EXERCISE SHEET NO. 13

Deadline: 03/02/2020.

You don't have to hand any of the following exercise. However, you might get 5 bonus points for each of them.

Theorem 0.1 (Gromov's Proportionality Principle). *Let M and N be n-manifolds which share the same Riemannian universal covering. Then,*

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)} \ .$$

Exercise 0.2 (10 bonus points). Prove Gromov's Proportionality Principle 0.1 in the case in which the isometry group of $\widetilde{M} \cong \widetilde{N}$ is discrete.

If X is a topological space and $\mathcal{S}_n(X)$ is the space of singular n-simplices in X, we endow $\mathcal{S}_n(X)$ with the compact open topology (i.e. that topology whose basis is given by the set of functions sending a compact set in Δ^n to an open set of X).

Exercise 0.3 (5 bonus points). Let X be a metric space and let $\mathcal{S}_n(X)$ denote the space of singular n-simplices in X, then the compact-open topology on $\mathcal{S}_n(X)$ coincides with the topology induced by the metric of uniform convergence, i.e.

$$d(s_1, s_2) \coloneqq \sup\{d(s_1(x), s_2(x)) \,|\, x \in \Delta^n\} \ ,$$

for every s_1 and $s_2 \in \mathcal{S}_n(X)$.

Exercise 0.4 (10 bonus points). Given two topological spaces X and Y we denote by $\mathcal{C}(X, Y)$ the space of continuous functions from X to Y endowed with the compact-open topology. Prove that

(i) If $f \colon X \to Y$ and $g \colon Y \to Z$ are continuous maps between topological spaces, then also the induced maps

$$f^* \colon \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z) \ ,$$

and

$$g_* \colon \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

are so.

Notice that

(ii) There exists a bijection between the set of real singular cochains $C^n(X; \mathbb{R})$ and the set of functions $\{f \colon \mathcal{S}_n(X) \to \mathbb{R}\}$.

We say that a cochain $\varphi \in C^n(X; \mathbb{R})$ is *continuous* if the corresponding function $f \colon \mathcal{S}_n(X) \to \mathbb{R}$ is so. Let $C^n_c(X; \mathbb{R})$ denote the space of *continuous cochains on X*. Prove that

(iii) The cochain complex $(C^\bullet_c(X; \mathbb{R}), \delta^\bullet)$ endowed with the usual coboundary operator is well-defined.

Conclude that we have the following induced maps

(iv) $H^\bullet_{b,c}(X;\mathbb{R}) \to H^\bullet_b(X;\mathbb{R})$ and $H^\bullet_c(X;\mathbb{R}) \to H^\bullet(X;\mathbb{R})$, where $H^\bullet_{b,c}$ and $H^\bullet_c$ denote the *continuous bounded cohomology* (i.e. the cohomology of the continuous and bounded singular complex) and the *continuous cohomology* (i.e. the cohomology of the continuous singular complex).