Shape comparison by multidimensional size functions

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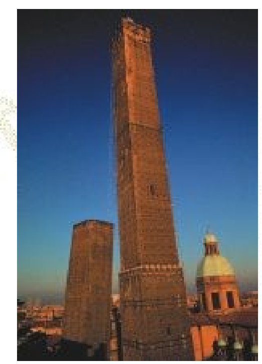


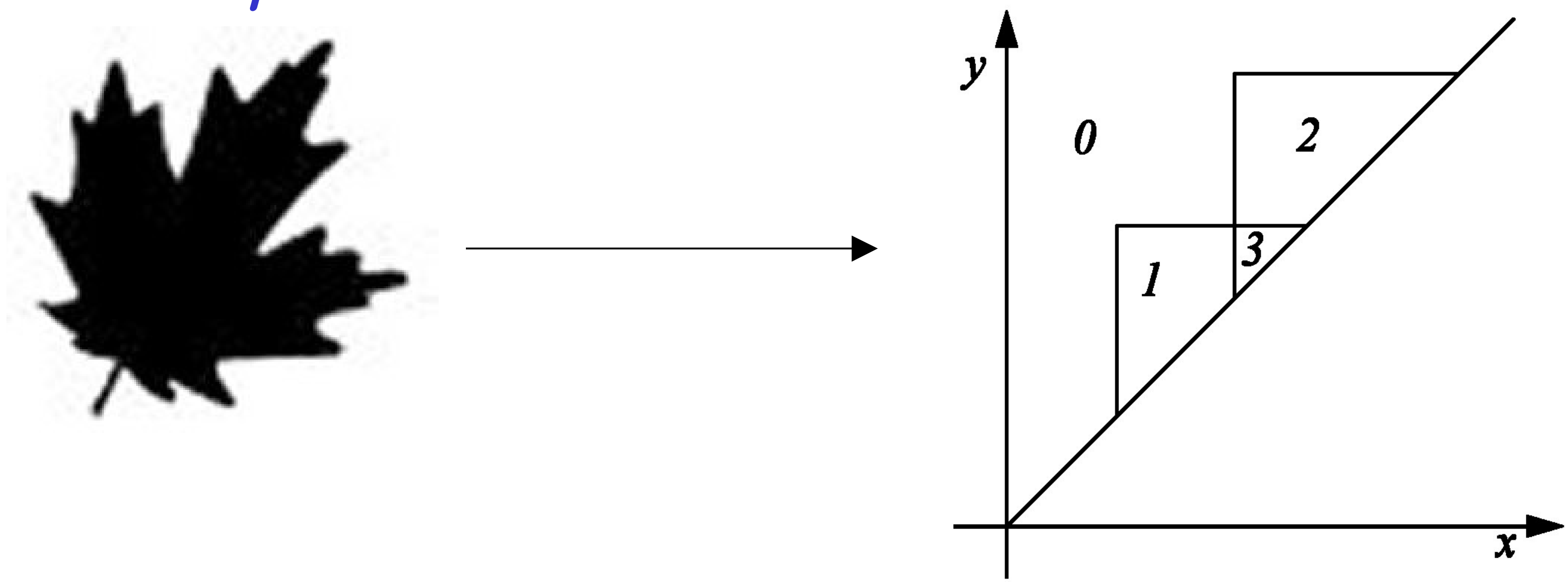
What are size functions?

Assume a size pair (M,φ) is given

(i.e. M=topological space, $\varphi : M \to \mathbb{R}^k$)

We want to take each size pair into a function describing the shape of M with respect to φ.

Instead of comparing manifolds, we shall compare these descriptors.



NOTE: in the following of this talk we shall write $(x_1, \ldots, x_k) \leq (y_1, \ldots, y_k)$ *to mean* $x_i \leq y_i$ *for every index* i. *Analogously,* $(x_1, \ldots, x_k) < (y_1, \ldots, y_k)$ *will mean that* $x_i < y_i$ *for every index* i.

Definition of size function

Let M be a topological space and assume that a continuous function $\varphi : M \to \mathrm{IR}^k$ is given.

For each $y \in \mathrm{IR}^k$ we denote by $M\langle\varphi \le y\rangle$ the set of all points of M at which the measuring function φ takes a value not greater than y.

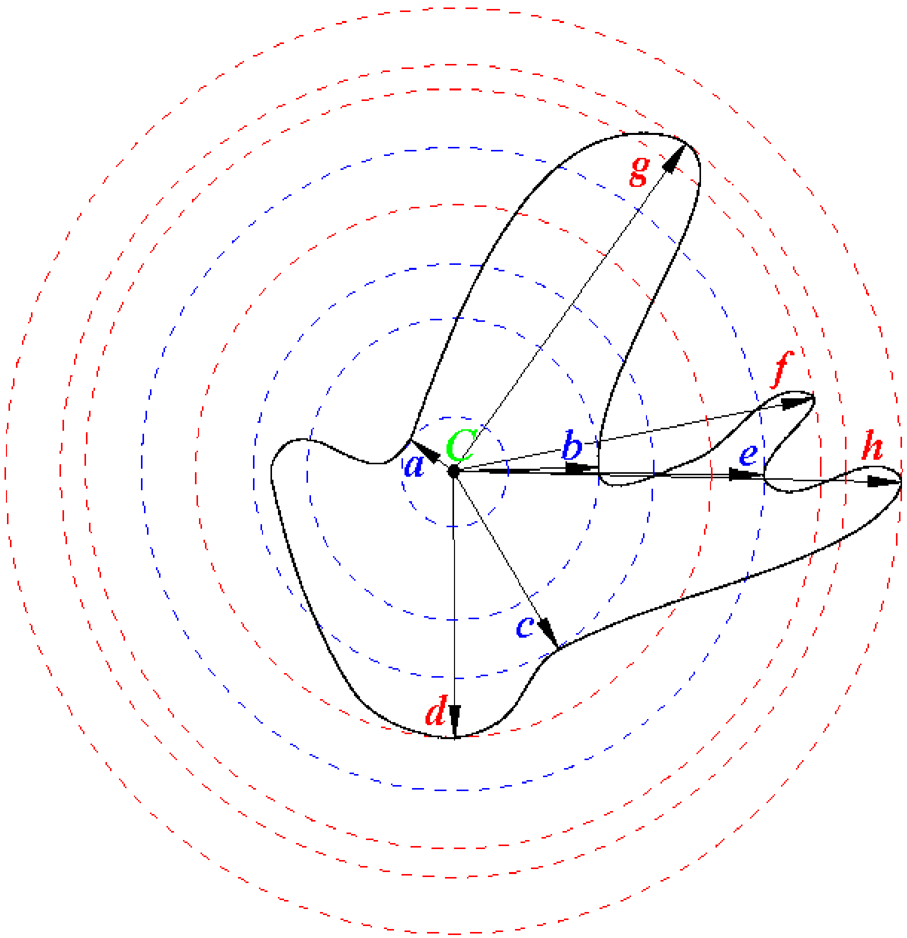
For each real vector y we say that two points P,Q are $\langle\varphi \le y\rangle$-connected if and only if they belong to the same component in $M\langle\varphi \le y\rangle$.

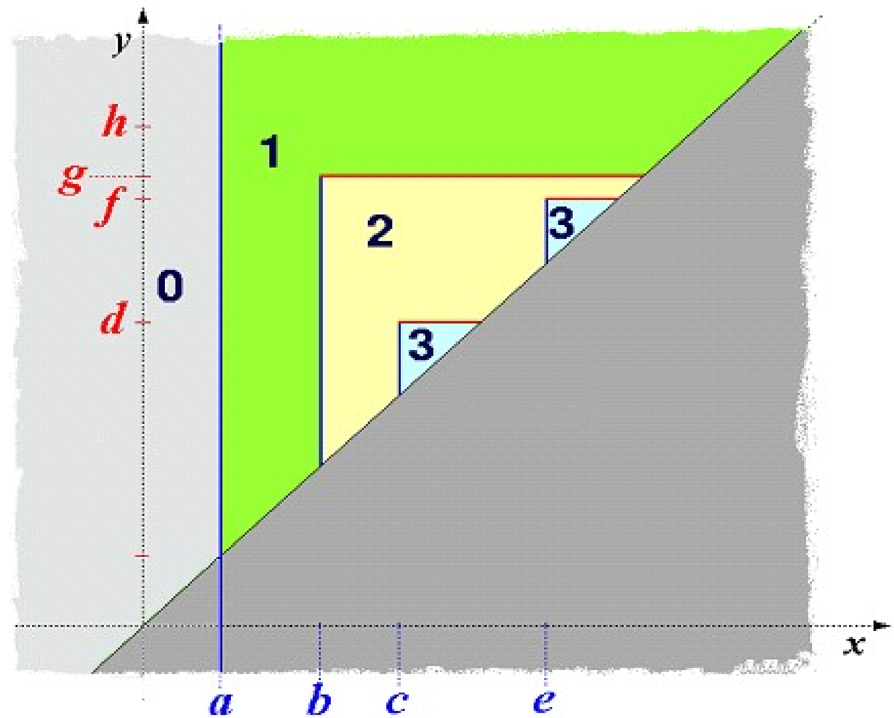
We call *size function* (of the size pair (M,φ)) the function $\ell_{(M,\varphi)}: \Delta^+ = \{(x,y) \in \mathbb{R}^{2k}: x < y\} \to \mathbb{N}$ that takes each point (x,y) to the number of equivalence classes of $M\langle\varphi \le x\rangle$ with respect to $\langle\varphi \le y\rangle$-connectedness.

