Image comparison via group invariant non-expansive operators

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Assumptions in our model

- The observer cannot usually choose the functions representing the images he/she is interested in, but can often choose the operators that will be applied to those functions.
- The choice of the operators reflects the invariances that are relevant for the observer.
- In some sense we could state that the observer can be represented as a collection of (suitable) operators, endowed with the invariance he/she has chosen.

In this talk we will consider the case of operators that act on a space Φ of continuous functions representing images, and take Φ to itself.

Observers are usually interested in invariances

The observer usually takes some invariance into account. We suggest that this invariance could be represented by a group of homeomorphisms. The reason is that, if the images are described by functions from a topological space X to $\mathbb{R}^k$, a natural way of stating the equivalence between two functions $\varphi_1, \varphi_2 : X \to \mathbb{R}^k$ consists in saying that $\varphi_1 \equiv \varphi_2 \circ g$ for a suitable homeomorphism g chosen in a given group G of self-homeomorphisms of X. The composition of φ_2 with g to obtain φ_1 can be seen as a kind of alignment of data, as happens in image registration. The choice of the group G corresponds to the selection of the alignments of data that are judged admissible by the observer.

These remarks justify the introduction of the G-invariant pseudo-metric that will be defined in the next slide.

Natural pseudo-distance associated with a group G

In our model data are compared by the following pseudo-metric. (pseudo-metric=metric without the property $d(x,y)=0 \implies x=y$).

Definition

Let X be a compact space. Let G be a subgroup of the group $\mathrm{Homeo}(X)$ of all homeomorphisms $f : X \to X$. The pseudo-distance $d_G : C^0(X,\mathbb{R}^k) \times C^0(X,\mathbb{R}^k) \to \mathbb{R}$ defined by setting

$$d_G(\varphi,\psi) = \inf_{g\in G} \max_{x\in X} \|\varphi(x) - \psi(g(x))\|_\infty$$

is called the natural pseudo-distance associated with the group G.

In plain words, the definition of d_G is based on the attempt of finding the best correspondence between the functions φ,ψ by means of homeomorphisms belonging to the chosen group G.

Difficulty in computing d_G

$\boxed{\text{The natural pseudo-distance } d_G \text{ represents our ground truth.}}$

Unfortunately, d_G is usually difficult to compute.

Nevertheless, in this talk we will show that d_G can be approximated with arbitrary precision by means of a **DUAL** approach based on persistent homology and G-invariant non-expansive operators.

References:

- P. Frosini, G. Jabłoński, *Combining persistent homology and invariance groups for shape comparison*, Discrete & Computational Geometry, vol. 55 (2016), n. 2, pages 373–409.
- Patrizio Frosini, *Towards an observer-oriented theory of shape comparison*, Proceedings of the 8th Eurographics Workshop on 3D Object Retrieval, 2016, pages 5–8.

G-invariant non-expansive operators (GINOs)

Let us consider the following objects:

- A triangulable space X with nontrivial homology in degree m.
- A set Φ of continuous functions from X to $\mathbb{R}^k$, that contains the set of all constant functions (Φ is the set of images).
- A topological subgroup G of $\mathrm{Homeo}(X)$ that acts on Φ by composition on the right (G represents the invariances according to the observer).
- A subset $\mathscr{F}$ of the set $\mathscr{F}^{\mathrm{all}}(\Phi, G)$ of all G-invariant non-expansive operators from Φ to Φ (GINOs) ($\mathscr{F}$ represents the observer).

The operator space $\mathscr{F}^{\mathrm{all}}(\Phi, G)$ of all GINOs

In plain words, $F \in \mathscr{F}^{\mathrm{all}}(\Phi, G)$ means that

1. $F : \Phi \longrightarrow \Phi$
2. $F(\varphi \circ g) = F(\varphi) \circ g$. ($F$ is a G-operator)
3. $\|F(\varphi_1) - F(\varphi_2)\|_\infty \leq \|\varphi_1 - \varphi_2\|_\infty$. ($F$ is non-expansive)

The operator F is not required to be linear.

