Gromov's Mapping Theorem via multicomplexes

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Higher Invariants
Interactions between Arithmetic Geometry and Global Analysis
SFB 1085 · Funded by the DFG

International Young Seminars on Bounded Cohomology and Simplicial Volume

Motivation - Historical background

- Gromov '82: **Bounded cohomology:** Topological spaces; **Techniques:** Multicomplexes; **Simplicial volume:** Both compact and non-compact manifolds.

Motivation - Historical background

- Ivanov '87: **Bounded cohomology:** Countable CW-complexes; **Techniques:** Homological algebra and resolutions; **Simplicial volume:** Closed manifold.

Motivation - Historical background

- Ivanov '17: **Bounded cohomology:** Topological spaces; **Techniques:** Invariance of bounded cohomology under weak homotopy equivalences; **Simplicial volume:** Closed manifolds.

Motivation - Historical background

- FM '18: **Bounded cohomology:** Topological spaces; **Techniques:** Multicomplexes; **Simplicial volume:** Both compact and non-compact manifolds.

Bounded cohomology of spaces

Let X be a path-connected space and let $f \in C^\bullet(X; \mathbb{R})$.

- The **ℓ^∞-norm** of f is defined as

$$\|f\|_\infty = \sup\{|f(\sigma)| \text{, where } \sigma \in \mathcal{S}_\bullet(X)\} \ .$$

- f is said to be **bounded** if $\|f\|_\infty < +\infty$.

- $(C_b^\bullet(X; \mathbb{R}) = \{f \in C^\bullet(X; \mathbb{R}) \mid \|f\|_\infty < +\infty\}, \delta^\bullet)$.

The **bounded cohomology** of X, $H_b^\bullet(X; \mathbb{R})$, is the cohomology of $(C_b^\bullet(X; \mathbb{R}), \delta^\bullet)$.

Bounded cohomology of groups

Given a discrete group Γ, a $K(\Gamma, 1)$-**space** is a path-connected CW-complex such that

- $\pi_1(X) = \Gamma$;
- $\pi_{k \geq 2}(X) = 1$.

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Johnson '72 : If Γ is amenable, then

$$H_b^{k \geq 1}(\Gamma; \mathbb{R}) = 0 \ .$$

Gromov's Mapping Theorem

Mapping Theorem (Gromov '82, Ivanov '17, FM '18): Let $f\colon X \to Y$ be a continuous map such that

- $\pi_1(f)\colon \pi_1(X) \to \pi_1(Y)$ is surjective;
- $\ker(\pi_1(f))$ is amenable.

Then, $\forall\, n \in \mathbb{N}$
$$H_b^n(f)\colon H_b^n(Y) \to H_b^n(X)$$
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Corollary : All spaces X with amenable fundamental group have
$$H_b^{k\geq 1}(\Gamma; \mathbb{R}) = 0 \ .$$

Gromov's Mapping Theorem - Simplified version

Mapping Theorem (Simplified version): Let $f\colon X \to Y$ be a continuous map between CW-complexes such that $\pi_1(f)$ is an isomorphism. Then, $\forall\, n \in \mathbb{N}$

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Strategy : Given a CW-complex X, we want to construct a classifying map $X \rightsquigarrow K(\pi_1(X), 1)$ inducing an isometric isomorphism in bounded cohomology.

Multicomplexes

Gromov 1982 - *"A* ***multicomplex*** K *is a set divided into the union of closed affine simplices* Δ_i, $i \in I$, *such that the intersection of any two simplices* $\Delta_i \cap \Delta_j$ *is a (simplicial) subcomplex in* Δ_i *as well as* Δ_j*"*.