EQUIVALENT DEFINITION

$\ell_{(M,\varphi)}(x,y)$ is the number of connected components of $M\langle\varphi \le y\rangle$ that contain at least one point of $M\langle\varphi \le x\rangle$.

Let us make the definition clear by some example in the case $k=1$

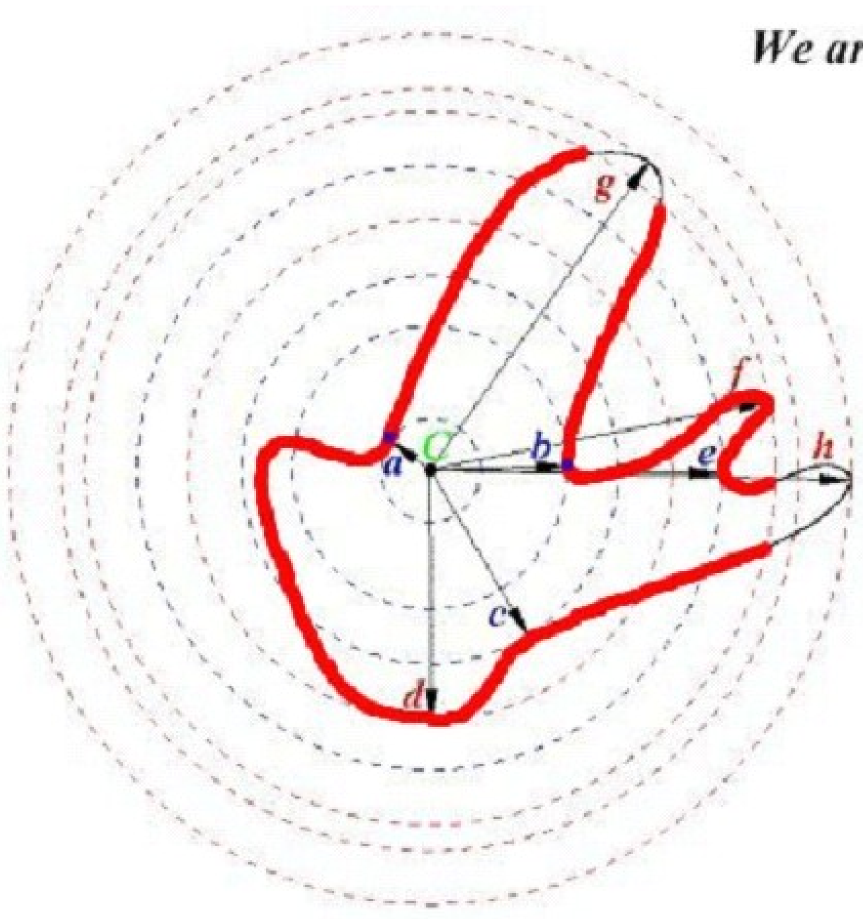


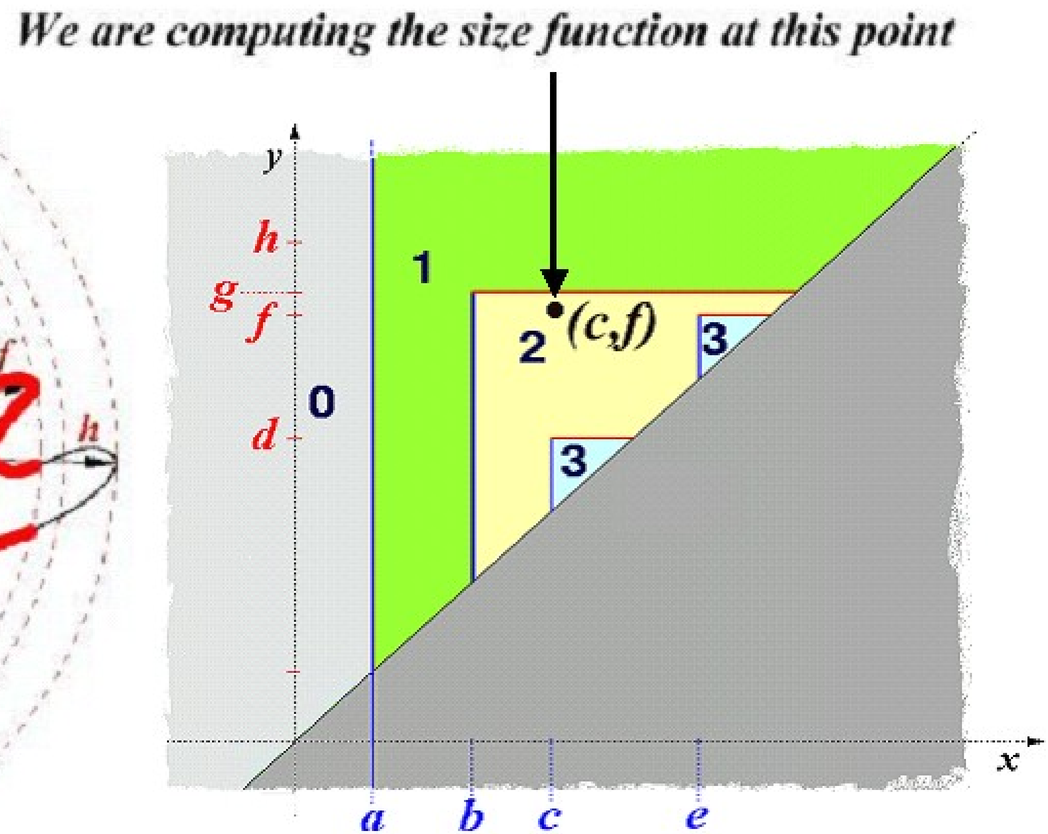


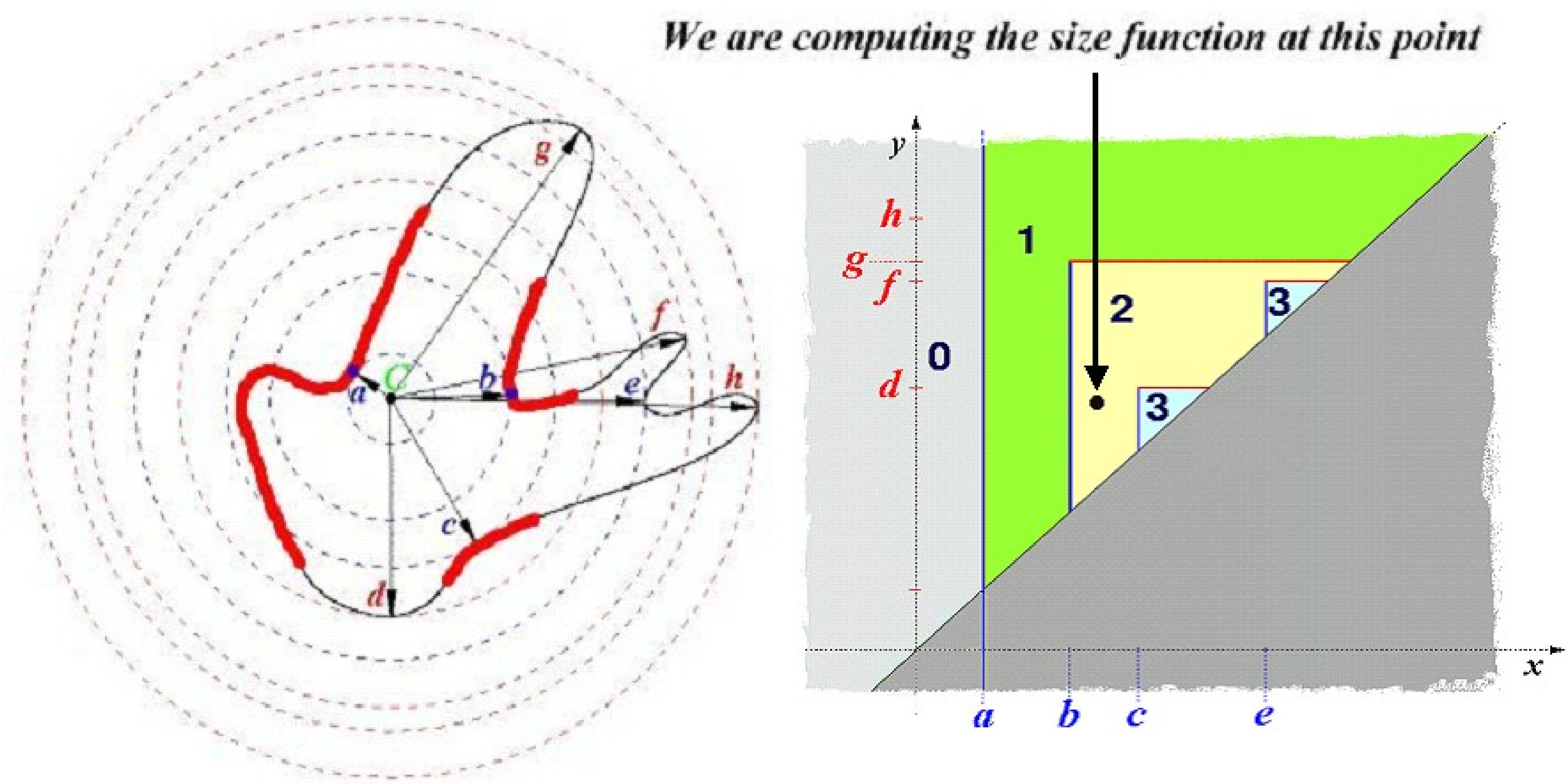
This is the size function $\ell_{(M,\varphi)}$.

