoints with multiplicities, one can see that d_{match} verifies all the properties of a distance.

We will show in Th. 5.1 that the inf and the sup in the definition of matching distance are actually attained, that is to say $d_{match}(\ell_1^*, \ell_2^*) = \min_\sigma \max_i d(a_i, b_{\sigma(i)})$. In other words, an optimal matching exists.

4. Stability of the matching distance. In this section we shall prove that if φ and ψ are two measuring functions on $\mathcal{M}$ whose difference on the points of $\mathcal{M}$ is controlled by ϵ (namely $\max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| \leq \epsilon$), then the matching distance between $\ell^*_{(\mathcal{M},\varphi)}$ and $\ell^*_{(\mathcal{M},\psi)}$ is also controlled by ϵ (namely $d_{match}(\ell^*_{(\mathcal{M},\varphi)}, \ell^*_{(\mathcal{M},\psi)}) \leq \epsilon$).

For the sake of clarity, we will now give a sketch of the proof that will lead to this result, stated in Th. 4.6. We begin by proving that each cornerpoint of $\ell^*_{(\mathcal{M},\varphi)}$ with multiplicity m admits a small neighborhood, where we find exactly m cornerpoints (counted with multiplicities) for $\ell^*_{(\mathcal{M},\psi)}$, provided that on $\mathcal{M}$ the functions φ and ψ take close enough values (Prop. 4.2). Next, this local result is extended to a global result by considering the convex combination $\Phi_t = \frac{t}{\epsilon}\psi + \frac{\epsilon - t}{\epsilon}\varphi$ of φ and ψ. Following the paths traced by the cornerpoints of $\ell^*_{(\mathcal{M},\Phi_t)}$ as t varies in $[0, \epsilon]$, in Prop. 4.3 we show that, along these paths, the displacement of the cornerpoints is not greater than ϵ (displacements are measured using the distance d, and cornerpoints are counted with their multiplicities). Thus we are able to construct an injection f from the set of the cornerpoints of $\ell^*_{(\mathcal{M},\varphi)}$ to the set of the cornerpoints of $\ell^*_{(\mathcal{M},\psi)}$ (extended to the points of the diagonal), that moves points less than ϵ (Prop. 4.4). Repeating the same argument backwards, we construct an injection g from the set of the cornerpoints of $\ell^*_{(\mathcal{M},\psi)}$ to the set of the cornerpoints of $\ell^*_{(\mathcal{M},\varphi)}$ (extended to the points of the diagonal) that moves points less than ϵ. By using an extension of the Cantor-Bernstein theorem (see Prop. 4.5), we prove that there exists a bijection from the set of the cornerpoints of $\ell^*_{(\mathcal{M},\psi)}$ (extended to the points of the diagonal) to the set of the cornerpoints of $\ell^*_{(\mathcal{M},\varphi)}$ (extended to the points of the diagonal) that moves points less than ϵ. This will be sufficient to conclude the proof. Once again, we recall that in the proof we have just outlined, cornerpoints are always counted with their multiplicities.

We first prove that the number of cornerpoints contained in a sufficiently small square can be computed in terms of jumps of reduced size functions.

PROPOSITION 4.1. *Let $(\mathcal{M}, \varphi)$ be a size pair. Let $\bar{p} = (\bar{x}, \bar{y}) \in \Delta^+$ and let $\eta > 0$ be such that $\bar{x}+\eta < \bar{y}-\eta$. Also let $a = (\bar{x}+\eta, \bar{y}-\eta)$, $b = (\bar{x}-\eta, \bar{y}-\eta)$, $c = (\bar{x}+\eta, \bar{y}+\eta)$, $e = (\bar{x}-\eta, \bar{y}+\eta)$. Then*

$$\ell^*_{(\mathcal{M},\varphi)}(a) - \ell^*_{(\mathcal{M},\varphi)}(b) - \ell^*_{(\mathcal{M},\varphi)}(c) + \ell^*_{(\mathcal{M},\varphi)}(e)$$

is equal to the number of cornerpoints for $\ell^*_{(\mathcal{M},\varphi)}$, counted with their multiplicities, contained in the semi-open square $\hat{Q}$, with vertexes at a, b, c, e, given by

$$\hat{Q} := \{(x, y) \in \Delta^+ : \bar{x} - \eta < x \leq \bar{x} + \eta,\ \bar{y} - \eta < y \leq \bar{y} + \eta\}.$$

Proof. It immediately follows from Th. 2.7. □

We now show that small changes in the measuring functions produce small displacements of cornerpoints.

