Vanishing results of topological volumes via open covers

Marco Moraschini (Universität Regensburg)

Higher Invariants
Interactions between Arithmetic Geometry and Global Analysis
SFB 1085 · Funded by the DFG

Real and Complex Manifolds –
The mathematical heritage of Edoardo Vesentini

Topological volumes

Gauss-Bonnet's Theorem : If Σ_g is a genus g surface, we have

$$\int_{\Sigma_g} K dA = 2\pi\chi(\Sigma_g) \ .$$

If Σ_g is **hyperbolic**, i.e. $K = -1$, we have $\mathrm{Vol}(\Sigma_g) = 2\pi|\chi(\Sigma)| \ .$

Topological volumes

Gauss-Bonnet's Theorem : If Σ_g is a genus g surface, we have

$$\int_{\Sigma_g} K dA = 2\pi\chi(\Sigma_g) \ .$$

If Σ_g is **hyperbolic**, i.e. $K = -1$, we have $\mathrm{Vol}(\Sigma_g) = 2\pi|\chi(\Sigma)|$.

Warning : We cannot extend it to **all** higher dimensions, because $\chi(M^{2k+1}) = 0$.

Topological volumes

Gauss-Bonnet's Theorem : If Σ_g is a genus g surface, we have

$$\int_{\Sigma_g} K dA = 2\pi\chi(\Sigma_g) \ .$$

If Σ_g is **hyperbolic**, i.e. $K = -1$, we have $\mathrm{Vol}(\Sigma_g) = 2\pi|\chi(\Sigma)|$.

Warning : We cannot extend it to **all** higher dimensions, because $\chi(M^{2k+1}) = 0$.

Theorem (Gromov, Thurston $\sim$'80) : If M is an oriented closed and connected hyperbolic n-manifold, then

$$\|M\| = \frac{\mathrm{Vol}(M)}{v_n} \ .$$

Simplicial volume

Assumption : Our manifold M is assumed to be o.c.c., i.e. oriented, closed and connected.

Simplicial volume

Assumption : Our manifold M is assumed to be o.c.c., i.e. oriented, closed and connected.

Given a singular chain $c = \sum_{j=0}^{k} a_j \cdot \sigma_j \in C_n(M; \mathbb{R})$, we define the **$\ell^1$-norm** of c as

$$\|c\|_1 := \sum_{j=0}^{k} |a_j| \in \mathbb{R}_{\geq 0}.$$

Simplicial volume

Assumption : Our manifold M is assumed to be o.c.c., i.e. oriented, closed and connected.

Given a singular chain $c = \sum_{j=0}^{k} a_j \cdot \sigma_j \in C_n(M;\mathbb{R})$, we define the **$\ell^1$-norm** of c as

$$\|c\|_1 := \sum_{j=0}^{k} |a_j| \in \mathbb{R}_{\geq 0}.$$

The **simplicial volume** of an n-manifold M is

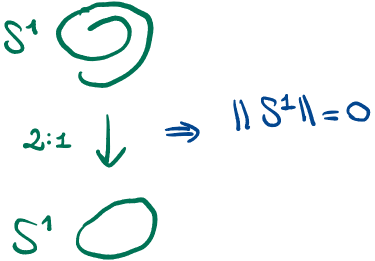
$$\|M\| := \inf\{\|c\|_1 \mid [c] = [M] \in H_n(M;\mathbb{R}) \cong \mathbb{R}\} \in \mathbb{R}_{\geq 0} .$$

Remark : Simplicial volume is a homotopy invariant.

Simplicial volume: Main properties

Multiplicativity w.r.t. finite coverings : If $f\colon N \to M$ is a finite covering of degree d between o.c.c. manifolds of the same dimension, then

$$\|N\| = d \cdot \|M\| \ .$$



Simplicial volume: Main properties

Multiplicativity w.r.t. finite coverings : If $f\colon N \to M$ is a finite covering of degree d between o.c.c. manifolds of the same dimension, then

$$\|N\| = d \cdot \|M\| \ .$$

Vanishing results :

- $\|S^1\| = 0$ and $\|T^n\| = 0$;
- **Gromov '82 :** If $\pi_1(M)$ is amenable (e.g. finite, Abelian, solvable groups, ...), then $\|M\| = 0$;
- All flat manifolds have zero simplicial volume.