Example: M is the displayed curve, φ is the distance from C.

More details about our example






We are computing the size function at this point
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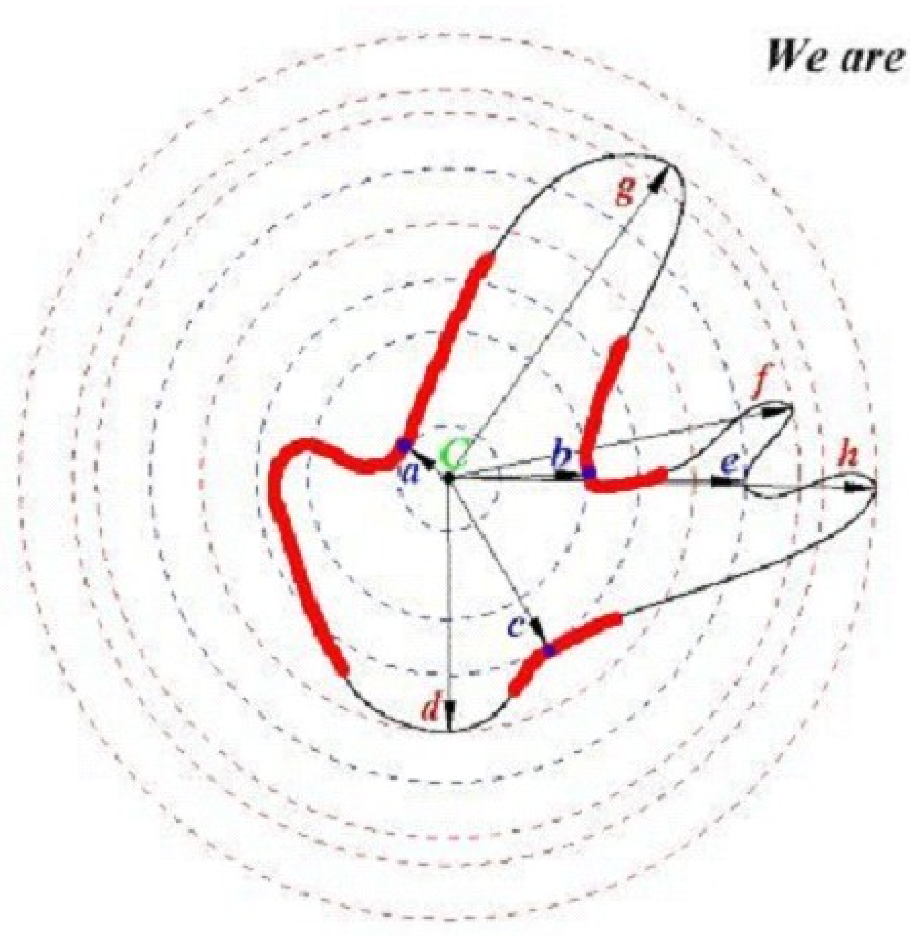
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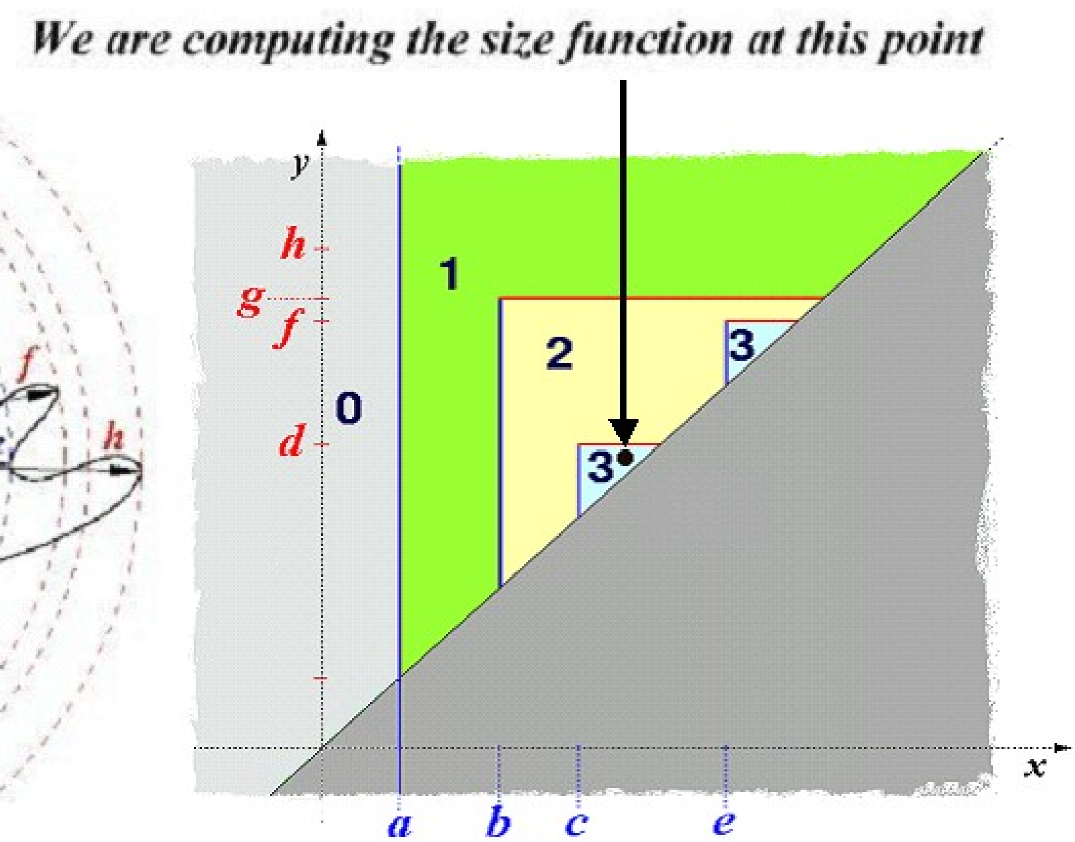
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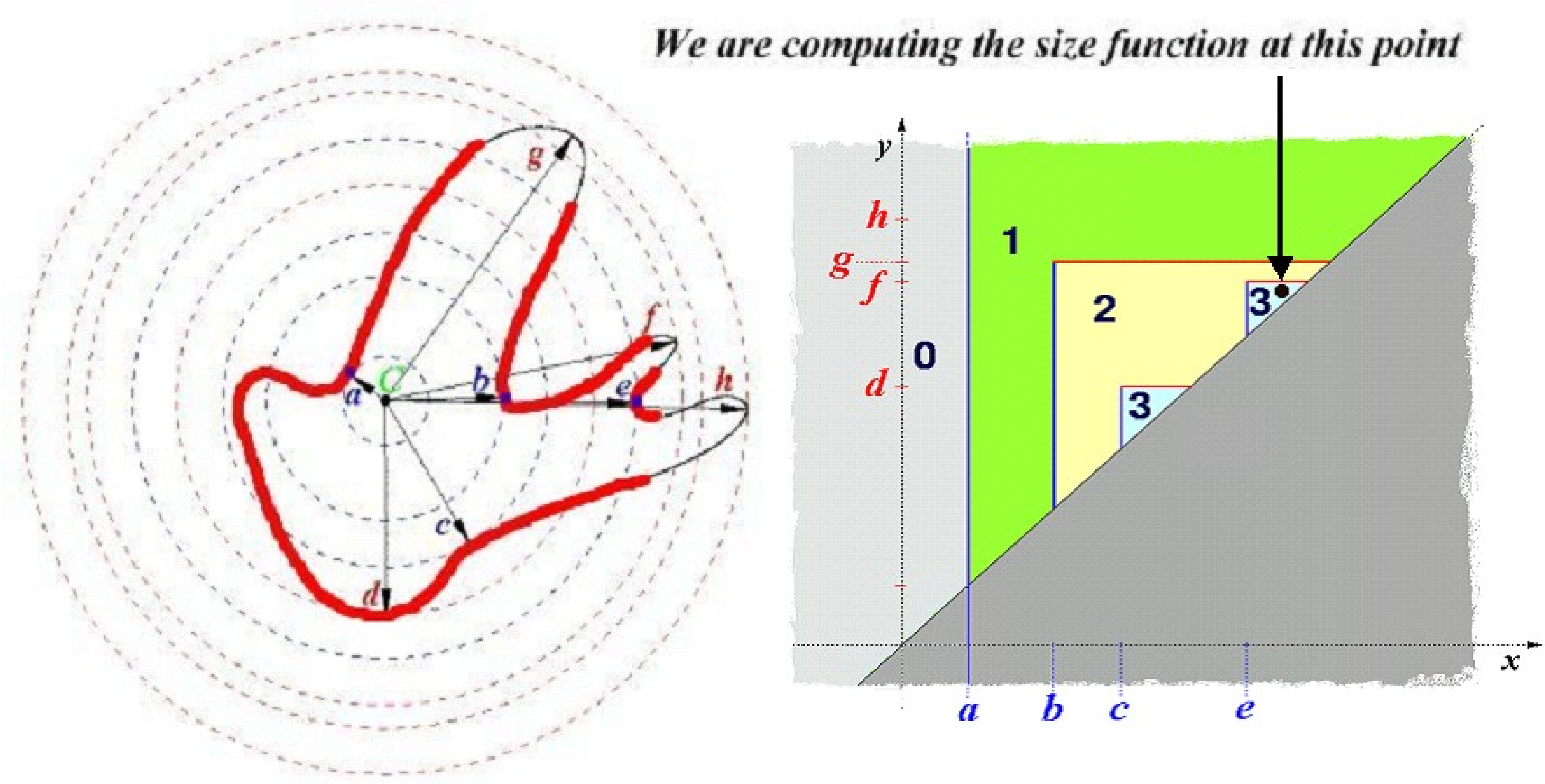
We are computing the size function at this point
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Size functions were introduced in