PROPOSITION 4.2. *Let $(\mathcal{M}, \varphi)$ be a size pair and let $\bar{p}$ be a point in Δ^+, with multiplicity $\mu(\bar{p})$ for $\ell^*_{(\mathcal{M},\varphi)}$ (possibly $\mu(\bar{p}) = 0$). Then, there is a real number $\bar{\eta} > 0$*

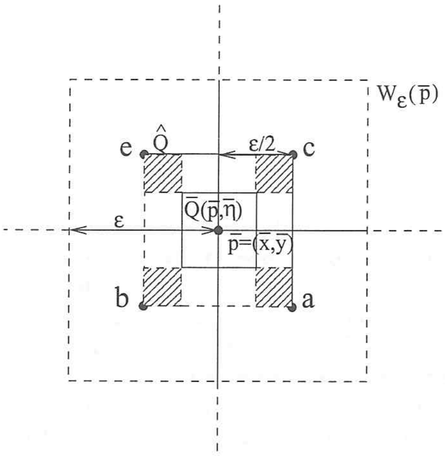


FIG. 4.1. *The sets $W_\epsilon(\bar{p})$, $\hat{Q}$ and $\bar{Q}(\bar{p},\bar{\eta})$ used in the proof of Prop. 4.2.*

such that, for any real number η with $0 \le \eta \le \bar{\eta}$, and for any measuring function $\psi : \mathcal{M} \to \mathbb{R}$ with $\max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| \le \eta$, the size function $\ell^*_{(\mathcal{M},\psi)}$ has exactly $\mu(\bar{p})$ cornerpoints (counted with their multiplicities) in the closed square $\bar{Q}(\bar{p},\eta)$, centered at $\bar{p}$ with side 2η.

Proof. By Prop. 2.4, a sufficiently small real number $\epsilon > 0$ exists such that the set

$$W_\epsilon(\bar{p}) = \{(x,y) \in \mathbb{R}^2 : |\bar{x} - x| < \epsilon, |\bar{y} - y| < \epsilon, x \neq \bar{x}, y \neq \bar{y}\}$$

is contained in Δ^+, and does not contain any discontinuity point for $\ell^*_{(\mathcal{M},\varphi)}$.

Let us set $a = (\bar{x} + \epsilon/2, \bar{y} - \epsilon/2)$, $b = (\bar{x} - \epsilon/2, \bar{y} - \epsilon/2)$, $c = (\bar{x} + \epsilon/2, \bar{y} + \epsilon/2)$, and $e = (\bar{x} - \epsilon/2, \bar{y} + \epsilon/2)$ (see Fig. 4.1). By Prop. 2.6 and Prop. 4.1, the multiplicity of $\bar{p}$ for $\ell^*_{(\mathcal{M},\varphi)}$ is equal to $\ell^*_{(\mathcal{M},\varphi)}(a) - \ell^*_{(\mathcal{M},\varphi)}(b) - \ell^*_{(\mathcal{M},\varphi)}(c) + \ell^*_{(\mathcal{M},\varphi)}(e)$.

By Prop. 2.3, if $\psi : \mathcal{M} \to \mathbb{R}$ is a measuring function with $h := \max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| < \epsilon/2$, then

$$\ell^*_{(\mathcal{M},\varphi)}(\bar{x} + \frac{\epsilon}{2} - h, \bar{y} - \frac{\epsilon}{2} + h) \le \ell^*_{(\mathcal{M},\psi)}(\bar{x} + \frac{\epsilon}{2}, \bar{y} - \frac{\epsilon}{2}) \le \ell^*_{(\mathcal{M},\varphi)}(\bar{x} + \frac{\epsilon}{2} + h, \bar{y} - \frac{\epsilon}{2} - h).$$