In the example where Φ is the space of all normalized grayscale images and G is the group of rigid motions of the plane, a simple example of operator $F \in \mathscr{F}^{\mathrm{all}}(\Phi, G)$ is given by the Gaussian blurring filter, i.e. the operator F taking each $\varphi \in \Phi$ to the function

$$\psi(x) = \frac{1}{2\pi\sigma^2} \int_{\mathbb{R}^2} \varphi(y) e^{-\frac{\|x-y\|^2}{2\sigma^2}} \, dy.$$

Persistent homology

We recall that persistent homology is a theory describing the m-dimensional holes (components, tunnels, voids, ...) of the sublevel sets of a topological space X endowed with a continuous function $\varphi : X \to \mathbb{R}^k$. In the case $k = 1$, persistent homology is described by suitable collections of points called persistence diagrams or, equivalently, by particular functions called persistent Betti number functions. Two such diagrams (or functions) can be compared by a suitable metric d_{match}, called bottleneck (or matching) distance.

The research concerning k-dimensional persistent homology is still at an early stage of development for $k > 1$. Because of this fact, in the rest of this talk we will confine ourselves to consider the case $k = 1$, for which well-established results and algorithms are available.

The pseudo-metric $D^{\mathscr{F}}_{\mathrm{match}}$

For every $\mathscr{F} \subseteq \mathscr{F}^{\mathrm{all}}(\Phi, G)$ and every $\varphi_1, \varphi_2 \in \Phi$ we set

$$D^{\mathscr{F}}_{\mathrm{match}}(\varphi_1, \varphi_2) := \sup_{F \in \mathscr{F}} d_{match}(\rho_m(F(\varphi_1)), \rho_m(F(\varphi_2)))$$

where $\rho_m(\psi)$ denotes the persistent Betti number function of ψ in degree m, while d_{match} denotes the usual bottleneck distance that is used to compare the persistence diagrams associated with $\rho_m(F(\varphi_1))$ and $\rho_m(F(\varphi_2))$.

Proposition

$D^{\mathscr{F}}_{match}$ is a G-invariant and stable pseudo-metric on Φ.

The G-invariance of $D^{\mathscr{F}}_{match}$ means that for every $\varphi_1, \varphi_2 \in \Phi$ and every $g \in G$ the equality $D^{\mathscr{F}}_{\mathrm{match}}(\varphi_1, \varphi_2 \circ g) = D^{\mathscr{F}}_{\mathrm{match}}(\varphi_1, \varphi_2)$ holds.

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An equivalence result

We observe that the pseudo-distance $D^{\mathscr{F}}_{\text{match}}$ and the natural pseudo-distance d_G are defined in quite different ways.

In particular, the definition of $D^{\mathscr{F}}_{\text{match}}$ is based on persistent homology, while the natural pseudo-distance d_G is based on the group of homeomorphisms G.

In spite of this, the following statement holds:

Theorem

If $\mathscr{F} = \mathscr{F}^{\text{all}}(\Phi, G)$, then the pseudo-distance $D^{\mathscr{F}}_{\text{match}}$ coincides with the natural pseudo-distance d_G on Φ.

Our main idea

The previous theorem suggests to study $D^{\mathscr{F}}_{\mathrm{match}}$ instead of d_G.

To this end, let us choose a finite subset $\mathscr{F}^*$ of $\mathscr{F}$, and consider the pseudo-metric $D^{\mathscr{F}^*}_{\mathrm{match}}$.

Obviously, $D^{\mathscr{F}^*}_{\mathrm{match}} \leq D^{\mathscr{F}}_{\mathrm{match}}$.

We observe that if $\mathscr{F}^*$ is dense enough in $\mathscr{F}$, then the new pseudo-distance $D^{\mathscr{F}^*}_{\mathrm{match}}$ is close to $D^{\mathscr{F}}_{\mathrm{match}}$.

In order to make this point clear, we need the next theoretical result.

Compactness of $\mathscr{F}^{\mathrm{all}}(\Phi, G)$

The following result holds:

Theorem

If Φ is a compact metric space with respect to the sup-norm, then $\mathscr{F}^{\mathrm{all}}(\Phi, G)$ is a compact metric space with respect to the distance d defined by setting

$$d(F_1, F_2) := \max_{\varphi \in \Phi} \|F_1(\varphi) - F_2(\varphi)\|_\infty$$

for every $F_1, F_2 \in \mathscr{F}$.