P. Frosini, *A distance for similarity classes of submanifolds of a Euclidean space*, Bull. Austral. Math. Soc. 42, 3 (1990), 407-416 *as a tool for comparing manifolds endowed with measuring functions.*

Size functions are strongly related to

Size homotopy groups (cf. P. Frosini, M. Mulazzani, *Size homotopy groups for computation of natural size distances*, Bull. Belg. Math. Soc. **6** (1999), 455-464).

Persistent homology groups (cf. H. Edelsbrunner, D.Letscher and A. Zomorodian, *Topological Persistence and Simplification*, Proc. 41st Ann. IEEE Sympos. Found Comput. Sci., 2000, 454-463).

Morse shape descriptor (cf. M. Allili, D. Corriveau, Djemel Ziou, *Morse Homology Descriptor for Shape Characterization*, Proc. 17^{th} Int. Conf. on Pattern Recognition, **4**, 2004, 27-30).

In particular, Allili, Corriveau and Ziou, have pointed out this relation between size functions and the Morse Shape Descriptor, in the case $k=1$:

$$\ell_{(M,\varphi)}(x,y) = Rank\big(H_0(M_y)\big) - R_\varphi(x,y,0)$$

where R_φ denotes the Morse Shape Descriptor.

An evolving curve and its evolving size function

We observe that, for $k=1$,

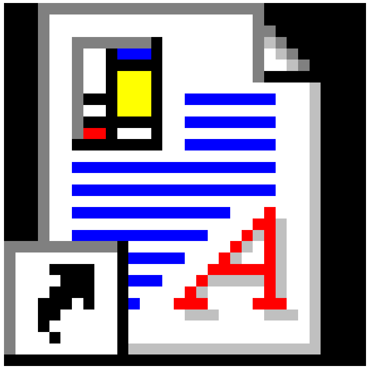
1) $\ell_{(M,\varphi)}(x,y)$ is non-decreasing in x and non-increasing in y;

2) Discontinuities in x propagate vertically towards the diagonal $\Delta=\{(x,y):\ x=y\}$, while discontinuities in y propagate horizontally towards the diagonal Δ.

The following result locates the discontinuity points of a size function, in the case $k=1$:

Theorem. Suppose that M is a closed C^2-manifold and the measuring function φ is C^1. If (x,y) is a discontinuity point for the size function, then either x or y or both are critical values for φ.

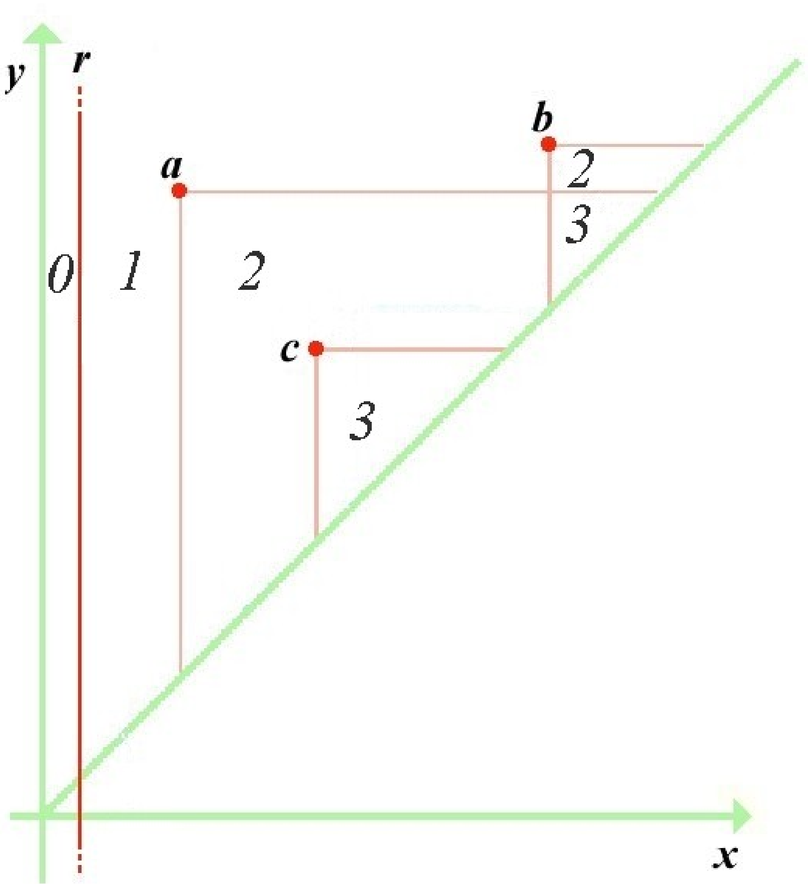
Size functions are easily computable (from the discrete point of view, we only have to count the connected components of a graph).



sizeshow

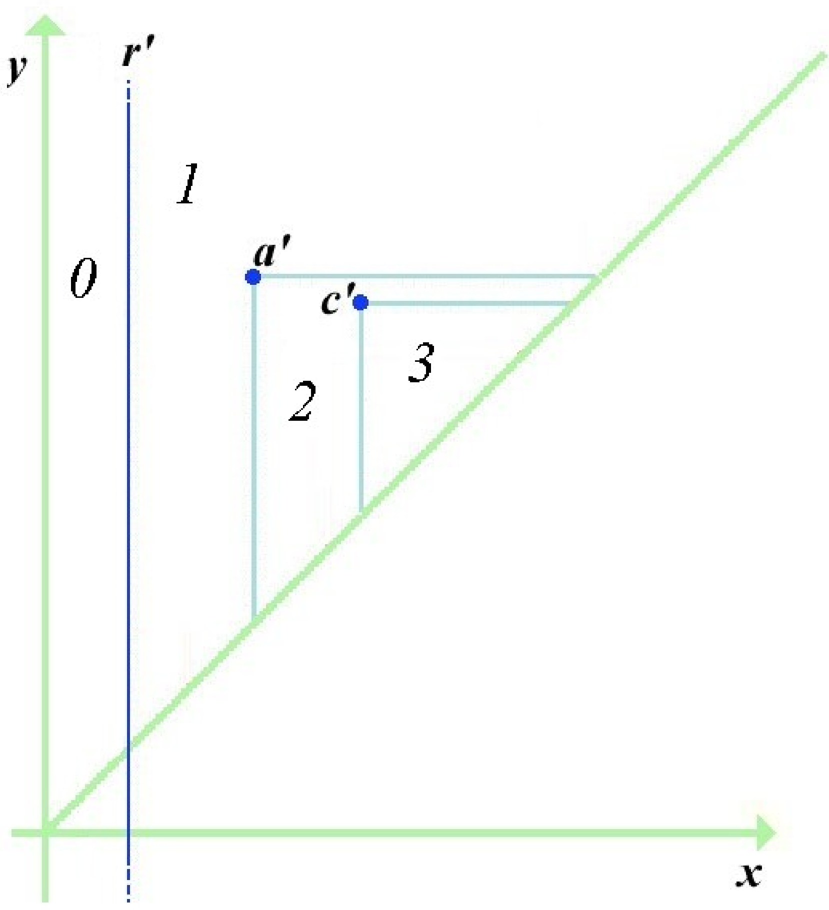
In the case $k=1$ a concise description of size functions is available.

The algebraic representation of a size function by a formal series of lines and points.