Since $\ell^*_{(\mathcal{M},\varphi)}$ is constant in each connected component of $W_\epsilon(\bar{p})$, and $h < \epsilon/2$, we have

$$\ell^*_{(\mathcal{M},\varphi)}(\bar{x} + \frac{\epsilon}{2} - h, \bar{y} - \frac{\epsilon}{2} + h) = \ell^*_{(\mathcal{M},\varphi)}(\bar{x} + \frac{\epsilon}{2} + h, \bar{y} - \frac{\epsilon}{2} - h) = \ell^*_{(\mathcal{M},\varphi)}(\bar{x} + \frac{\epsilon}{2}, \bar{y} - \frac{\epsilon}{2})$$

and hence $\ell^*_{(\mathcal{M},\psi)}(a) = \ell^*_{(\mathcal{M},\varphi)}(a)$. Analogously it holds that $\ell^*_{(\mathcal{M},\psi)}(b) = \ell^*_{(\mathcal{M},\varphi)}(b)$, $\ell^*_{(\mathcal{M},\psi)}(c) = \ell^*_{(\mathcal{M},\varphi)}(c)$, and $\ell^*_{(\mathcal{M},\psi)}(e) = \ell^*_{(\mathcal{M},\varphi)}(e)$. Thus, setting $\bar{\eta} = \epsilon/4$, for any $\eta \in \mathbb{R}$ with $0 \le \eta \le \bar{\eta}$, if $\max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| \le \eta$, then the multiplicity of $\bar{p}$ for $\ell^*_{(\mathcal{M},\varphi)}$ is equal to $\ell^*_{(\mathcal{M},\psi)}(a) - \ell^*_{(\mathcal{M},\psi)}(b) - \ell^*_{(\mathcal{M},\psi)}(c) + \ell^*_{(\mathcal{M},\psi)}(e)$.

Therefore, by Prop. 4.1, the multiplicity of $\bar{p}$ for $\ell^*_{(\mathcal{M},\varphi)}$ is equal to the number of cornerpoints for $\ell^*_{(\mathcal{M},\psi)}$ contained in the square $\hat{Q}$ with vertices at a, b, c, e. But this is equal to the number of cornerpoints for $\ell^*_{(\mathcal{M},\psi)}$ contained in the closed square $\bar{Q}(\bar{p},\eta)$, for any positive $\eta \leq \bar{\eta}$ and for any ψ with $\max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| \leq \eta$. Indeed, if $\ell^*_{(\mathcal{M},\psi)}$ had some cornerpoints in $\hat{Q} \setminus \bar{Q}(\bar{p},\eta)$, then, by Prop. 2.6, there would be either vertical or horizontal discontinuity points for $\ell^*_{(\mathcal{M},\psi)}$ in the set $\hat{Q} \cap \{(x,y) \in \mathbb{R}^2 : |x-\bar{x}| > \epsilon/4,\ |y-\bar{y}| > \epsilon/4\}$ (striped in Fig. 4.1). Thus, by Prop. 2.3, $\ell^*_{(\mathcal{M},\varphi)}$ would necessarily have discontinuity points in $W_\epsilon(\bar{p})$, giving a contradiction. □

The following result states that if two measuring functions φ and ψ differ less than ϵ in the L^∞-norm, then it is possible to match some finite sets of cornerpoints of $\ell^*_{(\mathcal{M},\varphi)}$ to cornerpoints of $\ell^*_{(\mathcal{M},\psi)}$, with a motion smaller than ϵ.

PROPOSITION 4.3. *Let $\epsilon \geq 0$ be a real number and let $(\mathcal{M},\varphi)$ and $(\mathcal{M},\psi)$ be two size pairs such that $\max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| \leq \epsilon$. Then, for each finite set K of cornerpoints for $\ell^*_{(\mathcal{M},\varphi)}$ with $d(K,\Delta) > \epsilon$, there exist (a_i) and (b_i) representative sequences for $\ell^*_{(\mathcal{M},\varphi)}$ and $\ell^*_{(\mathcal{M},\psi)}$ respectively, such that $d(a_i,b_i) \leq \epsilon$ for each i with $a_i \in K$.*

Proof. Let $\Phi_t = \frac{t}{\epsilon}\psi + \frac{\epsilon - t}{\epsilon}\varphi$ with $t \in [0,\epsilon]$. Then for every $t,t' \in [0,\epsilon]$ we have $\max_{P\in\mathcal{M}} |\Phi_t(P) - \Phi_{t'}(P)| \leq |t-t'|$.

