Exercise sheet No. 1

**Warning**: we will discuss these exercises in class.

**Exercise 0.1** (5 Points)**.** Prove that both

$$H_n\colon \mathbf{Top} \to \mathbf{Ab}$$

and

$$H^n\colon \mathbf{Top} \to \mathbf{Ab}$$

are functor.

**Definition 0.2.** Let $X$ be a topological space and let $\mathcal{R}$ be a coefficient ring (e.g. $\mathbb{R}$ or $\mathbb{Z}$). We define the $\mathcal{R}$-bilinear *cap product* to be

$$\begin{aligned}\frown\colon C^i(X;\mathcal{R}) \times C_j(X;\mathcal{R}) &\to C_{j-i}(X;\mathcal{R})\\ (\varphi,\sigma) &\mapsto \varphi \frown \sigma = \varphi(\sigma|[v_0,\cdots,v_i]) \cdot \sigma|[v_i,\cdots,v_j]\ ,\end{aligned}$$

for $i \le j$, $\sigma\colon \Delta^n \to X$ and $\varphi \in C^\ell(X;\mathcal{R})$.

**Exercise 0.3** (5 Points)**.** Prove that it induces a well-define map in homology and cohomology:

$$\frown\colon H^i(X;\mathcal{R}) \times H_j(X;\mathcal{R}) \to H_{j-i}(X;\mathcal{R})\ .$$

**Exercise 0.4** (Hurewicz Theorem)**.** Let $X$ be a topological space. Show that by regarding loops as singular 1-cycles, we obtain a well-defined homomorphism

$$h\colon \pi_1(X,x_0) \to H_1(X;\mathbb{Z})\ .$$

Discuss the situation in which $X$ is path-connected. Show that

(i) $h$ is a well-defined homomorphism;
(ii) $h$ is surjective;
(iii) *Bonus:* Show that $\ker h$ is the commutator subgroup of $\pi_1(X)$, that is $[\pi_1(X),\pi_1(X)]$.