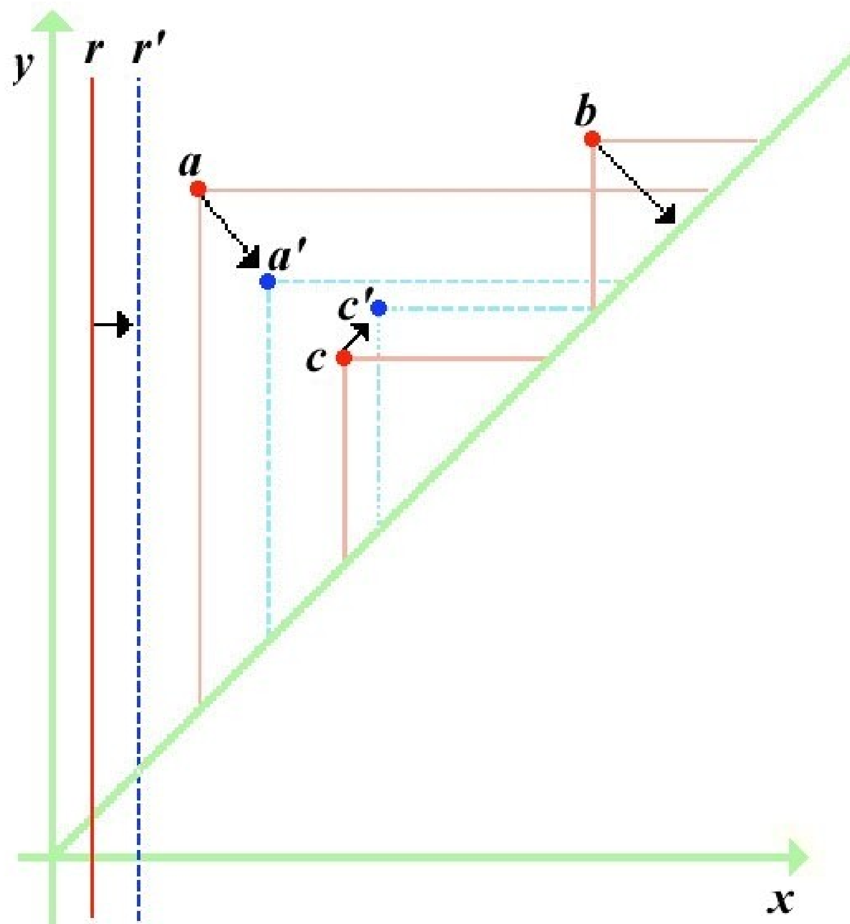
$\longleftrightarrow$ $r+a+b+c$

Another size function with its formal series of lines and points.



$\longleftrightarrow \quad r'+a'+c'$

The algebraic representation of size functions allows us to compare them by a matching distance:



A curvature driven plane curve evolution and the corresponding size function (w.r.t. the distance from the center of mass)

Thanks to Frederic Cao for the curvature evolution code and to Michele d'Amico for making this animation.

What are size functions for?

1) Finding lower bounds for the natural pseudodistance (i.e., the pseudodistance used for comparing manifolds endowed with measuring functions). The natural pseudodistance between the size pairs (M,φ), (N,ψ) is

$$\delta = \inf_{f \in H(M,N)} \max_{P \in M} \|\varphi(P) - \psi(f(P))\|_\infty$$

THEOREM: $\boxed{d_{match} \leq \delta}$

2) Comparing shapes in applications.

How can we manage the general case $k>1$?

Three main problems:

0) How can we compare size functions in the case $k>1$?

1) How can we extend the description by formal series of points and lines?

2) Is there stability with respect to computation?

Managing the case $k>1$ in a direct way is difficult

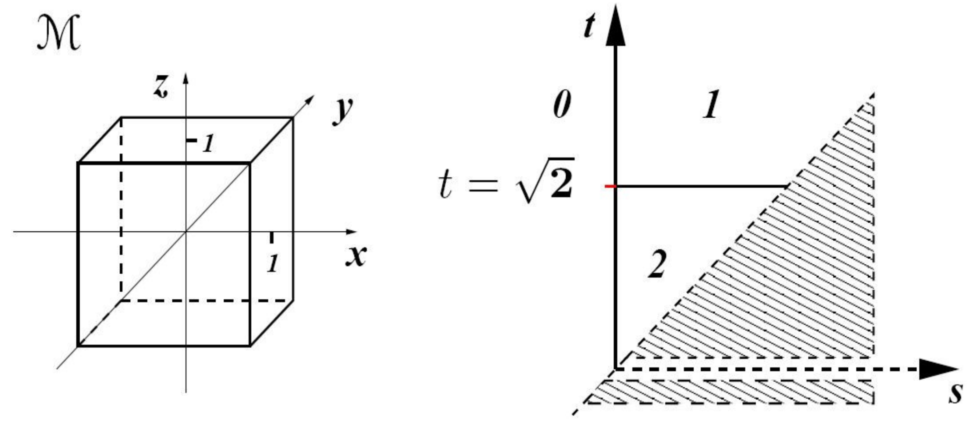
1) A satisfactory representation by formal series of points, lines, planes and hyperplanes seems not to exist (negative multiplicities and non-localities arise).

2) However, a direct approach involves working in a domain of IR^{2k}. This requires us to work with very big approximation graphs and matrices.

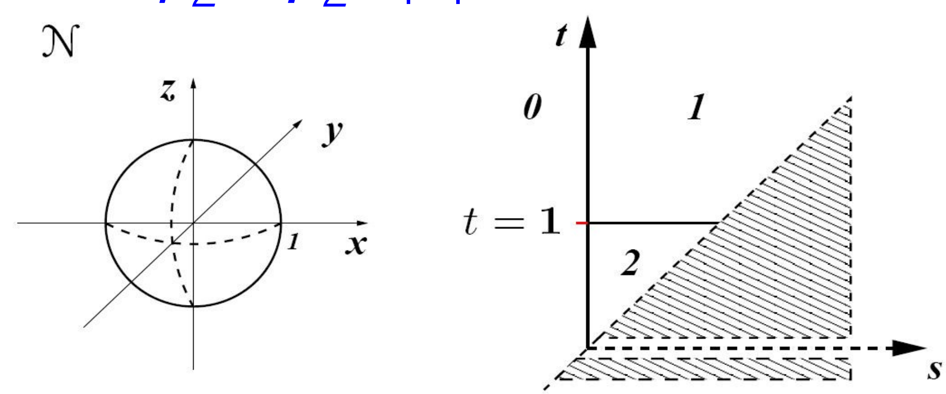
The answers to these questions come from recognizing that a parameterized family of half-planes in $\mathbb{R}^k \times \mathbb{R}^k$ exists such that the restriction of $\ell_{(M,\varphi)}$ to each of these planes can be seen as a particular 1-dimensional size function.

$$\boldsymbol{u}=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),\ \boldsymbol{b}=(0,0)$$

Important remark: in general, the size function with respect to $\varphi=(\varphi_1,\dots,\varphi_k)$ contains more information than the set of all size functions with respect to $\varphi_1,\dots,\varphi_k$, considered independently.



$$\varphi_1=\psi_1=|x|$$
$$\varphi_2=\psi_2=|z|$$



For every unit vector $\boldsymbol{u}=(u_1,\ldots,\ u_k\ )$ of IR^k such that $u_i>0$ for $i=1,\ldots,k$ and for every vector $\boldsymbol{b}=(b_1,\ldots,\ b_k)$ of IR^k such that $\sum_i b_i=0$, we shall say that the pair $(\boldsymbol{u},\boldsymbol{b})$ is *admissible*. Given an admissible pair $(\boldsymbol{u},\boldsymbol{b})$, we define the half-plane $\pi_{(\boldsymbol{u},\boldsymbol{b})}$ of $\mathrm{IR}^k \times \mathrm{IR}^k$ by the following parametric equations:

$$\begin{cases} x=s\boldsymbol{u}+\boldsymbol{b} \\ y=t\boldsymbol{u}+\boldsymbol{b} \end{cases} \qquad \boxed{s<t}$$

For each admissible pair $(\boldsymbol{u},\boldsymbol{b})$ we can consider a new measuring function $F^{\varphi}_{(\boldsymbol{u},\boldsymbol{b})}: M \to IR$.

$$F^{\varphi}_{(\boldsymbol{u},\boldsymbol{b})}(P) = \max_{i=1,\ldots,k} \left\{ \frac{\varphi_i(P) - b_i}{u_i} \right\}$$

Theorem. If $(\boldsymbol{x},\boldsymbol{y})=(s\boldsymbol{u}+\boldsymbol{b}, t\boldsymbol{u}+\boldsymbol{b})$ then

$$\ell_{(M,\varphi)}(\boldsymbol{x},\boldsymbol{y}) = \ell_{\left(M,F^{\varphi}_{(\boldsymbol{u},\boldsymbol{b})}\right)}(s,t)$$

In this way the theory in dimension k is reduced to the 1-dimensional case.

As a consequence, most of the results proved in the 1-dimensional case can be extended to the general k-dimensional case. In particular we can prove that also in the general k-dimensional case

0) size functions can be represented by using formal series of points and lines;

1) size functions are stable with respect to small changes of the measuring function;

2) size functions give lower bounds for the natural pseudodistance between size pairs.

EXAMPLES

The 3D models we are going to show are taken by

McGill 3D Shape Benchmark:
http://www.cim.mcgill.ca/~shape/benchMark/

MIRALab: http://www.miralab.unige.ch/

The data that we are going to display have been computed by Daniela Giorgi (CNR-IMATI, Genova)

A measuring function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ has been considered. In our examples φ_1 is the distance from the point $B + v/\|v\|$, where B is the center of mass of the model and v equals $\frac{1}{n}\sum_{i=1}^{n}(P_i - B)\cdot\|P_i - B\|$

(all functions are discretized).