Let $K = \{p_0, \ldots p_k\}$, let m_j be the multiplicity of p_j, for $j = 0,\ldots,k$, and $m = \sum_{j=0}^k m_j$. Then we can easily construct a representative sequence of points (a_i) for $\ell^*_{(\mathcal{M},\varphi)}$ such that

$$a_0 = p_0,\ \ldots,\ a_{m_0-1} = p_0,\ a_{m_0} = p_1,\ \ldots,\ a_{m_0+m_1-1} = p_1,\ \ldots,\ a_{m-1} = p_k.$$

Now we will consider the set A defined as

$$\{\delta \in [0,\epsilon] : \exists (a_i^\delta) \text{ representative sequence for } \ell^*_{(\mathcal{M},\Phi_\delta)} \text{ s.t. } d(a_i,a_i^\delta) \leq \delta,\ \forall a_i \in K\}.$$

In other words, if we think of the variation of t as the flow of time, A is the set of instants δ for which the cornerpoints in K move less than δ itself, when the homotopy Φ_t is applied to the measuring function φ.

A is non-empty, since $0 \in A$. Let us set $\bar{\delta} = \sup A$ and show that $\bar{\delta} \in A$. Indeed, let (δ_n) be a sequence of numbers of A converging to $\bar{\delta}$. Since $\delta_n \in A$, for each n there is a representative sequence $(a_i^{\delta_n})$ for $\ell^*_{(\mathcal{M},\Phi_{\delta_n})}$ such that $d(a_i,a_i^{\delta_n}) \leq \delta_n$, for each i such that $a_i \in K$. Since $\delta_n \leq \epsilon$, $d(a_i,a_i^{\delta_n}) \leq \epsilon$ for any i and any n. Thus, recalling that $d(K,\Delta) > \epsilon$, for each i such that $a_i \in K$, it holds that $a_i^{\delta_n} \in \bar{Q}(a_i,\epsilon)$ for any n. Hence, for each i with $a_i \in K$, possibly by extracting a convergent subsequence, we can define $a_i^{\bar{\delta}} = \lim_n a_i^{\delta_n}$. We have $d(a_i,a_i^{\bar{\delta}}) \leq \bar{\delta}$. Moreover, by Prop. 4.2, $a_i^{\bar{\delta}}$ is a cornerpoint for $\ell^*_{(\mathcal{M},\Phi_{\bar{\delta}})}$. Also, if r indexes $j_1,\ldots,j_r$ exist, such that $a_{j_1},\ldots,a_{j_r} \in K$ and $a_{j_1}^{\bar{\delta}} = \cdots = a_{j_r}^{\bar{\delta}} =: q$, then the multiplicity of q for $\ell^*_{(\mathcal{M},\Phi_{\bar{\delta}})}$ is not smaller than r. Indeed, since $\delta_n \to \bar{\delta}$, for each arbitrarily small $\eta > 0$ and for any sufficiently great n, the square $\bar{Q}(q,\eta)$ contains at least r cornerpoints for $\ell^*_{(\mathcal{M},\Phi_{\delta_n})}$, counted with their multiplicities. But Prop. 4.2 implies that, for each sufficiently small η, $\bar{Q}(q,\eta)$ contains exactly as many cornerpoints for $\ell^*_{(\mathcal{M},\Phi_{\delta_n})}$ as the multiplicity of q with respect to $\ell^*_{(\mathcal{M},\Phi_{\bar{\delta}})}$. Therefore, the multiplicity of q for $\ell^*_{(\mathcal{M},\Phi_{\bar{\delta}})}$ is greater than, or equal to, r.

In order to conclude that $\bar{\delta} \in A$, it is now sufficient to observe that $(a_0^{\bar{\delta}},\ldots,a_{m-1}^{\bar{\delta}})$ is easily extensible to a representative sequence for $\ell^*_{(\mathcal{M},\Phi_{\bar{\delta}})}$, simply by continuing the sequence with the remaining cornerpoints of $\ell^*_{(\mathcal{M},\Phi_{\bar{\delta}})}$ and with a countable collection of points of Δ. So we have proved that $\sup A$ is attained in A.