Multicomplexes

A **multicomplex** K is a set of simplices satisfying the following properties:

- Hereditary: If $\Delta \subset K$, then all the faces of Δ are in K;
- Intersection: If $\Delta_1 \cap \Delta_2 \neq \emptyset$, then $\Delta_1 \cap \Delta_2$ is a subcomplex of both of them;
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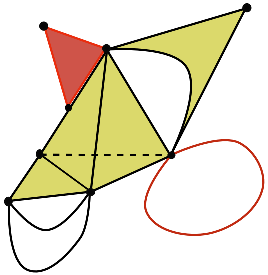

Geometric realization

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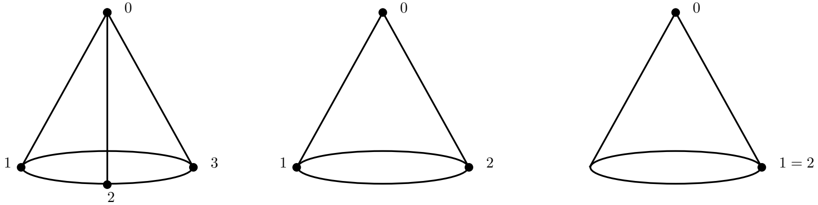

Figure: A cone as the geometric realization of a simplicial complex, a multicomplex and a Δ-complex, respectively.

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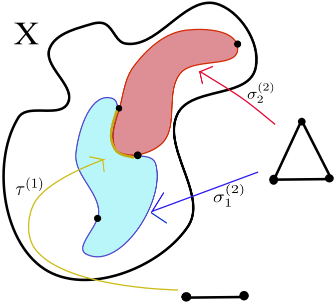

Example : $\mathcal{K}(X)$ contains (for instance):

- A 1-simplex corresponding to the singular 1-simplex $\tau^{(1)}$;
- Two 2-simplices $\sigma_1^{(2)}$ and $\sigma_2^{(2)}$ such that

$$\sigma_1^{(2)} \cap \sigma_2^{(2)} = \tau^{(1)} .$$

The natural projection

There exists a **natural projection** $S\colon |\mathcal{K}(X)| \to X$, defined as

$$S\Big(\left[\sigma^{(n)}\right], \lambda_1 x_1 + \cdots + \lambda_{n+1} x_{n+1} \Big) = \sigma^{(n)}(\lambda_1 e_1 + \cdots + \lambda_{n+1} e_{n+1}) \; .$$

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Theorem (Gromov '82, FM '18) : S is a homotopy equivalence.
In particular,

$$H_b^\bullet(S)\colon H_b^\bullet(X) \to H_b^\bullet(|\mathcal{K}(X)|)$$

is an isometric isomorphism in all degrees.

Complete multicomplexes

A multicomplex K is said to be **complete** if

for all continuous maps $f \colon \Delta^n \to |K|$ such that

$$f|_{\partial \Delta^n} \colon \partial \Delta^n \hookrightarrow |K| \text{ is a simplicial embedding,}$$

there exists a simplicial embedding $\iota \colon \Delta^n \hookrightarrow |K|$ s. t. $f \simeq_{\partial \Delta^n} \iota$.

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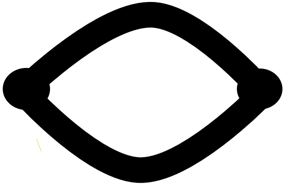

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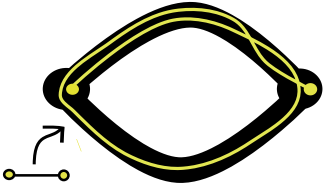

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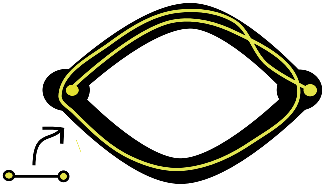

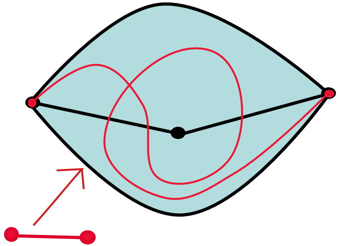

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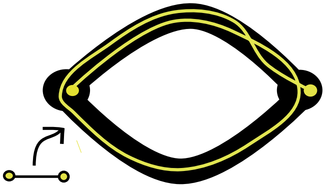