Simplicial volume: Main properties

Multiplicativity w.r.t. finite coverings : If $f\colon N \to M$ is a finite covering of degree d between o.c.c. manifolds of the same dimension, then

$$\|N\| = d \cdot \|M\| \ .$$

Vanishing results :

- $\|S^1\| = 0$ and $\|T^n\| = 0$;
- **Gromov '82 :** If $\pi_1(M)$ is amenable (e.g. finite, Abelian, solvable groups, ...), then $\|M\| = 0$;
- All flat manifolds have zero simplicial volume.

Inoue-Yano '82 : Manifolds with strictly negative sectional curvature have positive simplicial volume.

Simplicial volume vs. Euler characteristic

Gromov's question $\sim$'90 : If M is an o.c.c. *aspherical* n-manifold, then do we have

$$\|M\| = 0 \quad \Longrightarrow \quad \chi(M) = 0 \text{ ?}$$

ASPHERICITY IS NECESSARY! e.g. $\|S^2\| = 0$, but $\chi(S^2) = 2$.

Simplicial volume vs. Euler characteristic

Gromov's question $\sim$'90 : If M is an o.c.c. *aspherical* n-manifold, then do we have

$$\|M\| = 0 \quad \Longrightarrow \quad \chi(M) = 0 \text{ ?}$$

Some known examples :

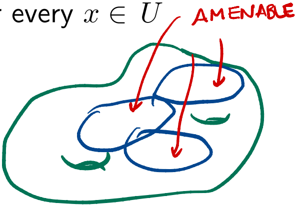
- S^1 and T^n;
- **Gromov $\sim$'90 :** If $\pi_1(M)$ is amenable, then $\|M\| = \chi(M) = 0$; ← M is aspherical!
- **Using multiplicativity w.r.t. finite coverings :** Every flat manifold M has both $\|M\| = \chi(M) = 0$.

Gromov's question via open covers

An open subset $U \subset M$ is called **amenable** if for every $x \in U$

$$\mathrm{im}\big(\pi_1(U,x) \hookrightarrow \pi_1(M,x)\big)$$

is amenable.



The **amenable category** of M is

$$\mathrm{cat}_{\mathrm{Am}}(M) := \min\left\{ n \in \mathbb{N} \,\middle|\, M = \bigcup_{i=1}^{n} U_i \text{ s. t. each } U_i \text{ is amenable} \right\}.$$

Vanishing theorems

Gromov '82, Ivanov '87, Frigerio-M. '19 : If M is an o.c.c. manifold with $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$, then $\|M\| = 0$.

Vanishing theorems

Gromov '82, Ivanov '87, Frigerio-M. '19 : If M is an o.c.c. manifold with $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$, then $\|M\| = 0$.

Sauer '09 : If M is an o.c.c. *aspherical* manifold with $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$, then $\chi(M) = 0$.

Vanishing theorems

Gromov '82, Ivanov '87, Frigerio-M. '19 : If M is an o.c.c. manifold with $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$, then $\|M\| = 0$.

Sauer '09 : If M is an o.c.c. *aspherical* manifold with $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$, then $\chi(M) = 0$.

Question : If M is an o.c.c. *aspherical* n-manifold, then do we have

$$\|M\| = 0 \Rightarrow \mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M) \ ?$$

(This implies Gromov's Question:

$$\|M\| = 0 \Rightarrow \mathrm{cat}_{\mathrm{Am}}(M) \leq \dim M \overset{[\text{Sauer}]}{\Rightarrow} \chi(M) = 0$$
)

Vanishing theorems

Gromov '82, Ivanov '87, Frigerio-M. '19 : If M is an o.c.c. manifold with $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$, then $\|M\| = 0$.

Sauer '09 : If M is an o.c.c. *aspherical* manifold with $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$, then $\chi(M) = 0$.

Question : If M is an o.c.c. *aspherical* n-manifold, then do we have

$$\|M\| = 0 \Rightarrow \mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M) \text{ ?}$$

Gómez-González-Heil '14 : The question is affirmative for all o.c.c. 3-manifolds.

Understanding $\mathrm{cat}_{\mathrm{Am}}(M)$: The monotinicity problem

Gromov '82 : If $f \colon M \to N$ is a degree-one map between o.c.c. n-manifolds, then we have

$$\|M\| \geq \|N\| \ .$$

Understanding $\mathrm{cat}_{\mathrm{Am}}(M)$: The monotinicity problem

Gromov '82 : If $f\colon M \to N$ is a degree-one map between o.c.c. n-manifolds, then we have

$$\|M\| \geq \|N\| \ .$$

Question 2 (Capovilla-Löh-M.) '21 : If $f\colon M \to N$ is a degree-one map between o.c.c. n-manifolds, then do we have

$$\mathrm{cat}_{\mathrm{Am}}(M) \geq \mathrm{cat}_{\mathrm{Am}}(N) \ ?$$

Ex: If $\exists\, M \geq_1 N$ s.t.
- $\mathrm{cat}_{\mathrm{Am}}(M) = \dim(M)$
- $\mathrm{cat}_{\mathrm{Am}}(M) < \mathrm{cat}_{\mathrm{Am}}(N)$