We end the proof by showing that $\max A = \epsilon$. In fact, if $\bar{\delta} < \epsilon$, by using Prop. 4.2, it is not difficult to show that there exists $\eta > 0$, with $\bar{\delta} + \eta < \epsilon$, and a representative sequence $(a_i^{\bar{\delta}+\eta})$ for $\ell^*_{(\mathcal{M},\Phi_{\bar{\delta}+\eta})}$, such that $d(a_i^{\bar{\delta}}, a_i^{\bar{\delta}+\eta}) \leq \eta$ for each $i \leq m-1$. Hence, by the triangular inequality, $d(a_i, a_i^{\bar{\delta}+\eta}) \leq \bar{\delta}+\eta$ for each $i \leq m-1$, implying that $\bar{\delta}+\eta \in A$. This would contradict the fact that $\bar{\delta} = \max A$. Therefore, $\epsilon = \max A$, and so $\epsilon \in A$. □

Now we prove that it is possible to injectively match all the cornerpoints of $\ell^*_{(\mathcal{M},\varphi)}$ to those of $\ell^*_{(\mathcal{M},\psi)}$ with a maximum motion not greater than the L^∞-norm between φ and ψ.

PROPOSITION 4.4. *Let $\epsilon \geq 0$ be a real number and let $(\mathcal{M},\varphi)$ and $(\mathcal{M},\psi)$ be two size pairs such that $\max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| \leq \epsilon$. Then there exist (a_i) and (b_i) representative sequences for $\ell^*_{(\mathcal{M},\varphi)}$ and $\ell^*_{(\mathcal{M},\psi)}$ respectively, and an injection $f : \mathbb{N} \to \mathbb{N}$ such that $d(a_i, b_{f(i)}) \leq \epsilon$.*

Proof. Let H be the set of all the cornerpoints for $\ell^*_{(\mathcal{M},\varphi)}$. We can write $H = K_1 \cup K_2$, where $K_1 = \{p \in H : d(p,\Delta) > \epsilon\}$ and $K_2 = \{p \in H : d(p,\Delta) \leq \epsilon\}$. We can write $\mathbb{N}$ as the disjoint union of I_1 and I_2, where $i \in I_1$ if $a_i \in K_1$, and $i \in I_2$ if $a_i \in K_2 \cup \Delta$.

The cardinality of K_1 is finite, according to Prop. 2.8. Therefore, Prop. 4.3 implies that there exist (a_i) and (b_i) representative sequences of points for $\ell^*_{(\mathcal{M},\varphi)}$ and $\ell^*_{(\mathcal{M},\psi)}$ respectively, such that $d(a_i, b_i) \leq \epsilon$ for each $a_i \in K_1$ (let us observe that necessarily $b_i \notin \Delta$). Thus, if $I_1 \neq \emptyset$, $\alpha : I_1 \to \mathbb{N}$, defined as the identity $\alpha(i) = i$, is an injection such that $d(a_i, b_{\alpha(i)}) \leq \epsilon$ for $i \in I_1$.

Now we observe that, by the definition of a representative sequence, there is a countably infinite collection of indexes j with b_j contained in Δ. Thus, there is an injection $\beta : I_2 \to \mathbb{N}$ such that $b_{\beta(i)} \in \Delta$. By construction, $d(a_i, b_{\beta(i)}) \leq \epsilon$ for $i \in I_2$, and $Im(\alpha) \cap Im(\beta) = \emptyset$. In conclusion, the function $f : \mathbb{N} \to \mathbb{N}$ that coincides with α on I_1, and with β on I_2, is the required injection. □

The following result is a slight modification of the well-known Cantor-Bernstein theorem (cf. [18]) and can be proved in a similar way.