Approximation of $\mathscr{F}$

This statement follows:

Corollary

Assume that the metric space Φ is compact with respect to the sup-norm. Let $\mathscr{F}$ be a subset of $\mathscr{F}^{\mathrm{all}}(\Phi, G)$. For every $\varepsilon > 0$, a finite subset $\mathscr{F}^*$ of $\mathscr{F}$ exists, such that

$$\left| D_{match}^{\mathscr{F}^*}(\varphi_1, \varphi_2) - D_{match}^{\mathscr{F}}(\varphi_1, \varphi_2) \right| \leq \varepsilon$$

for every $\varphi_1, \varphi_2 \in \Phi$.

This corollary implies that the pseudo-distance $D_{match}^{\mathscr{F}}$ can be approximated computationally, at least when Φ is compact.

Our idea in a nutshell

- The natural pseudo-metric d_G can be approximated with arbitrary precision by the pseudo-metric $D_{\mathrm{match}}^{\mathscr{F}^*}$.
- While d_G is usually difficult to compute, $D_{\mathrm{match}}^{\mathscr{F}^*}$ can be efficiently computed by algorithms developed for persistent homology.
- The set $\mathscr{F}$ of G-invariant non-expansive operators (GINOs) represents the observer. The subset $\mathscr{F}^* \subseteq \mathscr{F}$ represents an approximation of the observer.

In plain words, the metric model we have illustrated presents image comparison as a problem centered on the approximation of a given observer.

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GIPHOD (Group Invariant Persistent Homology On-line Demonstrator)

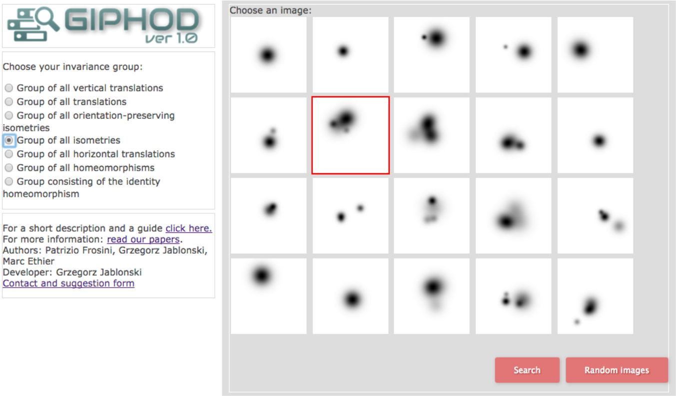
GIPHOD is an on-line demonstrator, allowing the user to choose an image and an invariance group. GIPHOD searches for the most similar images in the dataset, with respect to the chosen invariance group.
Purpose: to show the use of our theoretical approach for image comparison.
Dataset: 10.000 quite simple grey-level synthetic images obtained by adding randomly chosen bell-shaped functions. The images are coded as functions from $\mathbb{R}^2 \to [0,1]$.

GIPHOD can be tested at `http://giphod.ii.uj.edu.pl`.

Thanks to everyone that will give suggestions for improvement (please send them to `grzegorz.jablonski@uj.edu.pl`*)*

GIPHOD

Joint project with Grzegorz Jabłoński (Jagiellonian University and IST Austria) and Marc Ethier (Université de Saint-Boniface - Canada)



GIPHOD: Examples for the group of isometries

We will now show some results obtained by GIPHOD when the invariance group G is the group of isometries:

Some data about the pseudo-metric $D^{\mathscr{F}}_{match}$ in this case:

- Mean distance between images: 0.2984;
- Standard deviation of distance between images: 0.1377;
- Number of GINOs that have been used: 5.

GIPHOD: Examples for the group of isometries

Here are the five GINOs that we used for the invariance group of isometries:

- $F(\varphi) = \varphi$.
- $F(\varphi) :=$ constant function taking each point to the value $\int_{\mathbb{R}^2} \varphi(\mathbf{x})\ d\mathbf{x}$.
- $F(\varphi)$ defined by setting

