Exercise sheet No. 10

Deadline: 13/01/2020.

You should hand only exercises 0.2 and 0.4.

In the last lecture, we discussed the following proposition:

Proposition 0.1. *Let $\mathcal{R}$ be either $\mathbb{R}$ or $\mathbb{Z}$. Let X be a topological space admitting the universal covering and let $\Gamma = \pi_1(X)$. Let us consider the Γ-complex*

$$0 \longrightarrow \mathcal{R} \underset{\varepsilon}{\overset{k^0}{\rightleftarrows}} C_b^0(\tilde{X};\mathcal{R}) \underset{\delta^0}{\overset{k^1}{\rightleftarrows}} C_b^1(\tilde{X};\mathcal{R}) \underset{\delta^1}{\overset{k^2}{\rightleftarrows}} \cdots ,$$

where $\varepsilon(a)(\sum_{i=1}^s b_i\sigma_i^0) = a\sum_{i=1}^k b_i$. Then, if $\pi_i(X) = 0$ for every $i = 2, \cdots, n$, there exists a partial *contracting homotopy*

$$\mathcal{R} \xleftarrow{k^0} C_b^0(\tilde{X};\mathcal{R}) \xleftarrow{k^1} C_b^1(\tilde{X};\mathcal{R}) \xleftarrow{k^2} \cdots C_b^n(\tilde{X};\mathcal{R}) \xleftarrow{k^{n+1}} C_b^{n+1}(\tilde{X};\mathcal{R}) .$$

In particular, if X is aspherical (i.e. $\pi_i(X) = 0$ for all $i \geq 2$), we have that the resolution above is relatively injective and strong.

Exercise 0.2 (5 points)**.** Provide a complete proof of Proposition 0.1.

Definition 0.3. Let Γ be a discrete group and let $\mathcal{R}$ be either $\mathbb{R}$ or $\mathbb{Z}$.

A Γ-module V over $\mathcal{R}$ is *injective* if the following holds: whenever A, B are Γ-modules, $\iota\colon A \to B$ is an injective Γ-map and $\alpha\colon A \to V$ is a Γ-map, there exists a Γ-map $\beta\colon B \to V$ such that $\beta \circ \iota = \alpha$.

A Γ-map $\iota\colon A \to B$ between Γ-modules over $\mathcal{R}$ is *strongly injective* if there exists an $\mathcal{R}$-linear map $\sigma\colon B \to A$ such that $\sigma \circ \iota = Id_A$.

A Γ-module V over $\mathcal{R}$ is *injective* if the following holds: whenever A, B are Γ-modules, $\iota\colon A \to B$ strongly injective Γ-map and $\alpha\colon A \to V$ is a Γ-map, there exists a Γ-map $\beta\colon B \to V$ such that $\beta \circ \iota = \alpha$

Exercise 0.4 (5 points)**.** Let Γ be a discrete group and let $\mathcal{R}$ be either $\mathbb{R}$ or $\mathbb{Z}$. Show that

(i) The Γ-module $C^k(\Gamma, \mathcal{R})$ over $\mathcal{R}$ may be not injective;
(ii) Conclude that relative injectivity is weaker than injectivity in general;
(iii) What about $\mathcal{R} = \mathbb{R}$?
(iv) *Bonus:* Discuss the (possible) relations between relatively injectivity of Γ-modules and relatively injectivity of *normed* Γ-modules (according to the definition given in the last lecture).

Exercise 0.5. Show that a group Γ is amenable if and only if the trivial normed Γ-module $\mathbb{R}$ over $\mathbb{R}$ is relatively injective (according to the definition we discussed in the last lecture).

Hints:($\Rightarrow$)*: It may be easier to prove that every dual normed Γ-modules over $\mathbb{R}$ is relatively injective.* ($\Leftarrow$)*: It may be useful to use the characterization of amenability in terms of the existence of an invariant continuous non-trivial functional on $\ell^\infty(\Gamma)$.*