PROPOSITION 4.5. *Let A and B be two subsets of a set X endowed with a pseudo-distance δ. A real number $\epsilon \geq 0$ is given. If two injections $f : A \to B$ and $g : B \to A$ exist, such that, for each $p \in A$, $\delta(p, f(p)) \leq \epsilon$, and, for each $q \in B$, $\delta(q, g(q)) \leq \epsilon$, then there is a bijection h between A and B such that, for each $p \in A$, $\delta(p, h(p)) \leq \epsilon$.*

We are now ready to prove a key result of this paper.

THEOREM 4.6. (Matching stability theorem) *Let $(\mathcal{M},\varphi)$ be a size pair. For every real number $\epsilon \geq 0$ and for every measuring function $\psi : \mathcal{M} \to \mathbb{R}$ such that $\max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| \leq \epsilon$, the matching distance between $\ell^*_{(\mathcal{M},\varphi)}$ and $\ell^*_{(\mathcal{M},\psi)}$ is smaller than or equal to ϵ.*

Proof. Prop. 4.4 implies that there exist (a_i) and (b_i) representative sequences for $\ell^*_{(\mathcal{M},\varphi)}$ and $\ell^*_{(\mathcal{M},\psi)}$ respectively, and an injection $f : \mathbb{N} \to \mathbb{N}$ such that $d(a_i, b_{f(i)}) \leq \epsilon$. Analogously, there is an injection $g : \mathbb{N} \to \mathbb{N}$, such that for each b_i it holds that $d(b_i, a_{g(i)}) \leq \epsilon$. Then the claim follows immediately from Prop. 4.5. □

5. The connection between the matching distance and the natural pseudo-distance. As a corollary of Th. 4.6 we obtain the following Th. 5.2, stating that the matching distance between reduced size functions furnishes a lower bound for the natural pseudo-distance between size pairs. We recall that, given two size pairs $(\mathcal{M},\varphi)$ and $(\mathcal{W},\psi)$ with $\mathcal{M}$ and $\mathcal{W}$ homeomorphic, their natural pseudo-distance is a measure of their shape dissimilarity defined as $\inf_h \max_{P\in\mathcal{M}} |\varphi(P) - \psi(h(P))|$, where h varies among all the homeomorphisms between $\mathcal{M}$ and $\mathcal{W}$. For more details about natural pseudo-distances between size pairs, the reader is referred to [6] and [5].

Before stating this result, in Th. 5.1 we show that the inf and the sup in the definition of matching distance are actually attained, that is to say $d_{match}(\ell_1^*, \ell_2^*) = \min_\sigma \max_i d(a_i, b_{\sigma(i)})$.

THEOREM 5.1. *Let (a_i) and (b_i) be two representative sequences of points for the reduced size functions ℓ_1^* and ℓ_2^* respectively. Then the matching pseudo-distance between ℓ_1^* and ℓ_2^* is equal to the number $\min_\sigma \max_i d(a_i, b_{\sigma(i)})$, where i varies in $\mathbb{N}$ and σ varies among all the bijections from $\mathbb{N}$ to $\mathbb{N}$.*

Proof. Let us first see that, for any bijection σ from $\mathbb{N}$ to $\mathbb{N}$, $\sup_{i\in\mathbb{N}} d(a_i, b_{\sigma(i)}) = \max_{i\in\mathbb{N}} d(a_i, b_{\sigma(i)})$. This is true because the accumulation points for the set of cornerpoints of a reduced size function (if any) cannot belong to Δ^+, but only to Δ (see Prop. 2.8). By definition, for any p and p' in Δ, $d(p,p') = 0$, and hence the claim is proved.

Let us now prove that $\inf_\sigma \max_i d(a_i, b_{\sigma(i)}) = \min_\sigma \max_i d(a_i, b_{\sigma(i)})$. Let us set $s := \inf_\sigma \max_i d(a_i, b_{\sigma(i)})$. According to Prop. 2.8, if $s = 0$, then the cornerpoints of ℓ_1^* coincide with those of ℓ_2^*, and their multiplicities are the same, implying that the claim is true. Let us consider the case when $s > 0$. Let $J_1 := \{i \in \mathbb{N} : d(a_i, \Delta) > s\}$ and $J_2 := \{i \in \mathbb{N} : d(a_i, \Delta) \leq s\}$. By Prop. 2.8, J_1 contains only a finite number of elements (possibly $J_1 = \emptyset$) and therefore there exists a real positive number ϵ for which $d(a_i, \Delta) > s + \epsilon$ for each $i \in J_1$.

