Aut-invariant quasimorphisms on free products

Bastien Karlhofer

University of Aberdeen

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Definition (Aut-invariant quasimorphism)

- Let G be a group. A map $\psi \colon G \to \mathbb{R}$ is called a *quasimorphism* if there exists a constant $D \geq 0$ such that

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- If $\psi(\varphi(g)) = \psi(g)$ for all $g \in G$, $\varphi \in Aut(G)$, then ψ is called *Aut-invariant*.

Definition

Let $\psi\colon G \to \mathbb{R}$ be a quasimorphism. Then the *homogenisation* $\bar{\psi}\colon G \to \mathbb{R}$ of ψ is defined by $\bar{\psi}(g) = \lim_{n\in\mathbb{N}} \frac{\psi(g^n)}{n}$ for all $g \in G$.

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Remark

Any homogeneous quasimorphism is invariant under inner automorphisms.

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- $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$ admits no unbounded quasimorphism

Brooks' counting quasimorphisms

$G = A * B$. $z \in G$ reduced word. Define for $w \in G$ reduced

$$\theta_z(w) = \text{maximal number of } \textit{disjoint} \text{ occurences of } z \text{ in } w$$
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Let $G = G_1 * \cdots * \mathrm{G}_k$ where G_i freely indecomposable $\forall i$. $\mathrm{Aut}(G)$ *is generated by the following types of automorphisms:*

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4. *transvections: Assume $G_i \cong \mathbb{Z} = \langle s \rangle$ and $a \in G$. Send $s \to as$ or $s \to sa$ and all other letters to themselves.*

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- $\mathrm{Out}(G_1 * G_2)$ is generated by the images of $\mathrm{Aut}(G_1)$ and $\mathrm{Aut}(G_2)$ in $\mathrm{Out}(G_1 * G_2)$,
- If $\psi\colon G_1 * G_2 \to \mathbb{R}$ is a quasimorphism invariant under $\mathrm{Aut}(G_1)$ and $\mathrm{Aut}(G_2)$, then $\bar{\psi}$ is invariant under $\mathrm{Aut}(G_1 * G_2)$.

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2. if $A \cong B \ncong \mathbb{Z}/2$, then the sum $\bar{f}_z^A + \bar{f}_z^B$ is an unbounded Aut-invariant quasimorphism on G.

Theorem (K.)

Let $G = A * B$ be the free product of two non-trivial freely indecomposable groups A and B. Assume G is not the infinite dihedral group. Then G admits infinitely many linearly independent homogeneous Aut-invariant quasimorphisms, all of which vanish on single letters.

Examples

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- $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ for $p, q \geq 3$. Knot groups of torus knots if $gcd(p,q) = 1$.

Application: RAAGs and RACGs

Let $\Gamma = (V, E)$ be a graph. Define $W_\Gamma = (*_{v \in V} G_v)/N$, where N is the normal subgroup generated by all $[G_v, G_w]$ where $(v, w) \in E$ and $G_v \in \{\mathbb{Z}/2, \mathbb{Z}\}$.

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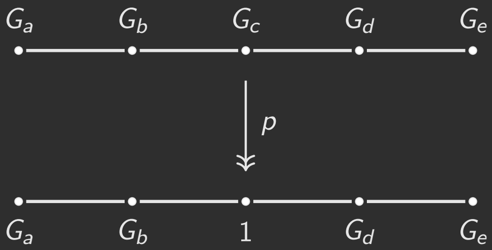

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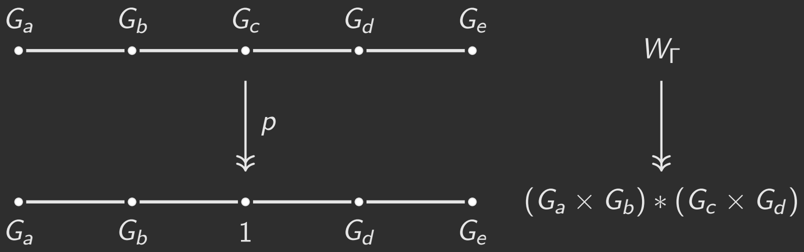

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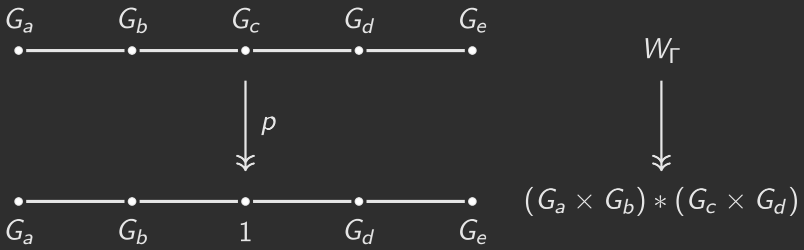

$\rightsquigarrow$ unbounded Aut-invariant quasimorphisms on W_Γ.

Outlook

- What about free products with more than one factor?
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Let $G \trianglelefteq \hat{G}$. Consider $[\hat{G}, G] \leq G$ generated by $[F, g]$ and their inverses where $F \in \hat{G}$ and $g \in G$. For $x \in [\hat{G}, G]$:

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Theorem (K.)

Let $G = A * B$ be a free product of freely indecomposable groups and assume that G is not the infinite dihedral group. Then there always exist elements $g \in G$ with positive Aut-invariant stable commutator length $\mathrm{scl}_{\mathrm{Aut}}(g) > 0$.

Thank you for your attention!