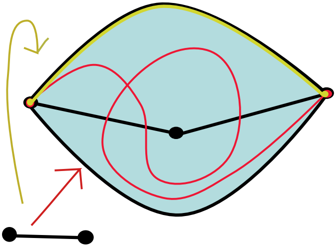

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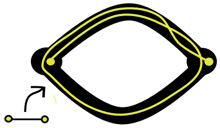

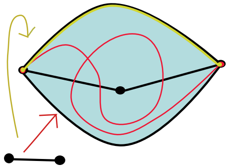

Theorem (Gromov '82, FM '18) : $\mathcal{K}(X)$ is complete.

Special spheres

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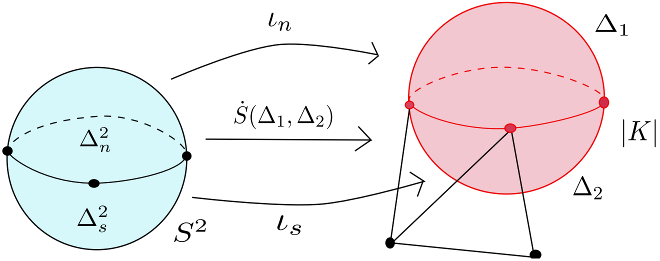
Let $\Delta_1^k, \Delta_2^k \subset K$ be two compatible simplices. A **special sphere**

$$\dot{S}(\Delta_1^k, \Delta_2^k) \colon \mathsf{S}^k \to |K|$$

is a continuous function defined as follows



Homotopy of multicomplexes

Two simplices $\Delta_1^k, \Delta_2^k \subset K$ are said to be **homotopic** if they are compatible and we have

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Let $\pi(\Delta) \subset K$ be the set of all simplices compatible with Δ.

Theorem (FM '18) : Let K be a complete multicomplex and let Δ be a k-simplex in K. Then,

$$\Theta \colon \pi(\Delta) \to \pi_k(|K|, x_0)$$

$$\Delta_1 \mapsto \dot{S}(\Delta, \Delta_1)$$

is surjective and

$$\Theta(\Delta_1) = \Theta(\Delta_2)$$

if and only if Δ_1 and Δ_2 are homotopic.

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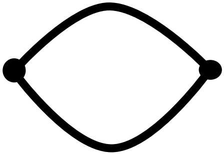

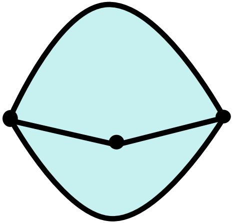

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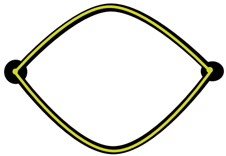

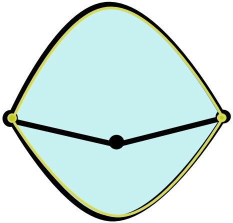

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Examples :

Corollary (FM '18) : Let K be a complete and minimal multicomplex and let $\Delta \subset K$ be a k-simplesso. Then,

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is a bijection.

Existence of minimal multicomplexes

Theorem (Gromov 82, FM '18) : Let K be a complete multicomplex. Then $\exists\, L \subset K$ such that

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Corollary : Given $X \rightsquigarrow \mathcal{L}(X) \subset \mathcal{K}(X)$ complete and minimal multicomplex such that

$$H_b^\bullet(X) \cong H_b^\bullet(|\mathcal{L}(X)|)$$

isometrically.

A useful simplicial action

Recap : $X \rightsquigarrow \mathcal{K}(X)$ complete $\rightsquigarrow \mathcal{L}(X)$ complete, minimal and homotopically equivalent to X.