Similarly, φ_2 is the distance from the point $B - v/\|v\|$.

Three admissible pairs have been chosen by setting

$$\boldsymbol{u}= \left(\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{\pi}{4}\right)\right), \boldsymbol{b}= \boldsymbol{0}$$

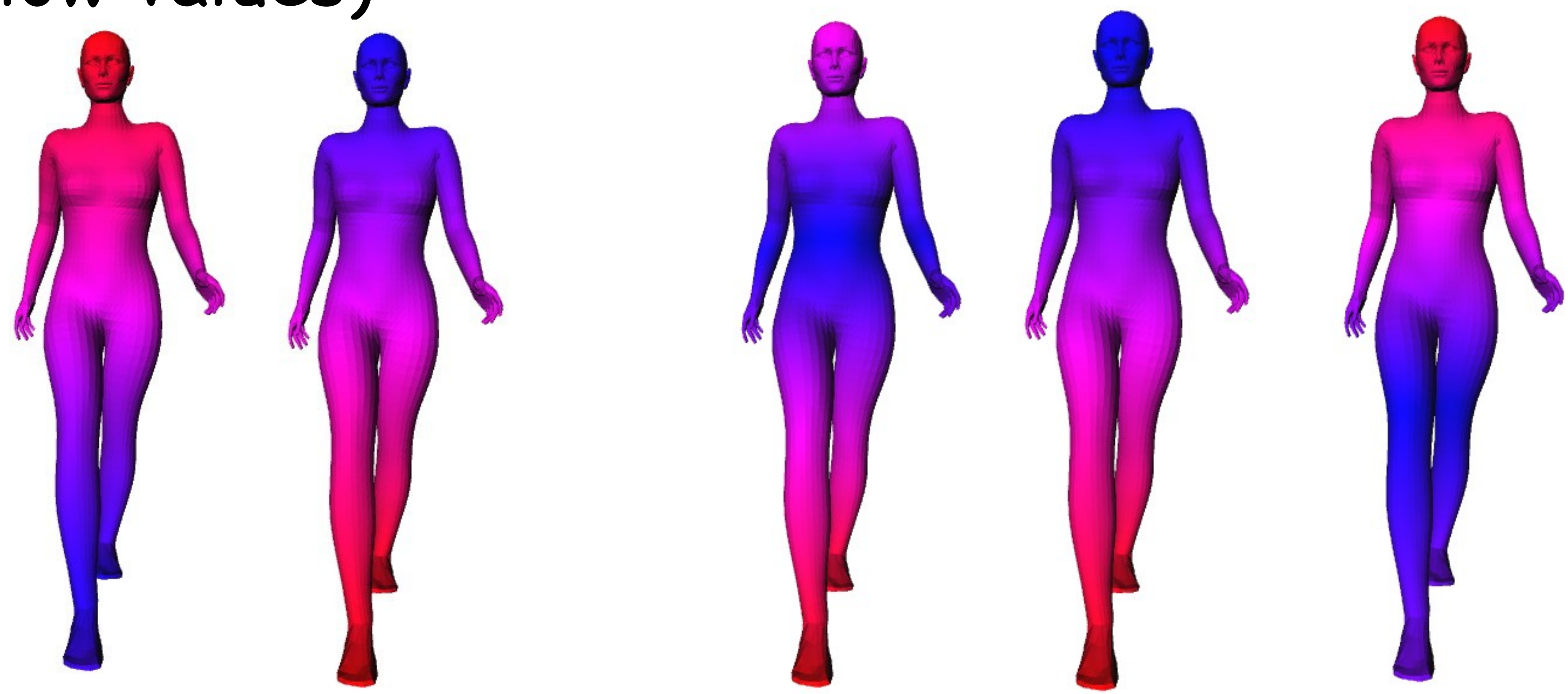
$$\boldsymbol{u}= \left(\sin\left(\frac{\pi}{8}\right), \cos\left(\frac{\pi}{8}\right)\right), \boldsymbol{b}= \boldsymbol{0}$$

$$\boldsymbol{u}= \left(\sin\left(3\frac{\pi}{8}\right), \cos\left(3\frac{\pi}{8}\right)\right), \boldsymbol{b}= \boldsymbol{0}$$

(Obviously, an arbitrary number of admissible pairs could be chosen. Each admissible pair corresponds to a plane slicing IR^{2k}).

The function $F^{\varphi}_{(\boldsymbol{u},\boldsymbol{b})}(P) = \max_{i=1,\ldots,k} \left\{ \frac{\varphi_i(P) - b_i}{u_i} \right\}$

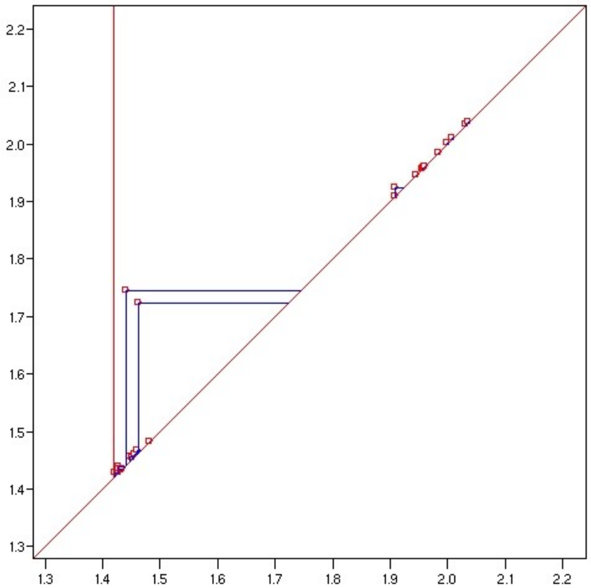
has been computed (red=high values, blue= low values):



φ_1 φ_2 $\pi/4$ $\pi/8$ $3\pi/8$

red=high values, blue= low values

φ_1 φ_2 $\pi/4$ $\pi/8$ $3\pi/8$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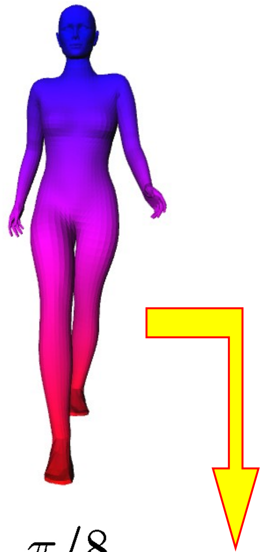
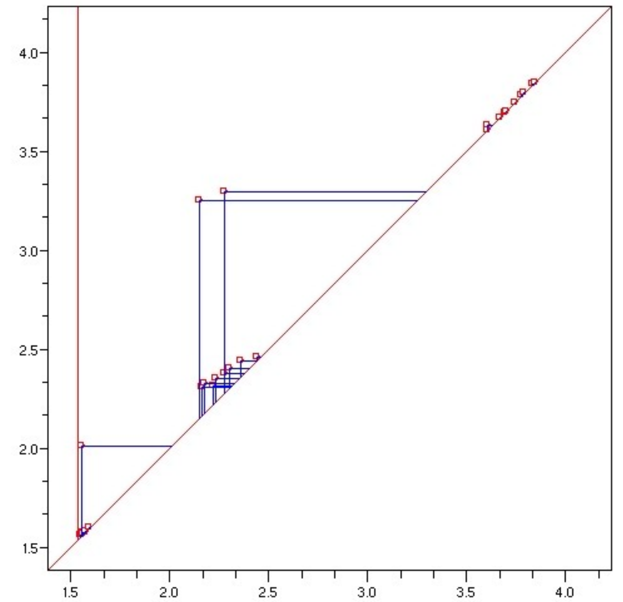


$\pi/8$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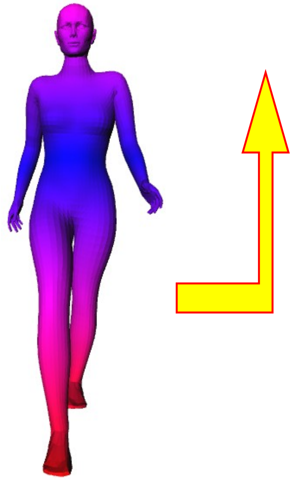
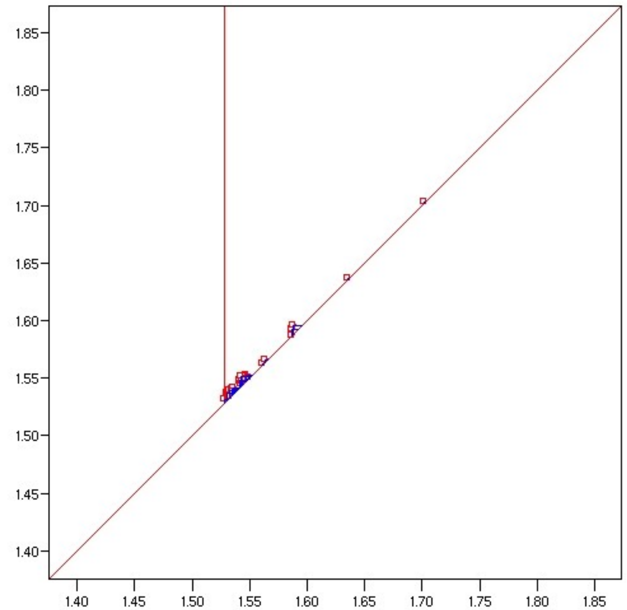


