Exercise sheet No. 2

Deadline: 28/10/2019.
You should hand only exercises 0.2 and 0.3.
Let as assume that $\mathcal{R}$ is either $\mathbb{Z}$ or $\mathbb{R}$.

Definition 0.1. Let X be a topological space. We define the *Kronecker product* to be the following bilinear form:

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon H^n(X;\mathcal{R}) \times H_n(X;\mathcal{R}) &\to \mathcal{R} \\ (\psi, \alpha) &\mapsto \langle \psi, \alpha \rangle = \varphi(c) \ , \end{aligned}$$

where $\psi = [\varphi]$ and $\alpha = [c]$.

Exercise 0.2 (5 points)**.** Show that

(i) The Kronecker product is a well-defined bilinear form.

Moreover,

(ii) if X is path-connected, discuss the relation between Kronecker product and cap product.

Recall that singular cohomology satisfies exactness axiom. This means that given a pair of spaces (X, A) we have that the short exact sequence

$$0 \to C^\bullet(X, A; \mathcal{R}) \to C^\bullet(X; \mathcal{R}) \to C^\bullet(A; \mathcal{R}) \to 0$$

induces a long exact sequence in cohomology

$$0 \to H^0(X, A; \mathcal{R}) \to H^0(X; \mathcal{R}) \to H^0(A; \mathcal{R}) \to H^1(X, A; \mathcal{R}) \to \cdots$$

Exercise 0.3 (5 points)**.** Let X be a topological space with k path-connected components. Using the definition of cohomology prove that

(i) $H^0(X;\mathcal{R}) \cong \mathcal{R}^k$.

Let us denote by $\mathbb{D}^n$, $n \geq 1$ be the n-dimensional disk. Let $X \subseteq \mathbb{D}^n$ be a subspace. Using only the Eilenberg-Steenrod axioms and the previous result, compute the following relative cohomology groups

(ii) $H^i(\mathbb{D}^n, X; \mathcal{R})$, $i \geq 0$ in terms of the ones of X.

Exercise 0.4. Let A be a retract of X. Then, show that

$$H^k(X;\mathcal{R}) \cong H^k(X, A;\mathcal{R}) \oplus H^k(A;\mathcal{R}) \ .$$

Exercise 0.5. Let X be a path-connected topological space. The aim of this exercise is to show that there exists a well-defined isomorphism

$$H^1(X;\mathcal{R}) \to \mathrm{Hom}(\pi_1(X), \mathcal{R}) \ .$$

Here, we interpret a cocycle $\varphi \in C^1(X;\mathcal{R})$ as a function from paths in X to $\mathcal{R}$. To this end it is convenient to prove the followings:

(i) $\varphi(\gamma * \eta) = \varphi(\gamma) + \varphi(\eta)$, where γ and η are to consecutive paths in X and $*$ denotes the concatenation of paths;

(ii) $\varphi(c) = 0$ for all constant paths c in X;

(iii) $\varphi(\gamma) = \varphi(\eta)$ if γ and η are homotopic relative to their endpoints;

(iv) φ is a coboundary if and only if $\varphi(\gamma)$ only depends on the endpoints of γ;
(v) $\mathrm{Hom}(\pi_1(X), \mathcal{R}) \cong \mathrm{Hom}(H_1(X; \mathbb{Z}), \mathcal{R})$.