If $J_1 \neq \emptyset$, let us consider the set Σ of all the injective functions $\sigma : J_1 \to \mathbb{N}$ such that $\max_{i\in J_1} d(a_i, b_{\sigma(i)}) < s + \epsilon/2$. This set is non-empty by the definition of s, and contains only a finite number of injections because J_1 is finite, and for each $i \in J_1$ the set $\{j \in \mathbb{N} : d(a_i, b_j) < s + \epsilon/2\}$ is finite. Thus we can take an injection $\bar\sigma : J_1 \to \mathbb{N}$ that realizes the minimum of $\max_{i\in J_1} d(a_i, b_{\sigma(i)})$ as σ varies in Σ. Thus, $\max_{i\in J_1} d(a_i, b_{\bar\sigma(i)}) \leq s$.

Moreover, we can take an injection $\hat\sigma : J_2 \to \mathbb{N}$ such that $\max_{i\in J_2} d(a_i, b_{\hat\sigma(i)}) \leq s$, because, for every $i \in J_2$, we can choose a different index j such that $b_j \in \Delta$. Since $Im(\bar\sigma) \cap Im(\hat\sigma) = \emptyset$, we can construct an injection $f : \mathbb{N} \to \mathbb{N}$ such that each displacement between a_i and $b_{f(i)}$ is not greater than s. Analogously, we can construct an injection $g : \mathbb{N} \to \mathbb{N}$ such that each displacement between b_i and $a_{g(i)}$ is not greater than s. Therefore, by Prop. 4.5, there is a bijection $h : \mathbb{N} \to \mathbb{N}$ such that $d(a_i, b_{h(i)}) \leq s$ for every index i, and so the theorem is proved. □

THEOREM 5.2. *Let $\epsilon \geq 0$ be a real number and let $(\mathcal{M}, \varphi)$ and $(\mathcal{W}, \psi)$ be two size pairs with $\mathcal{M}$ and $\mathcal{W}$ homeomorphic. Then*

$$d_{match}(\ell^*_{(\mathcal{M},\varphi)}, \ell^*_{(\mathcal{W},\psi)}) \leq \inf_h \max_{P\in\mathcal{M}} |\varphi(P) - \psi(h(P))|,$$

where h varies among all the homeomorphisms from $\mathcal{M}$ to $\mathcal{W}$.

Proof. We begin by observing that $\ell^*_{(\mathcal{W},\psi)} = \ell^*_{(\mathcal{M},\psi\circ h)}$, where $h : \mathcal{M} \to \mathcal{W}$ is any homeomorphism between $\mathcal{M}$ and $\mathcal{W}$. Moreover, for each homeomorphism h, by applying Th. 4.6 with $\epsilon = \max_{P\in\mathcal{M}} |\varphi(P) - \psi(h(P))|$, we have

$$d_{match}(\ell^*_{(\mathcal{M},\varphi)}, \ell^*_{(\mathcal{M},\psi\circ h)}) \leq \max_{P\in\mathcal{M}} |\varphi(P) - \psi(h(P))|.$$

Since this is true for any homeomorphism h between $\mathcal{M}$ and $\mathcal{W}$, it immediately follows that $d_{match}(\ell^*_{(\mathcal{M},\varphi)}, \ell^*_{(\mathcal{W},\psi)}) \leq \inf_h \max_{P\in\mathcal{M}} |\varphi(P) - \psi(h(P))|$. □

6. Conclusion. We have introduced the new concept of reduced size function and we have shown how a problem related to shape matching (that is, measuring the dissimilarity between size pairs by the natural pseudo-distance) can be more easily dealt with using this new tool. In fact, we have proved that an appropriate distance between reduced size functions, based on optimal matching, provides a stable and easily computable lower bound for the natural pseudo-distance between size pairs.

We are currently carrying out research aiming to clarify whether or not the result of Th. 5.2, giving a lower bound for the natural pseudo-distance between size pairs, can be improved by using some other distance between reduced size functions. Moreover, we are also studying the relationship between the estimate in Th 5.2 and another known estimate which is given in [5].

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