$\pi/4$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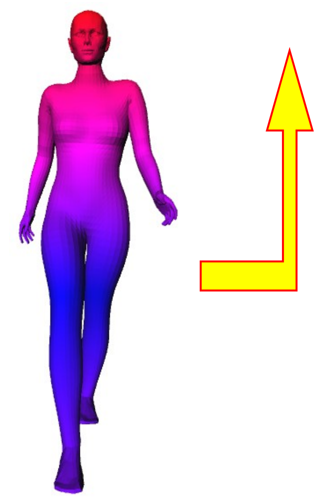
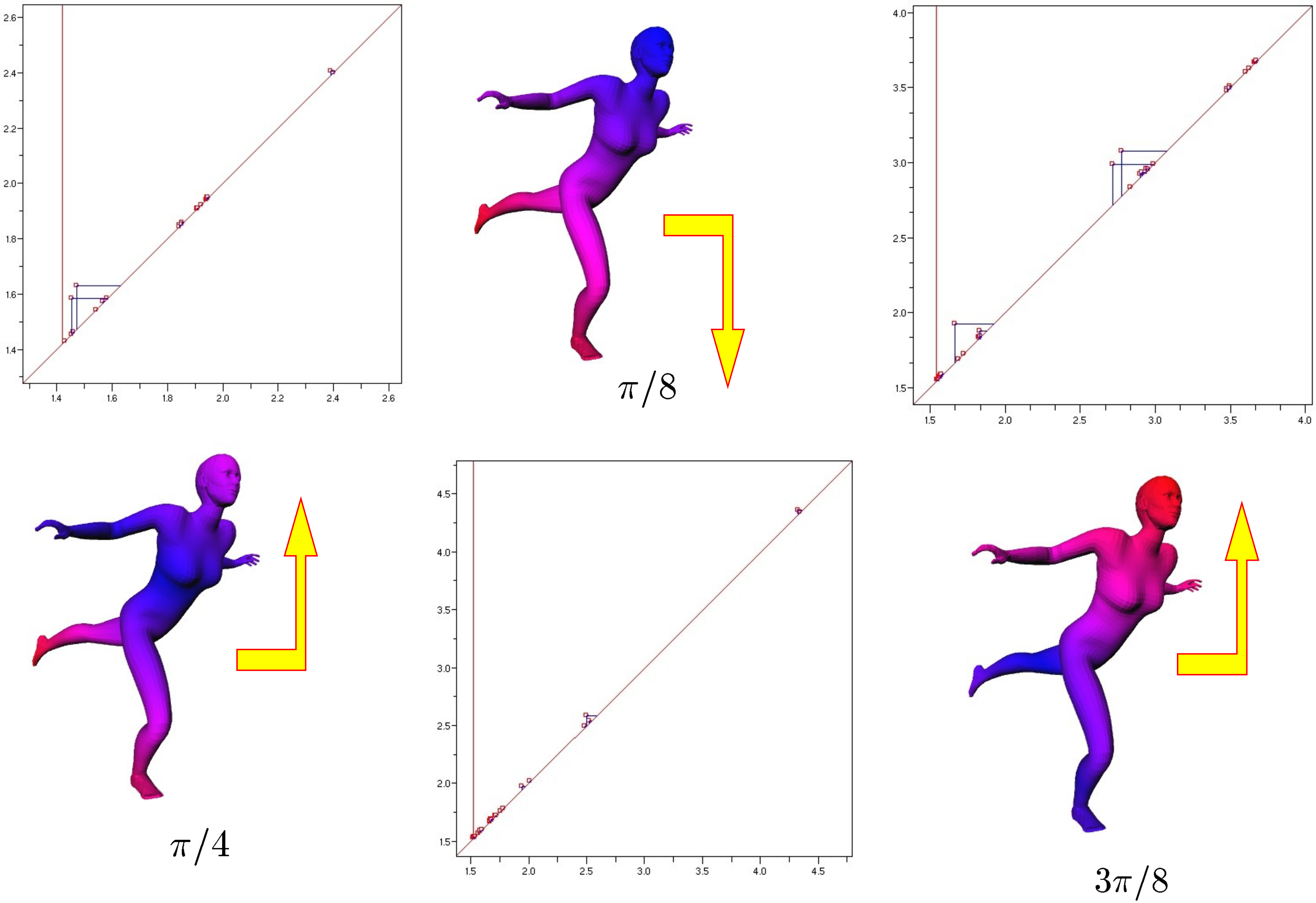
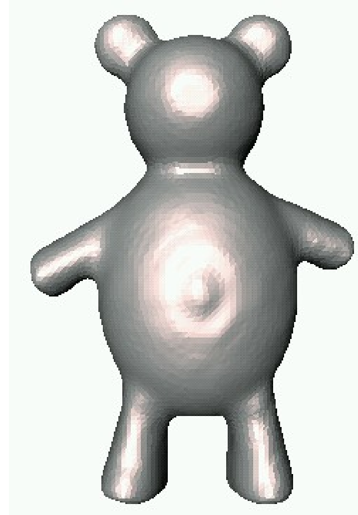


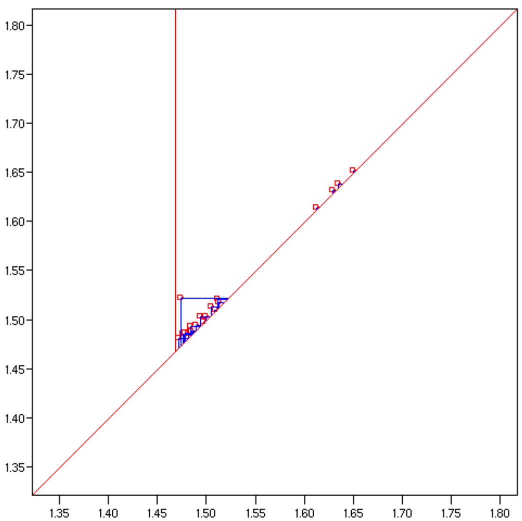
$3\pi/8$



$\pi/8$
$\pi/4$
$3\pi/8$

We have done the same for the other four 3D models

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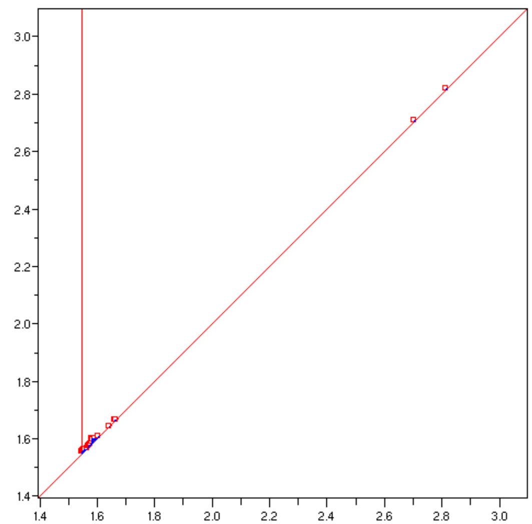
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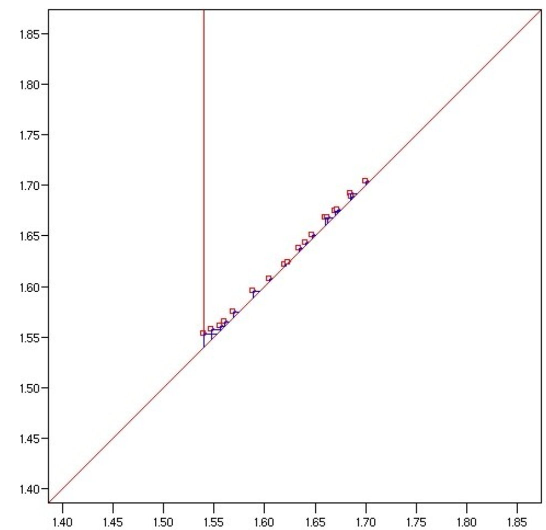
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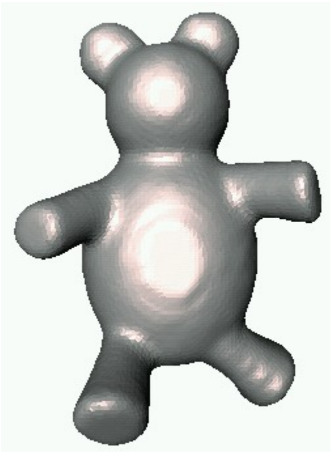

