Exercise sheet No. 11

Deadline: 20/01/2020.
You should hand only exercises 0.1 and 0.2.
Here $\mathcal{R}$ is either $\mathbb{Z}$ or $\mathbb{R}$.

Exercise 0.1 (5 points)**.** Show that $H_b^3(\mathbb{S}^2; \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$. Hence, the bounded cohomology of simply connected spaces with with integral coefficients in not trivial (in general) in positive degree.

Exercise 0.2 (5 points)**.** Let V be a normed Γ-module over $\mathcal{R}$ and let $(V, V^\bullet, \delta^\bullet)$ be any strong resolution of V. Then, the identity over V can be extended to a chain map $\alpha^\bullet$ between $V^\bullet$ and the standard resolution of V in such a way that $\|\alpha^n\| \leq 1$ for every $n \geq 0$.

As we mentioned in the lecture the extension can be constructed as follows: for every $v \in V^n$ and $g_0, \cdots, g_n \in \Gamma$ we set

$$\alpha^n(v)(g_0, \cdots, g_n) = \alpha^{n-1}(g_0(k^n(g_0^{-1}(v))))(g_1, \cdots, g_n) \ .$$

(i) Prove that it is a chain Γ-map (recall that we have already proved during the lecture that it is norm non-increasing).

Exercise 0.3. Let V be a relatively injective normed Γ-module over $\mathcal{R}$, let $j \colon A \to B$ be a Γ-map (i.e. bounded and Γ-equivariant) and suppose that there exists an $\mathcal{R}$-linear map $\sigma \colon B \to A$ such that $\|\sigma\| \leq 1$ and $j \circ \sigma \circ j = j$. Let also $\alpha \colon A \to V$ be a Γ-map and suppose that $\ker(j) \subset \ker(\alpha)$.

(i) Prove that there exists a Γ-map $\beta \colon B \to V$ such that $\beta \circ j = \alpha$ and $\|\beta\| \leq \|\alpha\|$.

Exercise 0.4. Let $\alpha \colon E \to F$ be a Γ-map between normed Γ-modules over $\mathcal{R}$. Let $(E, E^\bullet, \delta_E^\bullet)$ be a strong resolution of E and suppose that $(F, F^\bullet, \delta_F^\bullet)$ is relatively injective resolution of F. Then, α extends to a chain Γ-map $\alpha^\bullet$ (as we proved in the lecture).

(i) Prove that any two such extensions of α to chain maps are Γ-homotopic.