Then
- $\|M\| = 0 \geq \|N\| \Rightarrow \|N\| = 0$
- $\mathrm{cat}_{\mathrm{Am}}(N) > \dim(N)$

Understanding $\mathrm{cat}_{\mathrm{Am}}(M)$: The monotinicity problem

Gromov '82 : If $f\colon M \to N$ is a degree-one map between o.c.c. n-manifolds, then we have

$$\|M\| \geq \|N\| .$$

Question 2 (Capovilla-Löh-M.) '21 : If $f\colon M \to N$ is a degree-one map between o.c.c. n-manifolds, then do we have

$$\mathrm{cat}_{\mathrm{Am}}(M) \geq \mathrm{cat}_{\mathrm{Am}}(N) \text{ ?}$$

Capovilla-Löh-M. '21 : Question 2 is affirmative for all o.c.c. 3-manifolds.

Sketch of the proof (prime 3-manifolds case)

A 3-manifold M is prime if

$$M = N_1 \# N_2 \Longrightarrow \text{ either } N_1 \cong S^3 \text{ or } N_2 \cong S^3 \ .$$

Sketch of the proof for prime 3-manifolds : Given a degree-1 map $f \colon M \to N$ we have the following cases:

Sketch of the proof (prime 3-manifolds case)

A 3-manifold M is prime if

$$M = N_1 \# N_2 \Longrightarrow \text{ either } N_1 \cong S^3 \text{ or } N_2 \cong S^3 \ .$$

Sketch of the proof for prime 3-manifolds : Given a degree-1 map $f \colon M \to N$ we have the following cases:

- $\mathrm{cat}_{\mathrm{Am}}(M) = 1$ ⇒ $\pi_1 M$ = amenable.
  - deg(f) = 1 ⇒ $\pi_1(f)$ is epi.
  - $\pi_1 N$ = amenable (Am is closed under quotients)
  - ⇒ $\mathrm{cat}_{\mathrm{Am}}(N) = 1$.

Sketch of the proof (prime 3-manifolds case)

A 3-manifold M is prime if

$$M = N_1 \# N_2 \Longrightarrow \text{ either } N_1 \cong S^3 \text{ or } N_2 \cong S^3 \ .$$

Sketch of the proof for prime 3-manifolds : Given a degree-1 map $f\colon M \to N$ we have the following cases:

- $\mathrm{cat}_{\mathrm{Am}}(M) = 1$
- $\mathrm{cat}_{\mathrm{Am}}(M) = 2$

[CLM]: We characterise mflds with $\mathrm{cat}_{\mathrm{Am}}(M) = 2$.

$\Rightarrow \nexists$ prime mflds with $\mathrm{cat}_{\mathrm{Am}}(M) = 2$.

Sketch of the proof (prime 3-manifolds case)

A 3-manifold M is prime if

$$M = N_1 \# N_2 \Longrightarrow \text{ either } N_1 \cong S^3 \text{ or } N_2 \cong S^3 \ .$$

Sketch of the proof for prime 3-manifolds : Given a degree-1 map $f \colon M \to N$ we have the following cases:

- $\mathrm{cat}_{\mathrm{Am}}(M) = 1$
- $\mathrm{cat}_{\mathrm{Am}}(M) = 2$
- $\mathrm{cat}_{\mathrm{Am}}(M) = 3 \Rightarrow \|M\| = 0 \Rightarrow \|N\| = 0 \Rightarrow \mathrm{cat}_{\mathrm{Am}}(N) \leq 3.$ [Gromov] (Monotonicity) [GGH]

Sketch of the proof (prime 3-manifolds case)

A 3-manifold M is prime if

$$M = N_1 \# N_2 \implies \text{either } N_1 \cong S^3 \text{ or } N_2 \cong S^3 \ .$$

Sketch of the proof for prime 3-manifolds : Given a degree-1 map $f \colon M \to N$ we have the following cases:

- $\mathrm{cat}_{\mathrm{Am}}(M) = 1$
- $\mathrm{cat}_{\mathrm{Am}}(M) = 2$
- $\mathrm{cat}_{\mathrm{Am}}(M) = 3$
- $\mathrm{cat}_{\mathrm{Am}}(M) = 4$ $\geq \mathrm{cat}_{\mathrm{Am}}(N)$.