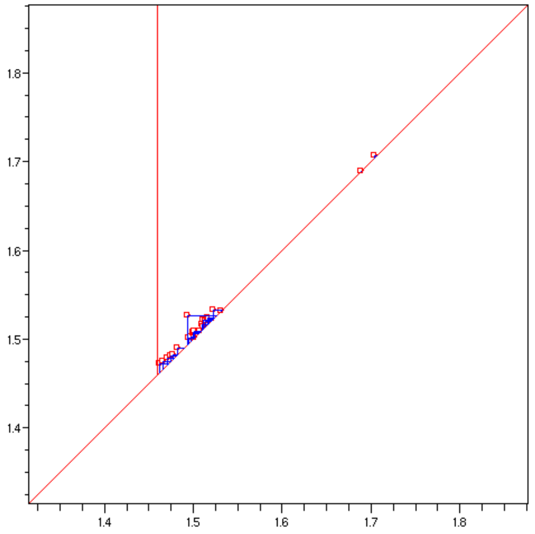

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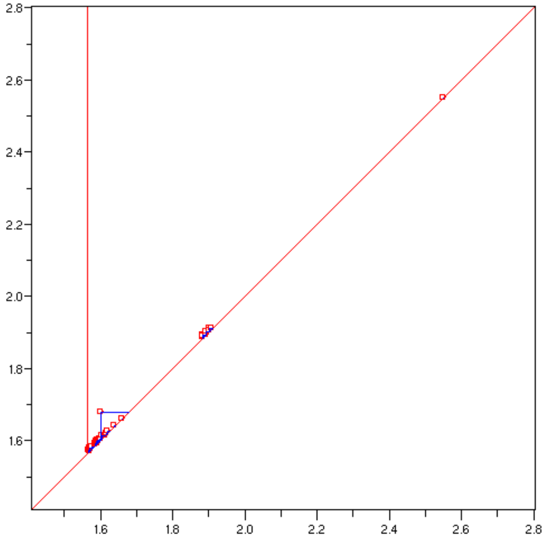

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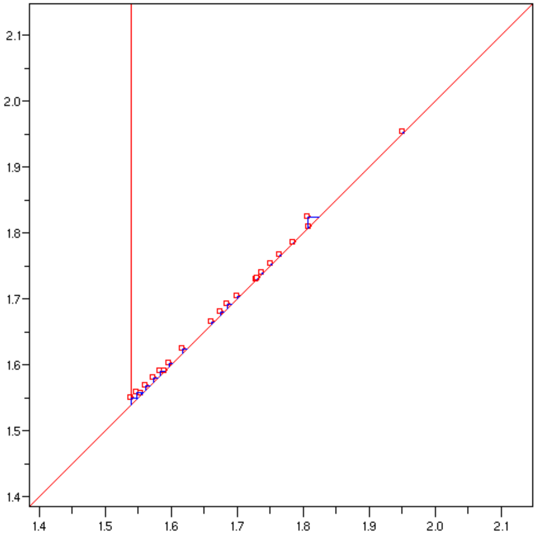

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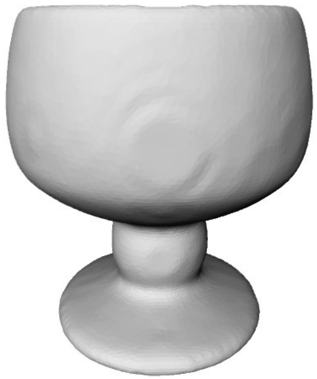
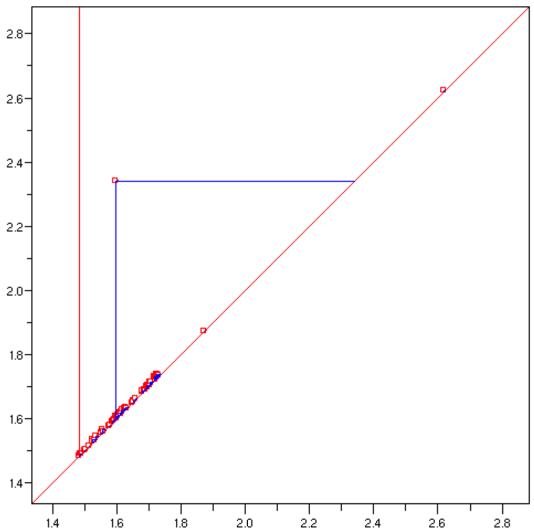
$3\pi/8$





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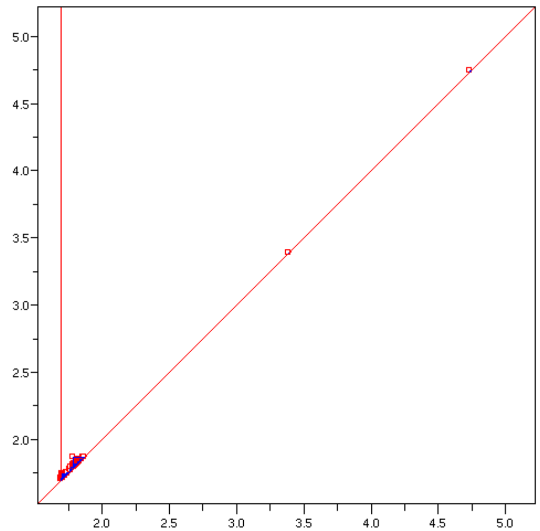

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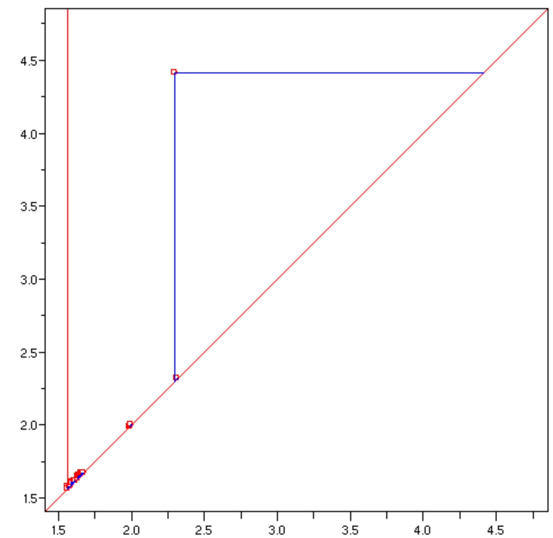
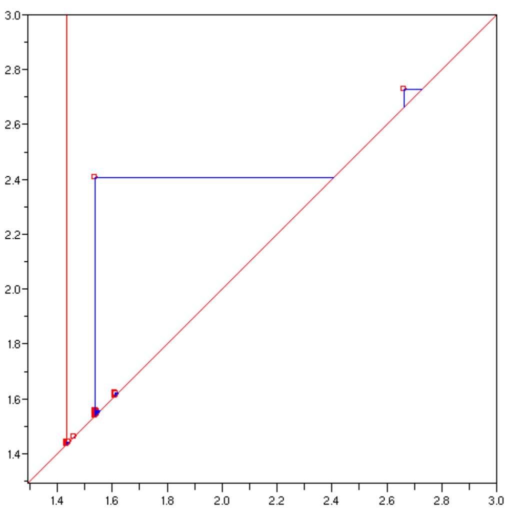


$3\pi/8$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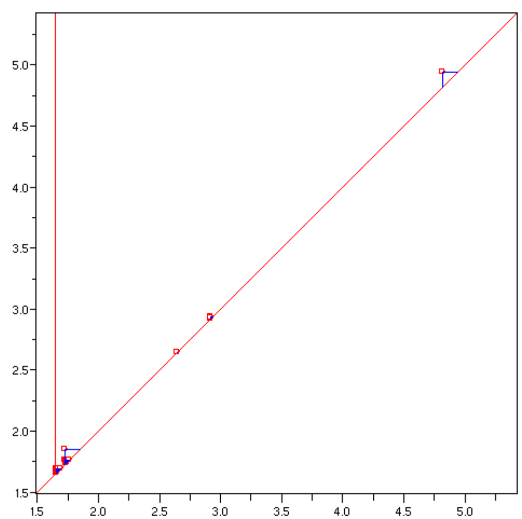
$\pi/4$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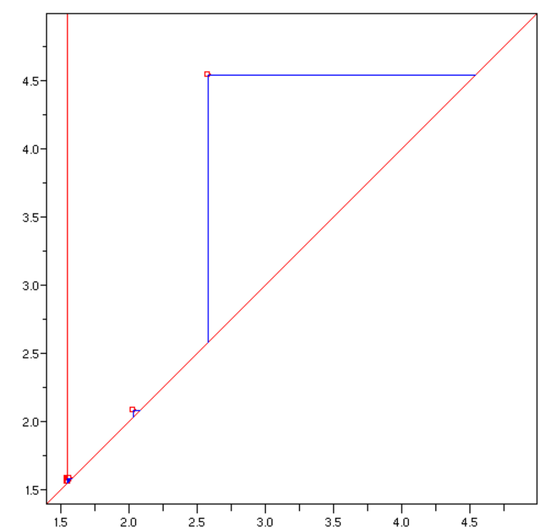
$\pi/8$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$3\pi/8$

Summary

1) Size functions are shape descriptors useful for comparing shapes of topological spaces and manifolds with respect to properties described by measuring functions.
2) The extension of the concept of size function to the case of measuring functions taking values in IR^k has been shown reducible to the case $k=1$.
3) This fact allows us to use the results known for the 1-dimensional case, and solves the problem of stability in the general case $k>1$.
4) As a consequence, new applications of size functions are available for pattern recognition.