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For every $i \geq 1$ we define Γ_i to be the group of simplicial automorphisms γ of $\mathcal{L}(X)$ such that

- $\gamma|_{|\mathcal{L}(X)|^i} = \mathrm{Id}_{\mathcal{L}(X)}$;
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Corollary : The quotient $\mathcal{A}(X) = \mathcal{L}(X)/\Gamma_1$ is a multicomplex without compatible simplices in dimension $i \geq 2$.

Proof of the (Simplified) Mapping Theorem

(Simplified) Mapping Theorem: Let $f \colon X \to Y$ be a continuous map between CW-complexes such that $\pi_1(f)$ is an isomorphism. Then $\forall n \in \mathbb{N}$, $H_b^n(f)$ is an isometric isomorphism.

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- $H_b^\bullet(X) \cong H_b^\bullet(|\mathcal{A}(X)|)$ is an isometric isomorphism.


"That's all Folks!"

How to show the amenability of Γ_1/Γ_i with $i \geq 1$

Remark : Given Γ_1/Γ_i, we have the following normal sequence:

$$\Gamma_1/\Gamma_i \trianglerighteq \Gamma_2/\Gamma_i \trianglerighteq \cdots \trianglerighteq \Gamma_i/\Gamma_i = \{1\} \ .$$

If we prove that for every $1 \leq k \leq i-1$ the group

$$\Gamma_k/\Gamma_{k+1}$$

is Abelian, then Γ_1/Γ_i is solvable.

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Strategy : We prove that Γ_{k+1}/Γ_k embeds into an Abelian group.

How to show that Γ_k/Γ_{k+1} is Abelian for all $k \geq 1$ (part 1)

Lemma 1 : Let J be the orbit set of $\Gamma_k \curvearrowright \mathcal{L}(X)$ and let $\Delta_\alpha \subset \mathcal{L}(X)$ be a representative for the orbit $\alpha \in J$. Then, the map

$$\begin{aligned} \varphi_\alpha \colon \Gamma_k &\to \pi_{k+1}(|\mathcal{L}(X)|, x_\alpha) \\ \gamma &\mapsto [\varphi_\alpha(\gamma)] = \left[\dot{S}(\Delta_\alpha, \gamma\Delta_\alpha)\right] \end{aligned}$$

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is a homomorphism.

Proof : We have the following computation for every $\gamma_1, \gamma_2 \in \Gamma_k$:

$$\begin{aligned} \varphi(\gamma_2\gamma_1) &= \left[\dot{S}(\Delta_\alpha, \gamma_2\gamma_1\Delta_\alpha)\right] \\ &= \left[\dot{S}(\Delta_\alpha, \gamma_2\Delta_\alpha) + \dot{S}(\gamma_2\Delta_\alpha, \gamma_2\gamma_1\Delta_\alpha)\right] \\ &= \varphi(\gamma_2) + \left[\gamma_2 \circ \dot{S}(\Delta_\alpha, \gamma_1\Delta_\alpha)\right] \\ &= \varphi(\gamma_2) + \varphi(\gamma_1)\ . \end{aligned}$$

How to show that Γ_k/Γ_{k+1} is Abelian for all $k \geq 1$ (part 2)

Lemma 2 : For every $k \geq 1$, the homomorphism into the direct product

$$\Phi\colon \Gamma_k \to \prod_{\alpha \in J} \pi_{k+1}(|\mathcal{L}(X)|, x_\alpha)$$

has kernel equal to Γ_{k+1}. In particular, Γ_k/Γ_{k+1} is Abelian.

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has kernel equal to Γ_{k+1}. In particular, Γ_k/Γ_{k+1} is Abelian.

Proof : Notice that:

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- Since $\ker(\Phi)$ is normal, this implies that it coincides with the stabilizers of each simplex in the orbit α and so:

$$\ker(\Phi) = \bigcap_{\alpha \in J} \ker(\varphi_\alpha) = \Gamma_{k+1} \ .$$