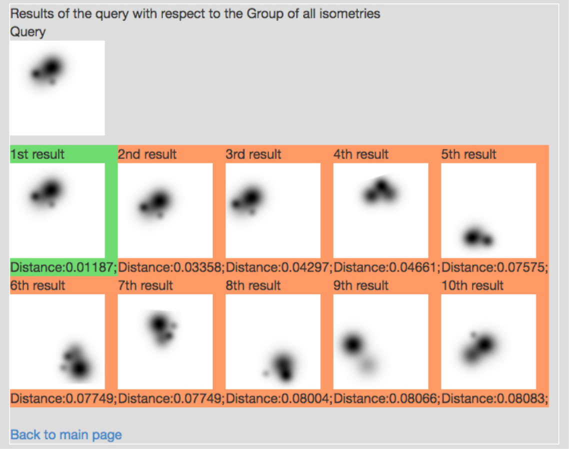
$$F(\varphi)(\mathbf{x}) := \int_{\mathbb{R}^2} \varphi(\mathbf{x}-\mathbf{y}) \cdot \beta\left(\|\mathbf{y}\|_2\right)\ d\mathbf{y}$$

where $\beta : \mathbb{R} \to \mathbb{R}$ is an integrable function with $\int_{\mathbb{R}^2} |\beta\left(\|\mathbf{y}\|_2\right)|\ d\mathbf{y} \leq 1$
(we have used three operators of this kind).

GIPHOD: Examples for the group of isometries

GIPHOD ver 1.0

For a short description and a guide click here.
For more information: read our papers.
Authors: Patrizio Frosini, Grzegorz Jablonski, Marc Ethier
Developer: Grzegorz Jablonski
Contact and suggestion form



Mean distance: 0.2984; Standard deviation of distance: 0.1377; Number of GINOs: 5.

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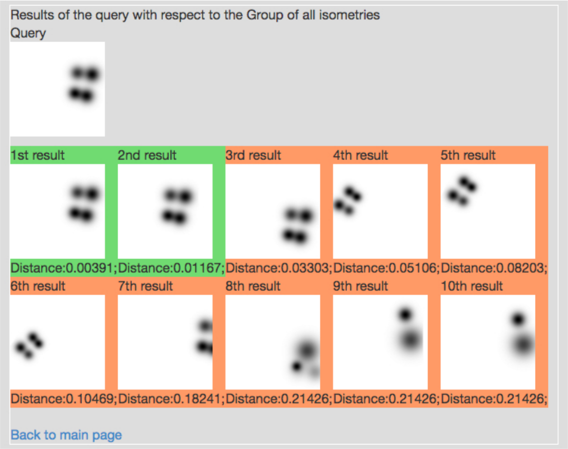

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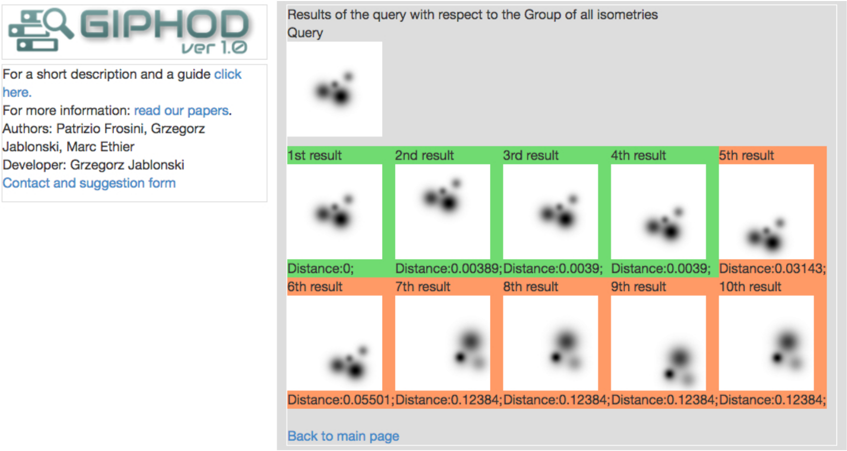

Mean distance: 0.2984; Standard deviation of distance: 0.1377; Number of GINOs: 5.

Conclusions

- Our model describes a way to compare images represented by functions from a topological space X to $\mathbb{R}^k$, via the natural pseudo-distance d_G. This pseudo-metric is based on the attempt of finding the best correspondence between the images by means of homeomorphisms belonging to the chosen group G.
- The pseudo-distance d_G is usually difficult to compute, but we have shown that it can be approximated with arbitrary precision by a new pseudo-metric $D^{\mathscr{F}^*}_{\mathrm{match}}$, based on persistent homology.
- The set $\mathscr{F}^*$ of G-invariant non-expansive operators (GINOs) used in defining $D^{\mathscr{F}^*}_{\mathrm{match}}$ represents the observer and the invariances he/she is interested in.
- Finally, we have presented the demonstrator GIPHOD, showing how our model can be applied in practice.


THANKS
FOR YOUR
ATTENTION!