Vanishing results of fibrations

Let M be an o.c.c. fiber bundle $p\colon M \to B$ with fiber F.

Gromov '82, Lück '02 : If $\pi_1(F)$ is amenable, then $\|M\| = 0$.

Vanishing results of fibrations

Let M be an o.c.c. fiber bundle $p\colon M \to B$ with fiber F.

Gromov '82, Lück '02 : If $\pi_1(F)$ is amenable, then $\|M\| = 0$.

Löh-M '21 : If $\mathrm{cat}_{\mathrm{Am}}(F) \leq \frac{\dim(M)}{\dim(B)+1}$, then

$$\|M\| = 0.$$

In particular, when M is aspherical, then M satisfies Gromov's question.

Vanishing results of fibrations

Let M be an o.c.c. fiber bundle $p\colon M \to B$ with fiber F.

Gromov '82, Lück '02 : If $\pi_1(F)$ is amenable, then $\|M\| = 0$.

Löh-M '21 : If $\mathrm{cat}_{\mathrm{Am}}(F) \leq \frac{\dim(M)}{\dim(B)+1}$, then

$$\|M\| = 0.$$

In particular, when M is aspherical, then M satisfies Gromov's question.

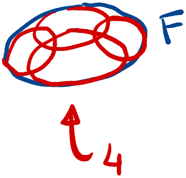
Example : If M is a hyperbolic n-manifold which fibers over S^1, then $2 \cdot \mathrm{cat}_{\mathrm{Am}}(F) > \dim(F) + 1$.

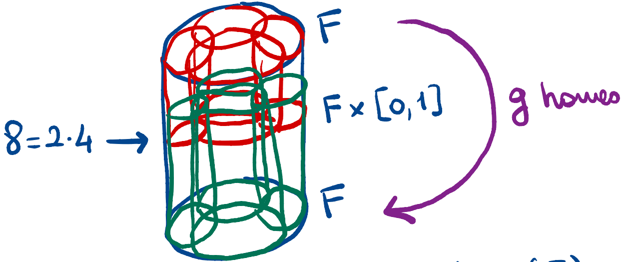
Sketch of the proof for mapping tori

If M fibers over S^1 with fiber F, then

$$M = \frac{F \times [0,1]}{(x,0) \sim (g(x),1)}$$

for every $x \in F$ and some given homeomorphism $g\colon F \to F$.





$\Rightarrow \mathrm{cat}_{Am}(M) \leq 8 = 2 \cdot \mathrm{cat}_{Am}(F).$

Sketch of the proof for mapping tori

If M fibers over S^1 with fiber F, then

$$M = \frac{F \times [0,1]}{(x,0) \sim (g(x),1)}$$

for every $x \in F$ and some given homeomorphism $g\colon F \to F$.

Then, if $2 \cdot \mathrm{cat}_{\mathrm{Am}}(F) \leq \dim(F) + 1$, we have

$$\mathrm{cat}_{\mathrm{Am}}(M) \leq 2 \cdot \mathrm{cat}_{\mathrm{Am}}(F) \leq \dim(F) + 1 = \dim(M) \ .$$

This shows that $\|M\| = 0$.

Concluding remarks: Why open cover?!

Which are the advantages of $\mathrm{cat}_{\mathrm{Am}}$?

Concluding remarks: Why open cover?!

Which are the advantages of $\mathrm{cat}_{\mathrm{Am}}$?

- It is a numerical invariant;

Concluding remarks: Why open cover?!

Which are the advantages of $\mathrm{cat}_{\mathrm{Am}}$?

- It is a numerical invariant;
- Related to the following classical invariants: Lusternik-Schnirelmann category, topological complexity, $\cdots$;

Concluding remarks: Why open cover?!

Which are the advantages of $\mathrm{cat}_{\mathrm{Am}}$?

- It is a numerical invariant;
- Related to the following classical invariants: Lusternik-Schnirelmann category, topological complexity, $\cdots$;
- Usually the results on $\mathrm{cat}_{\mathrm{Am}}$ also work for other class of groups, leading to the vanishing of other topological volumes:
- Example (Löh-M '21) : If Ab is the family of Abelian groups, then: If $\mathsf{cat}_{\mathsf{Ab}}(F) \leq \frac{\dim(M)}{\dim(B)+1}$, then

$$\mathsf{minvolent}(M) = 0.$$